Geometry of the noncommutative Darboux equations

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Outline

The Quadrilateral Lattice
   Geometric integrability
   The discrete Darboux equations
   The fundamental (binary Darboux) and Laplace transformations

Integrable reductions
   The B-(Moutard) quadrilateral lattice
   The cubic closest sphere packing (CCP) system
   The C-(symmetric) quadrilateral lattice
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Geometric Integrability Scheme

Given generic points \( x_0, x_1, x_2 \) and \( x_3 \) in a projective space, let \( x_{ij} \) be generic point of the plane \( \langle x_0, x_i, x_j \rangle \), \( 1 \leq i < j \leq 3 \).

Then there exists exactly one point \( x_{123} \) which belongs simultaneously to the planes \( \langle x_3, x_{13}, x_{23} \rangle \), \( \langle x_2, x_{12}, x_{23} \rangle \) and \( \langle x_1, x_{12}, x_{13} \rangle \).

Definition
A quadrilateral lattice is a map \( x : \mathbb{Z}^N \rightarrow \mathbb{D}^M, 3 \leq N \leq M \), whose all elementary quadrilaterals are planar.
Given generic points $x_0, x_1, x_2$ and $x_3$ in a projective space, let $x_{ij}$ be generic point of the plane $\langle x_0, x_i, x_j \rangle$, $1 \leq i < j \leq 3$.

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Division rings and the Desargues configuration

$(\mathbb{D}, +, 0, \cdot, 1)$ - division ring (skew field)

Theorem: In projective spaces over division rings the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point $O$ iff they are perspective from the line $l$. 
The vertices of the hexagon $ABCDEF$ lie alternately on two coplanar lines $k$ and $l$.

The three points $K$, $L$, $M$ of intersection of pairs of opposite sides are collinear.

**Theorem:** $[(i) \implies (ii)] \iff \mathbb{D}$ is commutative
Multidimensional consistency of the geometric integrability scheme
The linear problem

Notation:
\[ x^{(i)}(n_1, \ldots, n_i, \ldots, n_N) = x(n_1, \ldots, n_i + 1, \ldots, n_N), \]
\[ \Delta_i x = x^{(i)} - x. \]

\[ x^i_{(j)} = x^i P^{i j} + x^j Q^{i j}, \quad i \neq j \]

\[ P^{i j}, Q^{i j} : \mathbb{Z}^N \rightarrow \mathbb{D} \]
The discrete Darboux equations

The compatibility condition, \( i \neq j \neq k \neq i \)

\[
P^{ik} P^{ij}_{(k)} = P^{ij} P^{ik}_{(j)}, \tag{1}
\]

\[
P^{ik} Q^{ij}_{(k)} = Q^{ij} P^{ik}_{(j)} + Q^{kj} Q^{ik}_{(j)} \tag{2}
\]

Eqn. (1) allows to introduce potentials \( G^i : \mathbb{Z}^N \to \mathbb{D} \), such that \( P^{ij} = (G^i)^{-1} G^j_{(i)} \). Then in terms of \( \tilde{Q}^{ij} = G^i Q^{ij} (G^j_{(j)})^{-1} \) we have

\[
\tilde{Q}^{ij}_{(k)} = \tilde{Q}^{ij} + \tilde{Q}^{kj} \tilde{Q}^{ik}_{(j)}
\]

L. V. Bogdanov, B. G. Konopelchenko, 1995

which give

\[
\tilde{Q}^{ij}_{(k)} = (1 - \tilde{Q}^{kj} \tilde{Q}^{ik})^{-1} (\tilde{Q}^{ij} + \tilde{Q}^{kj} \tilde{Q}^{ik})
\]

\[
\Delta_j \tilde{X}^i = \tilde{X}^j \tilde{Q}^{ij}, \quad \text{where} \quad \tilde{X}^i = X^i (G^i)^{-1}
\]
The tetrahedron equation picture

\[ \psi'_\alpha = a\psi_\alpha + b\psi_\beta \]
\[ \psi'_\beta = c\psi_\alpha + d\psi_\beta \]

I. Korepanov, 1995
The tetrahedron equation picture

\[ \psi'_\alpha = a\psi_\alpha + b\psi_\beta \]
\[ \psi'_\beta = c\psi_\alpha + d\psi_\beta \]

I. Korepanov, 1995
\[ \Delta_j X^i = X^i Q^{ij}, \quad i \neq j, \quad (3) \]
\[ \Delta_k Q^{ij} = Q^{kj} Q^{ik}, \quad i \neq j \neq k \neq i, \quad (4) \]

In the natural continuous limit give the Darboux equations (describing multidimensional conjugate nets) and the corresponding linear problem

\[ \partial_j X^i = X^i q^{ij}, \quad i \neq j, \quad (5) \]
\[ \partial_k q^{ij} = q^{kj} q^{ik}, \quad i \neq j \neq k \neq i \quad (6) \]

G. Darboux, 1878
V. E. Zakharov, S. V. Manakov, 1985

There exist potentials \( \rho^i : \mathbb{Z}^N \rightarrow \mathbb{D} \)

\[ \rho^i_{(j)} = \rho^i (1 - Q^{ij} Q^{ji}), \quad i \neq j \]

which in the commutative case can be integrated once more

\[ \rho^i = \frac{\tau(i)}{\tau} \]
\( \Delta_j X^i = X^i Q^{ij}, \quad i \neq j, \)  
\( \Delta_k Q^{ij} = Q^{kj} Q^{ik}, \quad i \neq j \neq k \neq i \)  

in the **natural continuous limit** give the Darboux equations (describing multidimensional conjugate nets) and the corresponding linear problem

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There exist potentials \( \rho^i : \mathbb{Z}^N \to \mathbb{D} \)

\( \rho^i_{(j)} = \rho^i(1 - Q^{ji} Q^{jj}), \quad i \neq j \)

which in the **commutative case** can be integrated once more

\( \rho^i = \frac{\tau(i)}{\tau} \)
The vectorial fundamental transformation

Given the solution \( \xi^i : \mathbb{Z}^N \rightarrow \mathbb{D}^K \) of the linear problem
\[
\Delta_j \xi^i = \xi^j Q^{ij}, \quad i \neq j,
\]
and given the (row-vector) solution \( \xi^{*i} : \mathbb{Z}^N \rightarrow K \mathbb{D} \) of its adjoint
\[
\Delta_i \xi^{*j} = Q^{ij} \xi^{*i}, \quad i \neq j,
\]
they allow to construct matrix-valued potentials
\[
\Omega[\xi, \xi^{*}] : \mathbb{Z}^N \rightarrow M^K_K(\mathbb{D}) \quad \text{and} \quad \Omega[X, \xi^{*}] : \mathbb{Z}^N \rightarrow M^M_K(\mathbb{D})
\]
defined by
\[
\begin{align*}
\Delta_i \Omega[\xi, \xi^{*}] &= \xi^i \xi^{*i} \\
\Delta_i \Omega[X, \xi^{*}] &= X^i \xi^{*i}.
\end{align*}
\]

If \( \Omega[\xi, \xi^{*}] \) is invertible then the functions \( \hat{X}^i : \mathbb{Z}^N \rightarrow \mathbb{D}^M \)
\[
\hat{X}^i = X^i - \Omega[X, \xi^{*}] \Omega[\xi, \xi^{*}]^{-1} \xi^i
\]
provide a new solution of the linear problem with
\[
\hat{Q}^{ij} = Q^{ij} - \xi^{*j} \Omega[\xi, \xi^{*}]_{(j)}^{-1} \xi^{i}, \quad i \neq j.
\]
The Laplace transformation and the Toda system

The Laplace sequence of 2D quadrilateral lattices

\[ \ldots \rightarrow \mathcal{L}_H^{-1}(x) \rightarrow x \rightarrow \mathcal{L}_H(x) \rightarrow \ldots \mathcal{L}_H^k(x) \rightarrow \ldots \]

The corresponding functions \( Q^{(k)} = \mathcal{L}_H^k(Q^{12}) \) satisfy

\[
\Delta_2 \frac{\Delta_1 Q^{(k)}}{Q^{(k)}} = \left( \frac{Q^{(k)}_{(2)}}{Q^{(k-1)}} \right) - \frac{Q^{(k+1)}_{(2)}}{Q^{(k)}}, \quad k \in \mathbb{Z},
\]

which using the \( \tau \)-function (commutative case) is

\[
\tau^{(k)} \tau^{(k)}_{(12)} = \tau^{(k)}_{(1)} \tau^{(k)}_{(2)} - \tau^{(k-1)}_{(1)} \tau^{(k+1)}_{(2)}
\]

R. Hirota, 1981

In the continuous limit

\[
\partial_1 \partial_2 \theta^{(k)} = e^{\theta^{(k)} - \theta^{(k-1)}} - e^{\theta^{(k+1)} - \theta^{(k)}}, \quad q^{(k)} = e^{\theta^{(k)}}.
\]
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An integrable reduction of the quadrilateral lattice is a constraint which propagates during the construction of the lattice via the Geometric Integrability Scheme.

Example: The Circular Lattice \((\mathbb{D} = \mathbb{R})\)
When vertices of the initial quadrilaterals are concircular, then the same holds for the vertices of the resulting quadrilaterals.

(Miquel theorem)
The B-quadrilateral lattice

The following theorem holds iff the division ring $\mathbb{D}$ is commutative (Pappus)

Under hypotheses of the Geometric Integrability Scheme, assume that $x_0$, $x_{12}$, $x_{13}$ and $x_{23}$ are coplanar.

Then the points $x_1$, $x_2$, $x_3$ and $x_{123}$ are coplanar as well.

Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{F})$, is called the B-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the points $x$, $x_{(ij)}$, $x_{(ik)}$ and $x_{(jk)}$ are coplanar.
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The B-constraint implies existence of gauge such that the homogeneous coordinates \( \mathbf{x} : \mathbb{Z}^N \to \mathbb{F}^{M+1} \) satisfy the system of the discrete Moutard equations

\[
x_{(ij)} - \mathbf{x} = f^{ij} (\mathbf{x}(i) - \mathbf{x}(j)), \quad 1 \leq i < j \leq N,
\]

\textit{J. J. C. Nimmo, W. K. Schief, 1997}

whose compatibility gives Miwa’s discrete BKP equation

\[
\tau^B \tau^B_{(ijk)} = \tau^B_{(ij)} \tau^B_{(k)} - \tau^B_{(ik)} \tau^B_{(j)} + \tau^B_{(jk)} \tau^B_{(i)}, \quad 1 \leq i < j < k \leq N,
\]

\textit{T. Miwa, 1982} where

\[
f^{ij} = \frac{\tau^B_{(i)} \tau^B_{(j)}}{\tau^B_{(ij)} \tau^B_{(k)}}, \quad (\tau^B)^2 = \tau
\]
The diagonal staircase section

Black points of the staircase section - the triangular lattice

\[ V(\mathbb{T}) = \{(n_1, n_2, n_2) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 = 0\} \]

Points of the honeycomb lattice

\[ V(\mathbb{H}_\pm) = \{(n_1, n_2, n_2) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 = \pm 1\} \]
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Discrete self-adjoint linear problem on the T-lattice

The 3D linear discrete Moutard system can be restricted to the triangular lattice

\[ a(x_I - x) + a_{-I}(x_{-I} - x) + b(x_{II} - x) + b_{-II}(x_{-II} - x) + s_I(x_{I,-II} - x) + s_{II}(x_{-I,II} - x) = 0, \]

\[ a = \frac{1}{f^{12}}, \quad b = \frac{1}{f^{23}}, \quad s = -f_{-1-2-3}^{13}, \]

\[ (I) = (12), \quad \text{and} \quad (II) = (23). \]
Foliation of the "even" part of $\mathbb{Z}^3$ lattice into triangular lattices

$$V(\mathbb{T}_K) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 = 2K\}$$

The corresponding relations between the coefficients $a, b, s$ (the sublattice discrete BKP system) read

$$\frac{a^K}{q^K} = \frac{a_{-I,II}^{K+1}}{q_{-II}^{K+1}}, \quad \frac{b^K}{q^K} = \frac{b_{-I}^{K+1}}{r_{II}^{K+1}}, \quad \frac{s^K}{q^K} = \frac{s_{-I}^{K+1}}{r_{-II}^{K+1}},$$

where

$$q^K = a^K b^K + a^K s_{I,II}^K + b^K s_{I,II}^K, \quad r^K = a^K b^K_{-II} + a^K s^K_{-I} + b^K_{-II} s^K.$$
Cubic closest sphere packing system

Foliation of the "even" part of $\mathbb{Z}^3$ lattice into triangular lattices

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The corresponding relations between the coefficients $a, b, s$ (the sublattice discrete BKP system) read

$$\frac{a^K_{-\|}}{q^K_{-\|}} = \frac{a^{K+1}_{-\|-\|}}{r^{K+1}_{\|-\|}}, \quad \frac{b^K_{-\|}}{q^K_{-\|-\|}} = \frac{b^{K+1}_{\|}}{r^{K+1}_{\|}}, \quad \frac{s^K}{q^K_{-\|,-\|-\|}} = \frac{s^{K+1}_{-\|}}{r^{K+1}_{-\|-\|}},$$

where

$$q^K = a^K b^K + a^K s^K_{\|-\|} + b^K s^K_{\|}, \quad r^K = a^K b^K_{-\|} + a^K s^K + b^K_{-\|} s^K.$$
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Foliation of the "even" part of $\mathbb{Z}^3$ lattice into triangular lattices

$$V(\mathbb{T}_K) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 = 2K\}$$

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$$\frac{a^K_{-II}}{q^K_{-II}} = \frac{a^{K+1}_{-II, II}}{r^{K+1}_{II}}, \quad \frac{b^K_{-I}}{q^K_{-I}} = \frac{b^{K+1}}{r^{K+1}_{-II}}, \quad \frac{s^K}{q^K_{-I,-II}} = \frac{s^{K+1}_{-I}}{r^{K+1}_{-I}},$$

where

$$q^K = a^K b^K + a^K s^K_{l, II} + b^K s^K_{l, II}, \quad r^K = a^K_{-l} b^K_{-II} + a^K_{-l} s^K + b^K_{-II} s^K.$$
The discrete (elliptic) Laplace transformation

\( \mathcal{L}_E \) – transition from solution of the self-adjoint linear system on \( \mathbb{T}_K \) to the corresponding solution on \( \mathbb{T}_{K+1} \)

\[
\mathcal{L}_E(x^K) = \frac{1}{q^K_{-\|}} \left( b^K_{-\|} s^K_I x^K + a^K_{-\|} b^K_{-\|} x^K_{-\|} + a^K_{-\|} s^K_I x^K_{-\|, -\|} \right)
\]

\[
\mathcal{L}^{-1}_E(x^K) = \frac{1}{r^K_{\|}} \left( b^K s^K_{\|} x^K + a^K_{\|} b^K x^K_{\|} + a^K_{\|} s^K_{\|} x^K_{\|, \|} \right).
\]

The CCP system can be considered as the elliptic discrete Toda system
The CQL constraint
The CQL constraint
The C-quadrilateral lattice

Definition
A quadrilateral lattice \( x : \mathbb{Z}^N \rightarrow \mathbb{A}^M(\mathbb{F}) = \mathbb{P}^M(\mathbb{F}) \setminus H_\infty \), is called the **C-quadrilateral lattice** if for any triple of different indices \( 1 \leq i < j < k \leq N \) the three intersection points of the common lines of the opposite planes of the corresponding hexahedron with the hyperplane at infinity are collinear.
The discrete CKP equation

The 4D consistency of the CQL constraint is equivalent (again) to the Pappus configuration, i.e., it holds only when the division ring is commutative.

Algebraic characterization of the C-quadrilateral lattice

A (commutative) quadrilateral lattice is subject to the C-reduction if and only if its rotation coefficients satisfy the constraint

\[ Q^{ij} Q^{jk} Q^{ki} = Q^{kj} Q^{ik} Q^{ji}, \quad i, j, k \quad \text{distinct.} \]
The discrete CKP equation

After appropriate rescaling we have $\rho^j Q^{ij} = \rho^i Q^{ij}$, which in the continuous limit gives $q^{ij} = q^{ji}$.

In terms of the $\tau$-function we have the discrete CKP system

$$
\left( \tau \tau(ijk) - \tau(i) \tau(jk) - \tau(j) \tau(ik) - \tau(k) \tau(ij) \right)^2 =
4(\tau(i) \tau(j) \tau(ik) \tau(jk) + \tau(i) \tau(k) \tau(ij) \tau(jk) + \tau(j) \tau(k) \tau(ik) \tau(ij) -
\tau(i) \tau(j) \tau(k) \tau(ijk) - \tau(i) \tau(ik) \tau(jk) - \tau(k) \tau(ij) \tau(ik)),
\quad i, j, k \text{ distinct}.
$$

E. Date, M. Jimbo and T. Miwa, 1983

W. K. Schief, 2003
The discrete CKP equation

After appropriate rescaling we have $\rho^j Q^{ij} = \rho^i Q^{ji}$, which in the continuous limit gives $q^{ij} = q^{ji}$.

In terms of the $\tau$-function we have the discrete CKP system

$$
\left( \tau \tau(ijk) - \tau(i) \tau(jk) - \tau(j) \tau(ik) - \tau(k) \tau(ij) \right)^2 = 
4 \left( \tau(i) \tau(j) \tau(ik) \tau(jk) + \tau(i) \tau(k) \tau(ij) \tau(jk) + \tau(j) \tau(k) \tau(ik) \tau(ij) - 
\tau(i) \tau(j) \tau(k) \tau(ijk) - \tau(i) \tau(ij) \tau(jk) \tau(ik) \right),
$$
i, j, k distinct.

E. Date, M. Jimbo and T. Miwa, 1983
W. K. Schief, 2003
Things we have not mentioned

- analytic and algebro-geometric techniques to solve the nonlinear systems (and to construct the corresponding lattices)
- other basic reductions and their mutual superpositions
- restrictions of the linear problems (and of the nonlinear equations) to other (less regular) lattices
- quantum reductions, i.e., additional commutation relations which are compatible with the geometric integrability scheme