Gaussian thermostats as geodesic flows of nonsymmetric linear connections

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Abstract

We establish that Gaussian thermostats are geodesic flows of special metric connections. We give sufficient conditions for hyperbolicity of geodesic flows of metric connections in terms of their curvature and torsion.

1 Introduction

Let M be a compact manifold with a riemannian metric g, whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$. Denote by ∇ the Levi–Ci-vita connection of the metric g. Let E be a vector field on M.

A Gaussian thermostat is the dynamical system defined by the equations

$$(1) \qquad \frac{du}{dt} = v, \quad \frac{Dv}{dt} = E - \frac{\langle E, v \rangle v}{\langle v, v \rangle}.$$

where $\frac{D}{dt} = \nabla_v$ is the covariant derivative, [G-R].

Since v^2 is a first integral of the system we can restrict our attention to one level set. Although the dynamics is quite different for different values of v^2, there is no loss of generality in considering the Gaussian thermostat on the unit sphere bundle $SM = \{v \in TM : |v| = 1\}$. Indeed the change in the value of v^2 is equivalent, up to parameterization, to the rescaling of E. On SM we can write the equations (1) as equations of a spray.

$$(2) \qquad \frac{du}{dt} = v, \quad \frac{Dv}{dt} = v^2 E - \langle E, v \rangle v.$$

Every spray can be viewed as a geodesic flow of a canonical symmetric linear connection ∇^s, [A-P-S], defined in this case as

$$\nabla^s_X Y = \nabla_X Y - \langle X, Y\rangle E + \frac{1}{2}\langle X, E\rangle Y + \frac{1}{2}\langle Y, E\rangle X.$$

It is not the only linear connection that can be used for that purpose. We want to argue that there are indeed two more useful linear connections with the same geodesics up to parameterization. First of all the trajectories of a Gaussian thermostat are geodesics of the Weyl connection, [W1],

$$\widehat{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle E + \langle X, E\rangle Y + \langle Y, E\rangle X.$$

The advantage of the Weyl connection over the spray connection ∇^s is that its parallel transport is conformal. However the parameterization of the trajectories of (1) are unrelated to the Weyl connection. Moreover the geodesic flow of a Weyl connection on TM is not in general complete. The geodesics on M can be extended indefinitely but their velocity may go to infinity in finite time. Dynamical systems obtained from geodesic flows of Weyl connections by the parameterization with the arclength of a background riemannian metric were called W-flows in [W1]. The starting point of the paper was that Gaussian thermostats are W-flows. In this paper we propose to consider the equations of a Gaussian thermostat as the geodesic flow of the linear connection $\widetilde{\nabla}$

$$\widetilde{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle E + \langle Y, E\rangle X.$$

This connection is nonsymmetric but it has isometric parallel transport, i.e., it is a metric connection. The torsion of $\widetilde{\nabla}$ is

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = \langle Y, E\rangle X - \langle X, E\rangle Y.$$

Metric connections are uniquely determined by their torsions. We prove the following

Theorem 1 *For a Gaussian thermostat system* (1) *on SM the connection $\widetilde{\nabla}$ is the only linear connection on TM satisfying*
i) the trajectories of the system are geodesics for $\widetilde{\nabla}$,
ii) parallel transport defined by $\widetilde{\nabla}$ is isometric,
iii) the torsion $\widetilde{T}(X,Y)$ of the connection has values in $\mathrm{span}\{X, Y\}$.

The use of $\widetilde{\nabla}$ rather than the Weyl connection $\widehat{\nabla}$ will allow us to obtain in a simpler, more transparent way the basic results of [W1],[W2] on hyperbolic properties of Gaussian thermostats.

The linearization of geodesic flows is provided by Jacobi equations. For nonsymmetric connections the Jacobi equations involve both the curvature tensor $\widetilde{R}(X,Y) = \widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]}$ and the torsion $\widetilde{T}$. With a chosen metric g we introduce sectional curvature $\widetilde{K}(\Pi)$ and sectional torsion $\widetilde{T}(\Pi)$ of a connection in the direction of a plane Π by the formulas

$$\widetilde{K}(\Pi) = \langle \widetilde{R}(X,Y)Y, X\rangle, \qquad \tilde{T}(\Pi) = \frac{1}{4}\left|\widetilde{T}(X,Y)\right|^2,$$

where (X,Y) is an orthonormal frame in the plane Π.

We get the following generalization of a result from [W1].

Theorem 2 *If for every plane the sum of the sectional curvature and the sectional torsion of a metric connection is negative then the geodesic flow is Anosov.*

This theorem is formulated for an arbitrary metric connection. For special metric connections related to Gaussian thermostats, where $\widetilde{T}(X,Y) = \varphi(Y)X - \varphi(X)Y$, for some linear form φ, the following theorem was proven in [W1].

Theorem 3 *If all the sectional curvatures of the metric connection are negative then the geodesic flow has a dominated splitting with exponential growth/decay of volumes.*

The dominated splitting (also called the exponential dichotomy) is the property of the linearized equations to have two subspaces of solutions $\mathcal{E}^+$ and $\mathcal{E}^-$ such that the exponential rates of growth in $\mathcal{E}^+$ dominate those in $\mathcal{E}^-$. We refer the reader to [M] and [W3] for detailed formulations and discussion. It is a much weaker property then Anosov. In our case we have the additional property that there is uniform growth in $\mathcal{E}^+$, and uniform decay in $\mathcal{E}^-$, of volumes. In particular in the case of two dimensional manifolds M the subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$ are one dimensional, the volume becomes length, and we obtain the Anosov property. This theorem applies to Gaussian thermostats studied by Bonetto, Gentile and Mastropietro, [B-G-M], and it gives Anosov property for electrical fields E of any strength. Dairbekov and Paternain, [D-P], showed recently that on two dimensional manifolds, if a Gaussian thermostat is Anosov then its SRB measure is singular, except when E has a global potential (i.e.,φ is exact).

It remains an open problem to decide if in higher dimensions one gets the Anosov property from negative sectional curvatures alone, either in general case of a metric connection or in the Gaussian thermostat case.

The plan of the paper is the following. In Section 2 we introduce the Jacobi equations of a metric connection $\widetilde{\nabla}$ and prove Theorems 2 and 3.

In Section 3 we study the class of linear connections on M whose geodesics coincide with the trajectories of the system (2), which leads us to the special role played by $\widetilde{\nabla}$ and the proof of Theorem 1.

In Section 4 we discuss the interpretation of Gaussian thermostats as geodesic flows and explore the role of the conformal class of the background metric.

Finally in Section 5 we show that the antisymmetric tensor $\widetilde{T}$ appears again in the interpretation of Gaussian thermostats as generalized hamiltonian systems, obtained in [W-L] and [W4].

2 The curvature, torsion and hyperbolic properties of geodesic flows of linear metric connections

We consider the geodesic flow $\Psi^t : SM \to SM$ of a metric connection $\widetilde{\nabla}$.

Let $u(s)$ be a fixed geodesic of the connection $\widetilde{\nabla}$, and let $u(s,a) \in M$ be a family of geodesics, $u(s) = u(s,0)$, where s is the arclength parameter on a geodesic, and a is some real parameter, taken from a small interval around 0. Define the unit velocity field v and the Jacobi field ξ by

$$v = \frac{du}{ds} \qquad\qquad \xi = \frac{du}{da}\Big|_{a=0} \in T_uM.$$

The velocity field v is a special Jacobi field.

Jacobi fields form a vector space which can be naturally identified with the tangent spaces of SM along the chosen geodesic.

If we introduce $\chi = \widetilde{\nabla}_\xi v \in T_uM$ we get the following Jacobi equations

$$\begin{aligned}\widetilde{\nabla}_v \xi &= \chi + \widetilde{T}(v,\xi),\\ \widetilde{\nabla}_v \chi &= \widetilde{R}(v,\xi)v,\end{aligned}$$

where $\widetilde{R}$ and $\widetilde{T}$ are the curvature and the torsion tensors respectively. These equations are completely general and they follow immediately from the definitions of the tensors. Indeed

$$\begin{aligned}\widetilde{\nabla}_v \xi &= \widetilde{\nabla}_\xi v + \widetilde{T}(v,\xi) = \chi + \widetilde{T}(v,\xi),\\ \widetilde{\nabla}_v \chi &= \widetilde{\nabla}_v \widetilde{\nabla}_\xi v = \widetilde{R}(v,\xi)v + \widetilde{\nabla}_\xi \widetilde{\nabla}_v v = \widetilde{R}(v,\xi)v.\end{aligned}$$

Due to the restriction of the geodesic flow to SM we have that $\langle \chi, v\rangle = 0$.

The dynamical significance of the Jacobi equations is that they provide a convenient, and geometrically meaningful, linearization of the geodesic flow.

More specifically a Jacobi field ξ is uniquely determined by the Cauchy data $(\xi(s), \chi(s))$ for the Jacobi equations. Hence the pair $(\xi(s), \chi(s))$ can be thought of as a tangent vector to the phase space SM, that is we have that $(\xi(s), \chi(s)) \in T_v(SM)$. With this identification we get that

$$D\Psi^s((\xi(0), \chi(0)) = (\xi(s), \chi(s)).$$

Thus the Jacobi equations give us a way to study the hyperbolic properties of the geodesic flow. In particular its Lyapunov exponents are the exponential rates of growth of the Jacobi fields.

We introduce a quadratic form $\mathcal{J}$ on SM by $\mathcal{J}(\xi) = \langle \xi, \chi\rangle$. For a fixed Jacobi field ξ the evaluation of $\mathcal{J}$ on ξ along the geodesic becomes the function of the arclength parameter s, namely $J(\xi)(s) = \langle \xi(s), \chi(s)\rangle$. Following [W3] we introduce the definition

Definition 1 *We say that the geodesic flow Ψ^s is*
a) strictly $\mathcal{J}$-monotone if for every Jacobi field ξ which is not colinear with the velocity field v we have $\frac{d}{ds}\mathcal{J}(\xi)(s) > 0$,
b) strictly $\mathcal{J}$-separated if for any Jacobi field ξ which is not colinear with the velocity field v, and for which $J(\xi)(0) = 0$, we have that $\frac{d}{ds}\mathcal{J}(\xi)(s)_{|s=0} > 0$.

Clearly if a geodesic flow is strictly $\mathcal{J}$-monotone then it is strictly $\mathcal{J}$-separated. The interest in this definition comes from the following

Theorem 4 ([W3]) *If a flow is strictly $\mathcal{J}$-separated than it has a dominated splitting. If a flow is strictly $\mathcal{J}$-monotone then it is Anosov.*

Using the fact that the connection $\widetilde{\nabla}$ is metric we get

$$\frac{d}{ds}\mathcal{J} = \frac{d}{ds}\langle \xi, \chi\rangle = \langle \widetilde{\nabla}_v \xi, \chi\rangle + \langle \xi, \widetilde{\nabla}_v \chi\rangle.$$

The Jacobi equations give us that the last expression is equal to

$$(3)\quad \chi^2 + \langle \widetilde{T}(v,\xi), \chi\rangle + \langle \xi, \widetilde{R}(v,\xi)v\rangle = \\ = \left|\chi + \frac{1}{2}\widetilde{T}(v,\xi)\right|^2 - \langle \xi, \widetilde{R}(\xi, v)v\rangle - \frac{1}{4}\widetilde{T}(v,\xi)^2.$$

In the special case of Weyl connections the equivalent of Theorem 2 was formulated in [W1]. The general case follows immediately from Theorem 4 and the second part of (3).

In the riemannian case there is a codimension one subspace of Jacobi fields which are orthogonal to the velocity field v. That is not the case in general, unless the values of $\widetilde{T}(v,\cdot)$ are orthogonal to v. For lack of invariant subspace transversal to the flow we need to consider the quotient of the tangent space to SM by the velocity field. One way to do it is to consider the projection of the Jacobi field on the subspace perpendicular to the velocity, that is for a given Jacobi field ξ we consider the field $\zeta = \xi - \langle \xi, v\rangle v$.

In particular the form $\mathcal{J}$ factors to the quotient space, that is its value depends only on ζ.

In the rest of the section we restrict ourselves to the special form of the torsion $\widetilde{T}(X,Y) = \varphi(Y)X - \varphi(X)Y$. In this case we get from the Jacobi equations the following *quotient equations* for the field ζ.

$$\begin{aligned} \widetilde{\nabla}_v \zeta &= \chi - \varphi(v)\zeta, \\ \widetilde{\nabla}_v \chi &= \widetilde{R}(v,\zeta)v. \end{aligned} \tag{4}$$

The proof of Theorem 3 can be extracted from [W1]. For completeness we provide it here in detail in the new setup.

Proof of Theorem 3.

For our special form of $\widetilde{T}$ we get $\langle \widetilde{T}(v,\xi),\chi\rangle = \varphi(\xi)\langle v,\chi\rangle - \varphi(v)\langle \xi,\chi\rangle$. We have always that $\langle v,\chi\rangle = 0$, because $\langle v,v\rangle = 1$. Under the assumption that $\mathcal{J}(\xi)(0) = \langle \xi(0),\chi(0)\rangle = 0$ we get from the first part of (3)

$$\frac{d}{ds}\mathcal{J}(\xi)(s)\Big|_{s=0} = \chi^2 - \langle \xi, \widetilde{R}(\xi,v)v\rangle. \tag{3}$$

We get that the geodesic flow is strictly $\mathcal{J}$-separated and the first part of our theorem follows from Theorem 4.

To prove the second part we choose an orthonormal frame at an initial point on our geodesic such that the first vector of the frame is the velocity vector. We transport the frame parallely along the geodesic using the connection $\widetilde{\nabla}$. With these frames fixed we can consider $\zeta(s)$ and $\chi(s)$ as vectors in $\mathbb{R}^{n-1}$ and the quotient equations (4) become

$$\begin{aligned} \frac{d}{ds}\zeta &= \chi - \varphi(v)\zeta, \\ \frac{d}{ds}\chi &= \widetilde{R}(v,\zeta)v. \end{aligned} \tag{5}$$

The dominated splitting property gives us two invariant subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$. We will establish exponential growth of volume on $\mathcal{E}^+$. The exponential decay of volume on $\mathcal{E}^-$ follows from the reversibility of the geodesic flow. To prove the exponential growth we will introduce a special volume element in $\mathcal{E}^+$.

We represent the subspace $\mathcal{E}^+ \subset \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ as a graph of an operator $U : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, that is $\mathcal{E}^+ = \{(\zeta, U\zeta) \mid \zeta \in \mathbb{R}^{n-1}\}$. The evolution of U follows from (5) and it is described by the following Riccati equation.

$$\frac{d}{ds}U = \varphi(v)U - U^2 - R, \qquad \text{where} \qquad R\zeta = \widetilde{R}(\zeta, v)v. \tag{6}$$

Since by the construction of $\mathcal{E}^+$ the quadratic form $\mathcal{J} = \langle \zeta, \chi \rangle$ is positive definite on $\mathcal{E}^+$ we get that the symmetric part of U is positive definite. In contrast to the riemannian case we are not guaranteed that U itself is symmetric, because the operator R is not in general symmetric. Let us split the operators $U = U_s + U_a$ and $R = R_s + R_a$ into symmetric and antisymmetric parts respectively. By the assumption of negative sectional curvatures everywhere we get that $-R_s$ is positive definite. (Using the formula for the curvature tensor from [W1] it can be calculated that $\langle R_a\zeta, \eta \rangle = \frac{1}{2}d\varphi(\zeta, \eta)$.) We also have that U_a^2 is negative semidefinite. We get from (6)

$$\frac{d}{ds}U_s = \varphi(v)U_s - U_s^2 - U_a^2 - R_s. \tag{7}$$

We introduce new linear coordinates $\kappa \in \mathbb{R}^{n-1}$ by the formula $\kappa = U_s\zeta$. We will show that in these coordinates the standard volume has uniform exponential growth. Indeed we get from (5) and (7)

$$\frac{d}{ds}\kappa = \left(-U_s + U_sUU_s^{-1} + (-U_a^2 - R_s)U_s^{-1}\right)\kappa.$$

Since $tr(U_sUU_s^{-1}) = tr\ U_s$ we get that the trace of the operator in the right hand side of the equation is equal to $tr(-U_a^2 - R_s)U_s^{-1}$. It is positive since a product of two positive definite operators has positive trace. It follows that the standard volume in the coordinates κ is uniformly exponentially expanded. $\square$

3 Linear connections determined by the family of geodesics

Let us recall that two linear connections on a manifold differ by a tensor. We consider two such connections ∇^1 and ∇^2,

$$\nabla^2_X Y = \nabla^1_X Y + A(X,Y) + B(X,Y)$$

where A is a symmetric and B an antisymmetric tensor. Clearly the equations of geodesics are not effected by the antisymmetric tensor B. The following Proposition is the classical theorem of H. Weyl.

Proposition 1 *The linear connections ∇^1 and ∇^2 have the same geodesics up to parameterizations if and only if $A(X,Y) = \alpha(X)Y + \alpha(Y)X$ for some linear form α.*

In the proof we will need the following elementary Lemma

Lemma 1 *For a bilinear map $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the following are equivalent*
(a) $C(X,Y) \in \mathrm{span}\{X,Y\}$, for every $X,Y \in \mathbb{R}^n$,
(b) there are linear forms α and β such that $C(X,Y) = \alpha(X)Y + \beta(Y)X$.

Proof. If C satisfies (a) then both its symmetric and antisymmetric parts satisfy (a). Hence it is enough to establish (b) separately for symmetric and antisymmetric maps C. We give the proof here only for the symmetric case. The antisymmetric case is somewhat more involved and we leave it to the reader.

Let us assume that a symmetric bilinear map C satisfies (a). Then $C(X,X)$ must be colinear with X. Let $(x^1, x^2, ..., x^n)$ be linear coordinates in $\mathbb{R}^n$. If $x^1 = 0$ then we get that also the first coordinate of $C(X,X)$ has to vanish. It follows that the first coordinate of $C(X,X)$ is a quadratic form all of whose terms must contain x^1. The same can be repeated for other coordinates. Hence we obtain that $C(X,X) = (\alpha_1(X)x^1, \dots, \alpha_n(X)x^n)$ for some linear forms $\alpha_k, k = 1, \dots, n$. Since $C(X,X)$ must be colinear with X it follows immediately that the linear forms must coincide, which gives us (b). □

Proof of Proposition 1.

Let $\gamma(t)$ be a geodesic of ∇^1. And let us assume that there is a change of time $t = t(u)$ such that $\gamma(t(u))$ is a geodesic of ∇^2. We have

$\frac{d\gamma}{dt} = v$, $\nabla^1_v v = 0$ and $\frac{d\gamma}{du} = w$, $\nabla^2_w w = 0$.

We have $w = t'v$ and

$0 = \nabla^2_w w = t''v + (t')^2 \nabla^2_v v = t''v + (t')^2 A(v,v)$.

It follows that $A(v,v) \in \mathrm{span}\{v\}$. Using Lemma 1 we obtain the desired conclusion.

The converse is straightforward. □

As a corollary of Proposition 1 we obtain that the trajectories of a Gaussian thermostat can be obtained by integrating the system $\frac{dx}{dt} = v$, $\frac{Dv}{dt} = v^2 E$, in which v^2 is not preserved. This maybe simpler then the integration of the original equation as in the following example.

Example 1 Let M be the two dimensional flat torus with coordinates (x_1, x_2) and $E = (1, 0)$ be a constant vector field. The trajectories of the Gaussian thermostat satisfy $\frac{dv_1}{dt} = v_1^2 + v_2^2, \frac{dv_2}{dt} = 0$. Integrating the first equation for $v_2 \neq 0$ we get $v_1 = v_2 \tan(v_2 t + c)$ which yields trajectories $x_1 = -\ln\cos(x_2 + c_1) + c_2$. The remaining trajectories are horizontal lines $(v_2 = 0)$.

Proposition 2 *The linear connections ∇^1 and ∇^2 define the same parallel transport up to dilation if and only if $\nabla^2_X Y - \nabla^1_X Y = \alpha(X)Y$ for some linear form α.*

Proof. As we observed before $C(X,Y) = \nabla^2_X Y - \nabla^1_X Y$ is a tensor.

Let us assume that the parallel transports of the two connections differ only by dilations. Let Y be a vector field parallel in direction X with respect to ∇^2 and let f be a positive function such that fY is parallel in the same direction with respect to ∇^1. We have

$$C(X, fY) = \nabla^2_X(fY) - \nabla^1_X(fY) = df(X)Y.$$

In view of the arbitrariness of X and Y the claim follows now from Lemma 1.

The proof in the other direction is straightforward. □

In view of Proposition 1 we consider the family of all linear connections that share the same geodesics up to parameterization

$$\widetilde{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle E + \alpha(X)Y + \alpha(Y)X + B(X,Y), \tag{8}$$

where α is a linear form and $B(X,Y)$ is an antisymmetric tensor.

We will say that a linear connection from this family is *compatible with the conformal class* (of the riemannian metric) if the parallel transport along a geodesic of vectors perpendicular to the geodesic results in perpendicular vectors.

Proposition 3 *A linear connection from the family* (8) *is compatible with the conformal class if and only if there is a linear form β such that $B(X,Y) = \beta(X)Y - \beta(Y)X + B_1(X,Y)$, $B_1(X,Y)$ is perpendicular to* $\mathrm{span}\{X,Y\}$*, and $\alpha(Y) - \beta(Y) = \langle E, Y\rangle$.*

Proof. Let $\widetilde{\nabla}$ be a linear connection from the family (8) compatible with the conformal class and let Let $\gamma(t)$ be one of its geodesics. We have for $v = \frac{d\gamma}{dt}$,

$$0 = \widetilde{\nabla}_v v = \nabla_v v - v^2 E + 2\alpha(v)v.$$

Let further Y be a parallel vector field along $\gamma(t)$ so that

$$0 = \widetilde{\nabla}_v Y = \nabla_v Y - \langle v, Y\rangle E + \alpha(v)Y + \alpha(Y)v + B(v, Y).$$

If Y is perpendicular to the geodesic then

$$0 = \frac{d}{dt}\langle v, Y\rangle = \langle \nabla_v v, Y\rangle + \langle v, \nabla_v Y\rangle \quad = v^2\langle E, Y\rangle - v^2\alpha(Y) - \langle B(v, Y), v\rangle$$

We get

$$\langle B(v, Y), v\rangle = (\langle E, Y\rangle - \alpha(Y))v^2 = -\beta(Y)v^2, \tag{9}$$

where $\beta(Y) = \alpha(Y) - \langle E, Y\rangle$ is a linear form. We conclude that if the parallel transport takes perpendicular vectors into perpendicular vectors then (9) holds for any orthogonal vectors v, Y.

Let us consider the antisymmetric tensor $B_1(X, Y) = B(X, Y) - \beta(X)Y + \beta(Y)X$. It follows from (9) that if X and Y are orthogonal then

$$\langle B_1(X, Y), X\rangle = 0, \quad \text{and} \quad \langle B_1(X, Y), Y\rangle = -\langle B_1(Y, X), Y\rangle = 0.$$

It follows that $B_1(X, Y)$ is orthogonal to span$\{X, Y\}$ for any X and Y. The "only if" part of the Proposition is proven. The other implication is straightforward.

□

Guided by Proposition 3 we will restrict our attention to the family of linear connections

$$\widetilde{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle E + \langle E, Y\rangle X + \langle E, X\rangle Y - \gamma(X)Y, \tag{10}$$

where γ is a linear form. The fact that we dropped the antisymmetric tensor B_1 comes from our inability to make an advantageous choice different from zero.

In view of Proposition 2 all of these connections share the same parallel transport up to dilation and hence their curvature tensors $\widetilde{R}(X, Y) = \widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]}$ have the same antisymmetric part.

Let us note that we are using a riemannian metric to describe the family of connections. Let us examine the role of the conformal class of the metric. For a linear connection (10) and a parallel field Y defined along a path we have $\widetilde{\nabla}_X Y = 0$, that is $\nabla_X Y = \langle X, Y\rangle E - \langle E, Y\rangle X - \langle E, X\rangle Y + \gamma(X)Y$. This leads us to

$$\frac{d}{dt}\langle Y, Y\rangle = 2\langle \nabla_X Y, Y\rangle = 2\left(\gamma(X) - \langle E, X\rangle\right)\langle Y, Y\rangle. \tag{11}$$

We have established that the parallel transport with respect to any of the connections is conformal. For $\gamma = 0$ we obtain the symmetric connection $\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y = \nabla_X Y - \langle X, Y \rangle E + \langle E, Y \rangle X + \langle E, X \rangle Y$. A linear symmetric connection with conformal parallel transport is called a Weyl connection [F].

The only connection in the family (10) with isometric parallel transport is obtained for $\gamma(X) = \langle E, X \rangle$. We will reserve the notation $\widetilde{\widetilde{\nabla}}$ for the resulting metric connection.

We are ready to prove Theorem 1.

Proof of Theorem 1. By Proposition 1 the connection $\widetilde{\widetilde{\nabla}}$ satisfies i). Property ii) follows from the (11) and the property iii) is checked by direct calculation.

To prove the converse we need to invoke again Proposition 1 to get the restriction to connections of the form (8). The torsion of the connection (8) is equal to $2B(X, Y)$. If the connection is metric, that is with isometric parallel transport, than it clearly is compatible with the conformal class and hence is covered by Proposition 4. Now the property iii) implies that the tensor B_1 from Proposition 4 vanishes, so that our connection belongs to the family (7). Finally by (11) the only connection of that family with isometric parallel transport is the one with the form γ dual to the vector field E. $\square$

4 Gaussian thermostats as geodesic flows and the role of the conformal class of the metric

We have two ways of interpreting the Gausiann thermostat system (2) defined by a background metric g and a tangent vector field E. One way is to introduce the Weyl connection $\widehat{\nabla}$.

From the point of view of a Weyl connection there is nothing special about the background metric used in the definition of a Gaussian thermostat. Weyl connections are defined in terms of the conformal class of metrics, [F]. For a chosen metric g in the conformal class the Weyl connection $\widehat{\nabla}$ is uniquely determined by the linear form φ, $\varphi(X) = \langle E, X \rangle$, i.e., φ is the dual of the vector field E. The defining property of the Weyl connection is that $\widehat{\nabla}_X g = -2\varphi(X) g$. If we change the metric g to $e^{-2U} g$ we get that the Weyl connection defined by the pair (g, φ) is also defined by the pair $(e^{-2U} g, \varphi + dU)$.

For a fixed Weyl connection $\widehat{\nabla}$ as we change the metric and use it to parametrize the Weyl geodesics by the arc length we get a family of Gaussian thermostats which are flow equivalent by the obvious identification of the respective unit tangent bundles, via the rescaling. Certain hyperbolic properties, e.g. Anosov property or dominated splitting, are shared by flow equivalent systems.

Our new point of view is that the Gaussian thermostat defined by a pair (g, φ) is the geodesic flow of the unique metric connection $\widetilde{\nabla}$ with the torsion $\widetilde{T}(X,Y) = \varphi(X)Y - \varphi(Y)X$. The pairs (g, φ) and $(e^{-2U}g, \varphi + dU)$ define flow equivalent systems. Indeed as the torsion $\widetilde{T}$ changes, so does the connection $\widetilde{\nabla}_X Y = \widehat{\nabla}_X Y - \varphi(X)Y$. However by Proposition 1 all of these connections share the same geodesics, albeit parameterized differently by the respective arclength parameters.

Moreover by Proposition 2 the parallel transports of $\widehat{\nabla}$ and any $\widetilde{\nabla}$ differ by dilations only. The curvature operator of a linear connection represents infinitesimal parallel transport. Hence the curvature tensor of $\widetilde{\nabla}$ is equal to the antisymmetric part of the curvature tensor of the Weyl connection. In particular it does not change when we change the background metric g, in the conformal class, to $e^{-2U}g$. However the sectional curvature of the new metric connection does change; it is obviously equal to $e^{2U}\widetilde{K}(\Pi)$. We see that the negativity of the sectional curvature is the property of the Weyl connection alone and holds simultaneously for all the metric connections. It is consistent with Theorem 3 and the fact that the presence of a dominated splitting is not destroyed by the change of time in a flow.

For the special torsion $\widetilde{T}(X,Y) = \varphi(X)Y - \varphi(Y)X$ we get the sectional torsion $\widetilde{T}(\Pi) = \frac{1}{4}\left|\varphi_{|\Pi}\right|^2$. Hence as the metric g and the form φ change to $e^{-2U}g$ and $\varphi + dU$ respectively, the sectional torsion changes to

$$e^{2U}\frac{1}{4}\left|(\varphi + dU)_{|\Pi}\right|^2$$

where the norm $|\cdot|$ is determined by the old riemannian metric.

These formulas allow the optimization of the sufficient condition for the geodesic flow to be Anosov from Theorem 3. Again the Anosov property is not affected by the parameterization of geodesics so it is enough to establish it for a conveniently chosen metric. As we change the metric in the conformal class the sum of the sectional curvature and the sectional torsion is equal to

$$e^{2U}\left(\widetilde{K}(\Pi) + \frac{1}{4}\left|(\varphi + dU)_{|\Pi}\right|^2\right).$$

Hence we would like to minimize $\left|(\varphi + dU)_{|\Pi}\right|^2$ over all possible functions U. One way to fine tune the function U is to use the Hodge theory. It allows the minimization of the L^2 norm of $\varphi + dU$ over the whole manifold by the orthogonal projection of the form φ on the subspace of "divergence free" forms. If our original vector field E has zero divergence this optimization is void; we already have the optimal metric.

Let us note that in general the resulting optimal form $\varphi + dU$ does not have zero divergence with respect to the new riemannian metric. That brings us to another way of optimization by requesting that the new form is divergence free with respect to the new metric. By the result of Gauduchon ([G]), it can be achieved on a compact manifold and there is a unique way to do it.

Applications of Theorem 3 hinge on the understanding of sectional curvatures of the metric connection. Although the difference between the curvature tensors of the metric connection and the Levi-Civita connection of the underlying metric contains many terms, the difference between the respective sectional curvatures simplifies to the following transparent formula in terms of the vector field E ([W1])

$$\widetilde{K}(\Pi) = K(\Pi) - (E^2 - E_\Pi^2) - \langle \nabla_X E, X \rangle - \langle \nabla_Y E, Y \rangle,$$

where for any orthonormal basis (X, Y) of the plane Π the vector $E_\Pi = \langle X, E \rangle X + \langle Y, E \rangle Y$ is the orthogonal projection of E on the plane Π and $K(\Pi)$ denotes the Gaussian sectional curvature of the background metric. We have also

$$\widetilde{T}(X, Y)^2 = (\langle E, Y \rangle X - \langle E, X \rangle Y)^2 = E_\Pi^2.$$

In particular we see that in dimension 2 the curvature of a metric connection is negative if and only if the respective Gauduchon metric has negative curvature.

These formulas allow to obtain the Anosov property for the Gaussian thermostats with divergence free fields E on surfaces of negative curvature, studied in [B-G-M].

They were also used to study Weyl connections with nonpositive sectional curvatures on tori, [W4].

5 The torsion and the hamiltonian formulation

The torsion comes in an interesting way into the generalized hamiltonian formulation of the problem, [W-L],[W4].

Using the background metric g we will identify a tangent space to TM at (u,v) with $T_uM \oplus T_uM$. Namely for a tangent vector defined by a parameterized curve $(u(a), v(a)), |a| < \epsilon$, we use $(\xi, \eta) \in T_{u(0)}M \oplus T_{u(0)}M$

$$\xi = \frac{du}{da}_{|a=0}, \quad \eta = \nabla_\xi v_{|a=0},$$

as coordinates.

Take the hamiltonian function $H = \frac{v^2}{2}$ on TM and the symplectic form on TM given by

$$\omega\big((\xi_1,\eta_1),(\xi_2,\eta_2)\big) = \big\langle \eta_1, \xi_2 \big\rangle - \big\langle \eta_2, \xi_1 \big\rangle.$$

We are looking for an antisymmetric 2-form γ on TM, such that the Gaussian thermostat (2) becomes the generalized hamiltonian flow with respect to the form $\Omega = \omega - \gamma$. More precisely we want the vector field

$$\overrightarrow{H}(u,v) = \Big(v, v^2 E - \langle E, v\rangle v\Big).$$

to satisfy

$$\Omega(\overrightarrow{H}, \cdot) = -dH(\cdot).$$

It turns out that γ is given by

$$\gamma\big((\xi_1,\eta_1),(\xi_2,\eta_2)\big) = \big\langle \widetilde{T}(\xi_1,\xi_2), v \big\rangle.$$

Indeed:

$$\begin{aligned} -dH(\xi,\eta) &= -\langle \eta, v\rangle \quad \text{and} \\ \Omega(\overrightarrow{H},(\xi,\eta)) &= \omega(\overrightarrow{H},(\xi,\eta)) - \gamma(\overrightarrow{H},(\xi,\eta)) = \\ &= \Big\langle v^2 E - \langle E, v\rangle v, \xi \Big\rangle - \langle \eta, v\rangle \\ &- \Big\langle \langle E, \xi\rangle v - \langle E, v\rangle \xi, v \Big\rangle = -\langle \eta, v\rangle. \end{aligned}$$

The form Ω is always nondegenerate. It was shown in [W-L] that it is conformally symplectic if φ is closed.

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