The Hirota equation and its reductions from the point of view of root lattices

Adam Doliwa
doliwa@matman.uwm.edu.pl

Faculty of Mathematics and Computer Science
University of Warmia and Mazury (Olsztyn, Poland)

Integrable Systems in Newcastle
Newcastle upon Tyne, 26–27 September, 2014
The affine Weyl group symmetry of Desargues maps
- The $A_N$ root lattice and its affine $W(A_N)$ Weyl group
- Desargues maps of the $Q(A_N)$ root lattice
- The non-commutative Hirota system

Planar quadrilaterals lattices and their reductions
- The quadrilateral lattice
- $B$ and $C$ reductions of the Hirota system

Periodic reduction of Desargues maps
- Gel’fand–Dikii systems
- Yang–Baxter maps
- Self-similarity $(2, 2)$ reduction to $q - P_{VI}$
The affine Weyl group symmetry of Desargues maps

The $A_N$ root lattice and its affine $W(A_N)$ Weyl group
Desargues maps of the $Q(A_N)$ root lattice
The non-commutative Hirota system

Planar quadrilaterals lattices and their reductions

The quadrilateral lattice
$B$ and $C$ reductions of the Hirota system

Periodic reduction of Desargues maps

Gel’fand–Dikii systems
Yang–Baxter maps
Self-similarity $(2, 2)$ reduction to $q - P_{VI}$
The affine Weyl group symmetry of Desargues maps

The $A_N$ root lattice and its affine $W(A_N)$ Weyl group
The affine Weyl group symmetry of Desargues maps

The $A_N$ root lattice and its affine $W(A_N)$ Weyl group

Adam Doliwa (UWM Olsztyn)

The Hirota equation and root lattices

26–27 September, 2014

5 / 43
The $A_N$ root lattice

$Q(A_N)$ is the lattice generated by vectors along the edges of regular $N$-simplex. If we take the vertices of the simplex to be the vectors of the canonical basis in $\mathbb{R}^{N+1}$

$$e_i = (0, \ldots, 1, \ldots, 0), \quad 1 \leq i \leq N + 1$$

then the generators are

$$\varepsilon^i_j = e_i - e_j, \quad 1 \leq i \neq j \leq N + 1$$

$$Q(A_N) = \{ (n_1, \ldots, n_{N+1}) \in \mathbb{Z}^{N+1} | n_1 + \cdots + n_{N+1} = 0 \}$$

$\alpha_i = e_i - e_{i+1}$ - simple root vectors

$$(\omega_i | \alpha_j) = \delta_{ij}, \quad i, j = 1, \ldots, N$$

$\omega_i$ - fundamental weights
Tiles (Delunay polytopes) of the $A_N$ root lattice

Holes - points locally maximally distant from the lattice
Delaunay polytope - convex hull of the lattice points closest to the hole

The Delaunay polytopes of $Q(A_N)$: $P(k, N)$, $k = 1, \ldots, N$
truncations of order $k - 1$ of the regular $N$-simplex

$\omega_k + Q(A_N)$ - centers of tiles of type $P(k, N)$
The $A_N$ Weyl group

The Weyl group $W_0(A_N)$ is generated by the reflections $r_i$

$$r_i : \mathbf{v} \mapsto \mathbf{v} - \frac{2(\mathbf{v}|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i, \quad i = 1, \ldots, N$$

$W_0(A_N) \equiv S_{N+1}$, where $r_i$ is identified with transposition $\sigma_i = (i, i + 1)$
The affine Weyl group symmetry of Desargues maps

The $A_N$ affine Weyl group

The affine Weyl group $W(A_N)$ is generated by the reflections $r_i$, $1 \leq i \leq N$, and by the affine reflection $r_0$

$$r_0 : v \mapsto v - \left(1 - \frac{2(v|\tilde{\alpha})}{(\tilde{\alpha}|\tilde{\alpha})}\right)\tilde{\alpha}$$

$$\tilde{\alpha} = -\alpha_0 = \alpha_1 + \cdots + \alpha_N = e_1 - e_{N+1} \text{ - the highest root vector}$$

$$W(A_N) = Q(A_N) \rtimes W_0(A_N)$$

Theorem (Coxeter)

The affine Weyl group acts on the Delaunay tiling by permuting tiles within each class $P(k, N)$. 
Definition of Desargues maps

Maps $\phi : Q(A_N) \rightarrow \mathbb{P}^M$ such that for any translate of the $N$-simplex its vertices are mapped into collinear points \[ \text{[AD 2011]} \]

By the Coxeter theorem we have

Theorem

If $\phi : Q(A_N) \rightarrow \mathbb{P}^M$ is a Desargues map then for an arbitrary $w \in W(A_N)$ acting on $Q(A_N)$ the map $\phi \circ w$ is a Desargues map.
Identify $\mathbb{Z}^N = \sum_{i=1}^{N} \mathbb{Z}\varepsilon_i^{N+1} = Q(A_N)$

Maps $\phi : \mathbb{Z}^N \to \mathbb{P}^M$, such that the points $\phi(n)$, $\phi(i)(n)$ and $\phi(j)(n)$ are collinear, for all $n \in \mathbb{Z}^N$, $i \neq j$

Notation: $\phi(i)(n_1, \ldots, n_i, \ldots, n_N) = \phi(n_1, \ldots, n_i + 1, \ldots, n_N)$

Observation: There are $N + 1$ equivalent choices of $\mathbb{Z}^N$ coordinates in $Q(A_N)$ (with fixed origin) respecting geometrically the Desargues map condition!
Linear problem for Desargues maps

Algebraic description in homogeneous coordinates $\Phi : \mathbb{Z}^N \rightarrow \mathbb{D}^{M+1}$

$$\Phi + \Phi_{(i)} A_{ij} + \Phi_{(j)} A_{ji} = 0, \quad i \neq j, \quad A_{ij} : \mathbb{Z}^N \rightarrow \mathbb{D}^\times$$

$\mathbb{D}$ – arbitrary division ring (skew field)
The first part of the Desargues map equations

\[ A_{ij}^{-1} A_{ik} + A_{kj}^{-1} A_{ki} = 1, \quad i, j, k \text{ distinct} \]
The Veblen configuration and the second part of the Desargues map equations

\[ A_{ik(j)} A_{jk} = A_{jk(i)} A_{ik}, \quad i, j, k \text{ distinct} \]
Four dimensional consistency of Desargues maps

Desargues theorem

In projective space two triangles are in perspective from a point if and only if they are in perspective from a line.

Desargues configuration is the image of $P(3, 4)$ cell

Remark: The four dimensional consistency of the Hirota–Miwa equation or the discrete Schwarzian KP equation has combinatorics of the Desargues configuration \[\text{[Schief 2009]}\]
Gauge transformations: $\Phi = \tilde{\Phi} F$, where $F : \mathbb{Z}^N \rightarrow \mathbb{D}^\times$ - gauge function results in $\tilde{A}_{ij} = F(i)A_{ij}F^{-1}$

One can find homogeneous coordinates such that $A_{ji} = -A_{ij} = U_{ij}^{-1}$

$$\Phi(i) - \Phi(j) = \Phi U_{ij}, \quad 1 \leq i \neq j \leq N,$$

Fact to remember

The gauge functions which do not change the structure of the above linear problem are characterized by the condition $F(i) = F(j)$ for all pairs of indices, i.e. $F$ is a function of $n_\sigma = n_1 + n_2 + \cdots + n_N$.

$$U_{ij} + U_{ji} = 0, \quad U_{ij} + U_{jl} + U_{li} = 0,$$

$$U_{li} U_{lj(i)} = U_{lj} U_{li(j)} \quad \Rightarrow \quad U_{ij} = \rho_i^{-1} \rho_i(j)$$

[Nimmo 2006]
The Hirota equation

When $\mathbb{D} = \mathbb{F}$ is commutative then the functions $U_{ij}$ can be parametrized in terms of a single potential $\tau : \mathbb{Z}^N \rightarrow \mathbb{F}$

$$U_{ij} = \frac{\tau \tau(ij)}{\tau(i) \tau(j)}, \quad 1 \leq i < j \leq N$$

The nonlinear system reads [Hirota 1981], [Miwa 1982]

$$\tau(i) \tau(jl) - \tau(j) \tau(il) + \tau(l) \tau(ij) = 0, \quad 1 \leq i < j < l \leq N$$

Remark: Other gauges lead to

- the non-commutative discrete mKP system [Nijhoff-Capel 1990]
- the generalized lattice spin system [Nijhoff-Capel 1990]
- non-commutative Schwarzian discrete KP system [Konopelchenko-Schief 2005]
Back to the root lattice

In the $N + 1$th sector $\mathbb{Z}^N = \sum_{j=1}^{N} \varepsilon_j^{N+1} = Q(A_N)$

$$\phi^{N+1}(n + \varepsilon_i^{N+1}) - \phi^{N+1}(n + \varepsilon_j^{N+1}) = \phi^{N+1}(n) U_{ij}^{N+1}(n), \quad 1 \leq i \neq j \leq N$$

$$U_{ij}^{N+1}(n) = \left[ \rho_i^{N+1}(n) \right]^{-1} \rho_i^{N+1}(n + \varepsilon_j^{N+1})$$

In the $i$th sector $\mathbb{Z}^N = \sum_{j=1,j \neq i}^{N+1} \varepsilon_j = Q(A_N)$
The "rotated" linear problems

Theorem

The functions $\phi^i : \mathbb{Z}^N = \sum_{j=1, j \neq i}^{N+1} \mathbb{Z} \varepsilon_j \rightarrow \mathbb{D}^{M+1}_*$ given by

$$
\phi^i(n) = (-1)^{|n| \varepsilon_i N + 1} \phi^{N+1}(n) \left[ \rho^{N+1}_i(n) \right]^{-1}
$$

satisfy the linear system

$$
\phi^i(n + \varepsilon_j) - \phi^i(n + \varepsilon_k) = \phi^i(n) U^i_{jk}(n), \quad i, j, k \quad \text{distinct},
$$

where

$$
U^i_{jk}(n) = \left[ \rho^i_j(n) \right]^{-1} \rho^i_j(n + \varepsilon_k),
$$

$$
\rho^i_j(n) = \begin{cases} 
\rho^N_j(n) \left[ \rho^{N+1}_i(n) \right]^{-1}, & j \neq N + 1, \\
\left[ \rho^{N+1}_i(n) \right]^{-1}, & j = N + 1.
\end{cases}
$$
Outline

1. The affine Weyl group symmetry of Desargues maps
   - The $A_N$ root lattice and its affine $W(A_N)$ Weyl group
   - Desargues maps of the $Q(A_N)$ root lattice
   - The non-commutative Hirota system

2. Planar quadrilaterals lattices and their reductions
   - The quadrilateral lattice
   - $B$ and $C$ reductions of the Hirota system

3. Periodic reduction of Desargues maps
   - Gel’fand–Dikii systems
   - Yang–Baxter maps
   - Self-similarity $(2, 2)$ reduction to $q - P_{VI}$
Planar quadrilaterals lattices and their reductions

Adam Doliwa (UWM Olsztyn)

The Hirota equation and root lattices

26–27 September, 2014
Embedding of $B_K$ into $A_{2K-1}$

**Algebraic description of $Q(A_N)$**

$(e_i)_{i=1}^{N+1}$ – the standard orthonormal basis of $\mathbb{R}^{N+1}$

$\mathbb{R}^{N+1} \supset Q(A_N) \ni \sum_{i=1}^{N+1} x_i e_i$, $x_i \in \mathbb{N}$, such that $x_1 + x_2 + \cdots + x_{N+1} = 0$

$\varepsilon_i = e_{N+1} - e_i$, $i = 1, \ldots, N$ a parallelogram basis of $Q(A_N)$

Fix $N = 2K - 1$ then the vectors $E_i = e_{2i-1} - e_{2i}$, $i = 1, \ldots, K$ satisfy $(E_i|E_j) = 2\delta_{ij}$ and generate the $\mathbb{Z}^K = Q(B_K)$ sub-lattice in $Q(A_{2K-1})$

$$
2K-1 \sum_{i=1}^{2K-1} n_i \varepsilon_i = -\sum_{j=1}^{K} m_j E_j + \sum_{j=1}^{K} \ell_j e_{2j}, \quad \ell = \sum_{j=1}^{K} \ell_j e_{2j} \in Q(A_{K-1})
$$

$m$ — quadrilateral lattice variables, $[i]$ — shift by $E_i$

$\ell$ — Laplace transformation variables
Planar quadrilaterals lattices and their reductions

Discrete Darboux equations

Fix $\ell \in Q(A_{K-1})$ define $\psi^\ell : \mathbb{Z}^K \to \mathbb{P}^M$ by $\psi^\ell(m) = \phi(n)$

- the points $\psi^\ell$, $\psi^\ell_{[i]}$, $\psi^\ell_{[j]}$, and $\psi^\ell_{[ij]}$ are coplanar
- the functions $\beta^\ell_{ij} = \text{sgn}(j - i) \left( \frac{\tau^\ell + e^{2i} - e^{2j}}{\tau^\ell} \right)_{[j]}$, $i \neq j$, satisfy the discrete Darboux equations [Bogdanov, Konopelchenko 1995]

$$
\beta^\ell_{ij}[k] = \beta^\ell_{ij} + \beta^\ell_{ik}[j] \beta^\ell_{kj}, \quad i, j, k \text{ distinct}
$$

$$
\tau^\ell \tau^\ell + e_{2i} - e_{2j} = \tau^\ell \tau^\ell + e_{2i} - e_{2j} + \text{sgn}(j - i) \text{sgn}(k - j) \text{sgn}(i - k) \tau^\ell \tau^\ell + e_{2k} - e_{2j}
$$
Quadrilateral lattice is a map $\psi : \mathbb{Z}^K \to \mathbb{P}^M(\mathbb{D})$, $2 \leq K \leq M$, whose all elementary quadrilaterals are planar.

- 2D lattices of planar quadrilaterals — discrete conjugate nets $\text{[Sauer 1937]}$
- Laplace sequence of 2D discrete conjugate nets — geometric interpretation of the Hirota–Miwa equation in the 2D discrete Toda system form $\text{[AD 1997]}$
- Multidimensional quadrilateral lattices — geometric interpretation of the discrete Darboux equations $\text{[AD, Santini 1997]}$
- Laplace transformations of generic $K$-dimensional quadrilateral lattices are parametrized by points of the root lattice $Q(A_{K-1})$ $\text{[AD, Mañas, Martínez Alonso, Medina, Santini 1999]}$
- FCC = $Q(A_3)$ description of 2D quadrilateral lattice and its Laplace sequence $\text{[Schief 2007]}$
(20_3, 15_4) configuration as the image of $P(3, 5)$ cell, and the quadrilateral lattice construction.
The discrete $C$-KP system

**Problem**

Find constraints on $\tau$ which result in a single equation involving fixed $\ell$

\[
\tau^\ell C + e_{2i} - e_{2j} + \tau^\ell C + e_{2j} - e_{2i} = 0, \quad i \neq j,
\]

[AD, Santini 2000]

\[
(\tau[i] \tau[jk] - \tau[j] \tau[ik] + \tau[k] \tau[ij] - \tau^2 \tau[ijk])^2 - 4 (\tau[i] \tau[jk] \tau[k] \tau[ij] + \tau[j] \tau[ik] \tau^2 \tau[ijk]) + 4 \tau[i] \tau[j] \tau[k] \tau[ijk] + 4 \tau^2 \tau[i] \tau[jk] \tau[ik] = 0
\]

$\tau = \tau^\ell C$

[Kashaev 1996], [Schief 2003]

**Remark**

"Half" of the discrete KP variables is fixed
The discrete $B$-KP system

\[
\left( \tau_{[ij]}^\ell B + e_2^i - e_2^j - \tau_{[ij]}^\ell B + e_2^j - e_2^i \right)^2 = 4 \tau_{[i]}^\ell B \tau_{[j]}^\ell B, \quad i \neq j \tag{\star}
\]

\[
\left[ (\tau_{[ij]} \tau_{[jk]} - \tau_{[i]} \tau_{[ik]} + \tau_{[k]} \tau_{[ij]} - \tau \tau_{[ijk]})^2 - 4 \left( \tau_{[i]} \tau_{[jk]} \tau_{[k]} \tau_{[ij]} + \tau_{[j]} \tau_{[ik]} \tau \tau_{[ijk]} \right) \right]^2 = 64 \tau_{[i]} \tau_{[j]} \tau_{[k]} \tau_{[ij]} \tau_{[jk]} \tau_{[ik]} \tau_{[ijk]}, \quad \tau = \tau^\ell B \tag{\star\star}
\]

**Proposition**

One can consistently parametrize (\star) by $\mu : \mathbb{Z}^K \rightarrow \mathbb{F}$ such that $\mu^2 = \tau^\ell B$

\[
\begin{align*}
\tau_{[ij]}^\ell B + e_2^i - e_2^j &= -(-1)^{\sum_{i \leq k < j} m_k} (\mu \mu_{[ij]} + \mu_{[i]} \mu_{[j]}) \quad i < j \\
\tau_{[ij]}^\ell B + e_2^j - e_2^i &= -(-1)^{\sum_{i \leq k < j} m_k} (\mu \mu_{[ij]} - \mu_{[i]} \mu_{[j]})
\end{align*}
\]

Then equation (\star\star) gives \[\text{[Miwa 1982]}\]

\[
\mu \mu_{[ijk]} = \mu_{[i]} \mu_{[jk]} - \mu_{[j]} \mu_{[ik]} + \mu_{[k]} \mu_{[ij]}, \quad i < j < k
\]
Outline

1. The affine Weyl group symmetry of Desargues maps
   - The $A_N$ root lattice and its affine $W(A_N)$ Weyl group
   - Desargues maps of the $Q(A_N)$ root lattice
   - The non-commutative Hirota system

2. Planar quadrilaterals lattices and their reductions
   - The quadrilateral lattice
   - $B$ and $C$ reductions of the Hirota system

3. Periodic reduction of Desargues maps
   - Gel’fand–Dikii systems
   - Yang–Baxter maps
   - Self-similarity $(2, 2)$ reduction to $q - P_{VI}$
Periodic reduction of Desargues maps

The orthogonal projection of $Q(A_{N+1}) \subset \mathbb{E}^{N+1}$ onto the hyperplane of $Q(A_N)$ gives the weight lattice $P(A_N)$.
The non-commutative KP hierarchy

Replace $N \to N + 1$, and distinguish the last variable $k = n_{N+1}$, denote also

$$n = (n_1, \ldots, n_N), \quad \Phi(n, k) = \Psi_k(n), \quad U_{N+1,i}(n, k) = u_{i,k}(n)$$

which allows the rewrite a part (that with the distinguished variable) of the linear problem in the form

$$\Psi_{k+1} - \Psi_{k(i)} = \Psi_k u_{i,k}, \quad i = 1, \ldots, N.$$ 

[Kajiwara, Noumi, Yamada 2002]

The compatibility of the above linear system reads

$$u_{j,k} u_{i,k(j)} = u_{i,k} u_{j,k(i)}, \quad i \neq j,$$

$$u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}.$$ 

The first part allows to define potentials $r_k(n) = \rho_{N+1}(n, k)$ such that $u_{i,k} = r_{k}^{-1}(n) r_{k(i)}$, while the other equations give the system

$$(r_{k(j)}^{-1} - r_{k(i)}^{-1}) r_{k(i)} = r_{k+1}^{-1}(r_{k+1(i)} - r_{k+1(j)}), \quad i \neq j.$$
Periodic Desargues maps: $\phi_{k+P}(n) = \phi_k(n)$

\[
\Psi_{k+P}(n) = \Psi_k(n) \mu_k(n), \quad \mu_{k+1}(n) = \mu_{k(i)}(n), \quad r_{k+P} = r_k \mu_k
\]

Matrix linear problem

\[
\begin{pmatrix}
-u_{i,1} & 0 & \cdots & 0 & \mu_1 \\
1 & -u_{i,2} & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -u_{i,P-1} & 0 \\
0 & 0 & \cdots & 1 & -u_{i,P}
\end{pmatrix}
\]

where $\mu_1$ is a function of the variable $n_{\sigma} = n_1 + \cdots + n_N$.

The corresponding (lattice non-isospectral non-commutative modified Gel’fand–Dikii) system of non-linear equations

\[
(r_{k(j)}^{-1} - r_{k(i)}^{-1}) r_{k(ij)} = r_{k+1}^{-1} (r_{k+1(i)} - r_{k+1(j)}), \quad k = 1, \ldots, P - 1,
\]

\[
(r_{P(j)}^{-1} - r_{P(i)}^{-1}) r_{P(ij)} = \mu_1^{-1} r_1^{-1} (r_1(i) - r_1(j)) \mu_1(\sigma) \quad i \neq j.
\]

Comutative and iso-spectral case \[Nijhoff, Papageorgiou, Capel, Quispel 1992\]
Three dimensional consistency of the GD systems

\[ r = (r_k) \text{ where } k \in \mathbb{Z}/(P\mathbb{Z}) \text{ – periodic case, or } k \in \mathbb{Z} \text{ in the full KP case} \]

Multidimensional consistency of a discrete system — possibility of extending the number of independent variables of the system by adding its copies in different directions

Fact

The lattice non-isospectral non-commutative modified Gel’fand–Dikii system is three-dimensionally consistent.
Multidimensional consistency of the KP map

Theorem

The non-commutative KP map (edge system $u_{i,k} = r_k^{-1} r_k(i)$)

$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1} u_{i,k}(u_{i,k+1} - u_{j,k+1}), \quad 1 \leq i \neq j \leq N,$$

is multidimensionaly consistent

$$u_{i} = (u_{i,k}, k \in \mathbb{Z} \text{ or } k \in \mathbb{Z}/(P\mathbb{Z}), \quad u_{i,k+p} = \mu_k^{-1} u_{i,k} \mu_k(i)$$
A map \( R: \mathcal{X} \times \mathcal{X} \) is called Yang–Baxter map if

\[
R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}, \quad \text{in} \quad \mathcal{X} \times \mathcal{X} \times \mathcal{X}
\]

If moreover \( \pi \circ R \circ \pi \circ R = \text{Id}_{\mathcal{X} \times \mathcal{X}} \), where \( \pi \) is the transposition, then \( R \) is called reversible YB map.

\[
\begin{array}{c}
u_{i(j)} = \tilde{x} \\
u_{i(i)} = y \\
u_{j} = \tilde{y} \\
u_{i} = x
\end{array}
\]

\[
\begin{array}{c}
R_{12} \\
R_{13} \\
R_{23}
\end{array}
\]

\[
\begin{array}{c}
\text{=}
\end{array}
\]

\[
\begin{array}{c}
R_{12} \\
R_{13} \\
R_{23}
\end{array}
\]
Non-commutative rational Yang–Baxter maps

Theorem

Given two assemblies of non-commuting variables \( \mathbf{x} = (x_1, \ldots, x_P) \), \( \mathbf{y} = (y_1, \ldots, y_P) \) define polynomials

\[
P_k = \sum_{a=0}^{P-1} \left( \prod_{i=0}^{a-1} y_{k+i} \prod_{i=a+1}^{P-1} x_{k+i} \right), \quad k = 1, \ldots, P,
\]

where subscripts in the formula are taken modulo \( P \). If the products \( \alpha = x_1 x_2 \ldots x_P \) and \( \beta = y_1 y_2 \ldots y_P \) are central then the map

\[
R(\mathbf{x}, \mathbf{y}) = (\tilde{x}, \tilde{y}), \quad \tilde{x}_k = P_k x_k P_{k+1}^{-1}, \quad \tilde{y}_k = P_k^{-1} y_k P_{k+1},
\]

is reversible Yang–Baxter map

commutative case [Kajiwara, Noumi, Yamada 2001], [Etingov 2003]
Fact

The products $\alpha$ and $\beta$ are conserved (for arbitrary $P$)

The simplest case: $P = 2$ we put $x = x_1$, $y = y_1$ to get a parameter dependent reversible Yang–Baxter map $R(\alpha, \beta) : (x, y) \mapsto (\tilde{x}, \tilde{y})$

$$
\tilde{x} = \left(\alpha x^{-1} + y\right) x \left(x + \beta y^{-1}\right)^{-1}, \\
\tilde{y} = \left(\alpha x^{-1} + y\right)^{-1} y \left(x + \beta y^{-1}\right),
$$

which in the commutative case is equivalent to the $F_{III}$ map in the list of [Adler, Bobenko, Suris 2004]
Non-commutative Gel’fand–Dikii systems with centrality assumptions

Proposition

In the $P$-periodic reduction $u_{i,k+P} = \mu_k^{-1} u_{i,k} \mu_k(i)$ of the non-commutative KP system assume centrality of the monodromy factors $\mu_k$ and of the products $U_i = u_{i,1} u_{i,2} \ldots u_{i,P} \mu_1^{-1}$. Then $U_i$ is a function of $n_i$ only.

In particular, for $P = 2$ we obtain the non-autonomous, non-isospectral lattice modified KdV equation for non-commutative variable $r = r_1$

$$\left( r_{(j)}^{-1} - r_{(i)}^{-1} \right) r_{(ij)} = \left( r_{(i)}^{-1} U_i - r_{(j)}^{-1} U_j \right) r_{\mu_1} \quad (\text{nc-ni-na-l-mKdV})$$

iso-spectral case [Bobenko, Suris 2002]
Self-similarity \((2, 2)\) reduction to \(q - P_{VI}\)

In nc-ni-na-l-mKdV take \(N = 2\), \(x_{(1122)} = x\)

\[
\frac{U_{i(ii)}}{U_i} = \frac{\mu}{\mu(\sigma\sigma\sigma\sigma)}, \quad i = 1, 2
\]

By separation of variables there exists a non-zero central constant \(q\)

\[
\mu(n_{\sigma}) = \alpha_k q^{n_{\sigma}}, \quad k = n_{\sigma} \mod 4,
\]

\[
U_i(n_i) = \beta_{i,k} q^{-2n_i}, \quad k = n_i \mod 2, \quad i = 1, 2,
\]

for certain non-zero parameters \(\alpha_k, \beta_{i,k}\)

Remark: We will need only \(\alpha_{k+2} = \alpha_k\)
The repeating pattern for $q$-$P_{VI}$

\[ w_n^0 = x(n_1, n_2 - 1), \quad w_n^1 = x(n_1, n_2), \quad w_n^2 = x(n_1 + 1, n_2), \quad w_n^3 = x(n_1 + 1, n_2 + 1) \]

\[
\begin{align*}
&f_n = \frac{1}{q^{\alpha_0} \sqrt{\beta_{1,0} \beta_{2,0}}} w_n^0 (w_n^2)^{-1} U_1(n_1) \mu(n_\sigma), \\
g_n = \frac{1}{\alpha_0 \sqrt{\beta_{1,0} \beta_{2,1}}} w_n^1 (w_n^3)^{-1} U_2(n_2) \mu(n_\sigma)
\end{align*}
\]
A non-commutative $q$-$P_{VI}$ system

$$t_n = t_0 \lambda^n, \quad \lambda = q^4, \quad t_0 = \sqrt[\beta_{1,0}\beta_{1,1}]{\beta_{2,0}\beta_{2,1}},$$

$$c_1 = \alpha_0 \sqrt[\beta_{1,1}\beta_{2,0}]{\beta_{1,0}\beta_{2,1}}, \quad c_2 = \alpha_0 \sqrt[\beta_{1,0}\beta_{2,1}]{\beta_{1,1}\beta_{2,1}}, \quad c_3 = \alpha_1 \sqrt[\beta_{1,1}\beta_{2,1}]{\beta_{1,0}\beta_{2,0}}, \quad c_4 = \alpha_1 \sqrt[\beta_{1,0}\beta_{2,0}]{\beta_{1,1}\beta_{2,1}}.$$

nc $q$-$P_{VI}$

$$f_{n+1} = \frac{g_n + t_n c_1^{-1}}{g_n + c_2^{-1}} f_n^{-1} \frac{g_n + t_n c_1}{g_n + c_2}, \quad t_{n+1} = \lambda t_n,$$

$$g_{n+1} = \frac{f_{n+1} + t_n \sqrt[\lambda]{c_3^{-1}}}{f_{n+1} + c_4^{-1}} g_n^{-1} \frac{f_{n+1} + t_n \sqrt[\lambda]{c_3}}{f_{n+1} + c_4}.$$
we recalled (SIDE IX, Varna 2010) the A-type root lattice description of Desargues maps and of the Hirota equation

$K$ dimensional lattices of planar quadrilaterals can be described from the corresponding $Q(B_k) \subset Q(A_{2K-1})$ perspective

the discrete $C$-KP and $B$-KP equations were given as reductions of the discrete ($A$-)KP equation

periodicity in one direction of the lattice gives nc-ni-na-l-mGD systems and corresponding YB maps

self-similarity $(2,2)$ reduction of nc-ni-na-l-mKdV equation gives nc $q$-$P_{VI}$ system
References

THANK YOU!