6. I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser, Basel, 2003.
7. H. Hahn, Über Folgen linearer Operationen, *Monatsh. Math. Phys.* **32** (1922) 3–88. doi:10.1007/BF01696876
8. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer-Verlag, New York, 1982.
9. F. Hausdorff, Zur Theorie der linearen metrischen Räume, *J. Reine Angew. Math.* **167** (1932) 294–311. doi:10.1515/crll.1932.167.294
10. J. Hennefeld, A nontopological proof of the uniform boundedness theorem, *Amer. Math. Monthly* **87** (1980) 217. doi:10.2307/2321614
11. T. H. Hildebrandt, On uniform limitedness of sets of functional operations, *Bull. Amer. Math. Soc.* **29** (1923) 309–315. doi:10.1090/S0002-9904-1923-03730-0
12. S. S. Holland Jr., A Hilbert space proof of the Banach-Steinhaus theorem, *Amer. Math. Monthly* **76** (1969) 40–41. doi:10.2307/2316786
13. B. D. MacCluer, *Elementary Functional Analysis*, Springer-Verlag, New York, 2009.
14. R. E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.
15. V. Müller and J. Vršovský, Orbits of linear operators tending to infinity, *Rocky Mountain J. Math.* **39** (2009) 219–230. doi:10.1216/RMJ-2009-39-1-219
16. A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007.
17. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, vol. I, *Functional Analysis*, Academic Press, New York, 1972.
18. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 1955.
19. H. H. Schaefer and M. P. Wolff, *Topological Vector Spaces*, 2nd ed., Springer-Verlag, New York, 1999.
20. C. Swartz, *Infinite Matrices and the Gliding Hump*, World Scientific, River Edge, NJ, 1996.
21. J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1980.

Department of Physics, New York University, New York, NY 10003 and
Department of Mathematics, University College London, London WC1E 6BT, England
sokal@nyu.edu

On the Group of Automorphisms of a Group

Marian Deaconescu and Gary L. Walls

Abstract. This note gives a generalization of the classical result asserting that if the center of a group G is trivial, then so is the center of its automorphism group $\mathrm{Aut}(G)$.

Let G be a group and let $Z(G)$ denote its center. We will denote the automorphism group of G by $\mathrm{Aut}(G)$. One of the classic results in group theory is the following theorem.

Theorem 1. *If* $Z(G) = 1$*, then* $Z(\mathrm{Aut}(G)) = 1$.

We use the notation $G = 1$ to mean that G is the appropriate trivial group, i.e., a group with one (necessarily trivial) element. This is a convenient abuse of notation, as the trivial element of $\mathrm{Aut}(G)$ is actually the identity automorphism Id_G.

Theorem 1 is surprising, being one of the first results establishing a relationship between the structure of a group and that of its automorphism group. It is a simple but powerful result, which was the starting point in H. Wielandt's proof of his elegant

doi:10.4169/amer.math.monthly.118.05.452

automorphism tower theorem [**2**]. It appears already in H. Zassenhaus's book—see [**3**] for a newer, more popular edition—but we were not able to determine its original source.

These days, Theorem 1 is usually presented as an exercise in a first course in group theory, as its proof is quite elementary. We give a proof here solely for the convenience of the reader. The proof presented could actually be much shorter, but we insert some details which will be useful later.

To start with, people who work in this part of group theory generally use a peculiar functional notation. For an arbitrary group G, $g \in G$, and $\alpha \in \mathrm{Aut}(G)$, they denote the image $\alpha(g)$ by exponentiation; that is, by g^α. For the benefit of readers interested in checking the few statements given without proof in the next paragraph we should mention that function composition is applied the other way around: that is, we write $g^{\alpha\beta} = (g^\alpha)^\beta = \beta(\alpha(g))$.

For $g \in G$, consider the *inner automorphism* T_g induced by g, that is, we define $x^{T_g} := g^{-1}xg$, and observe that the mapping $T : G \to \mathrm{Aut}(G)$ defined by $T(g) := T_g$ is a group homomorphism satisfying the identity $T(g^\alpha) = T_{g^\alpha} = \alpha^{-1}T_g\alpha$ for all $g \in G$ and all $\alpha \in \mathrm{Aut}(G)$. This shows at once that the image $T(G)$ is a normal subgroup of $\mathrm{Aut}(G)$—this group is usually denoted by $\mathrm{Inn}(G)$ and is called the group of inner automorphisms of G. Since the kernel of T is just the center of G, the fundamental isomorphism theorem applies to give $\mathrm{Inn}(G) = T(G) \cong G/Z(G)$ and therefore $G \cong \mathrm{Inn}(G)$ whenever $Z(G) = 1$.

Let now $\mathrm{Aut}_c(G) := C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$ denote the centralizer of $\mathrm{Inn}(G) = T(G)$ in $\mathrm{Aut}(G)$. A short calculation shows that $\alpha \in \mathrm{Aut}_c(G)$ if and only if $g^\alpha g^{-1} \in Z(G)$ for all $g \in G$. In particular, when $Z(G) = 1$, we see that $\mathrm{Aut}_c(G) = 1$ and thus $Z(\mathrm{Aut}(G)) = 1$.

A few years ago, the authors showed in [**1**] that a result similar to Theorem 1 holds if one replaces $Z(G)$ with the subgroup $R_2(G)$ of all right 2-Engel elements of G. The subgroup $R_2(G)$ consists of those elements $x \in G$ satisfying the identity $[[x, g], g] = 1$ for all $g \in G$. Note that if $a, b \in G$, then the commutator $[a, b]$ is defined by $[a, b] = a^{-1}b^{-1}ab$. More specifically it is shown that, if G is a group and if $R_2(G) = 1$, then $R_2(\mathrm{Aut}(G)) = 1$. The proof of this result about $R_2(G)$ was rather technical, involving a lot of commutator calculations.

Both $Z(G)$ and the mentioned $R_2(G)$ are what we might call *generic* subgroups of G. Actually we begin with an *operator*, X, which for each group G picks out a subgroup $X(G)$ of G with the property that if $f : G_1 \to G_2$ is an isomorphism, then $f(X(G_1)) = X(G_2)$. We then refer to such an $X(G)$ as a *generic* subgroup of G. It follows that generic subgroups of a group G must be characteristic subgroups of G. That is, they must be left invariant by every automorphism of G. Since a subgroup H of G is a normal subgroup, written $H \trianglelefteq G$, if it is left invariant by every inner automorphism, all characteristic subgroups must be normal. Further, note that if $X(G)$ is a generic subgroup of G, then so is its centralizer $C_G(X(G))$ in G.

There are many important examples of generic subgroups of arbitrary groups G. The simplest ones are 1 and G, but there are many others, of which we mention just a few. All terms of the ascending and descending central series of G, in particular the well-known commutator subgroup G', are generic. The Frattini subgroup $\Phi(G)$, defined as the intersection of all maximal subgroups of G (note that $\Phi(G) = G$ when G has no maximal subgroups), $S(G)$, the largest normal solvable subgroup of G, which is called the solvable radical of G, and $F(G)$, the largest normal nilpotent subgroup of G, called the Fitting subgroup, are also well-known examples of generic subgroups of G. Note that in order to assure that $S(G)$ and $F(G)$ exist, we need to assume that G is finite in these cases.

A natural question suggested by Theorem 1 and the result about $R_2(G)$ is: can we extend Theorem 1 to a statement with the *center* of G replaced by some kind of *generic* subgroup of G. That is, if for each group G, $X(G)$ is a generic subgroup of G, when will it be true that if for some particular group G, $X(G) = 1$, we must have $X(\mathrm{Aut}(G)) = 1$?

Consider first the case when we take $X(G) = G'$, the commutator subgroup of G. Our hypothetical extension of Theorem 1 would then read: "If G is a group and if $G' = 1$, then $\mathrm{Aut}(G)' = 1$." This statement is manifestly false—for a counterexample it suffices to take $G = C_2 \times C_2$, the Klein four group, since $\mathrm{Aut}(C_2 \times C_2)$ is isomorphic to the symmetric group S_3. Of course, there are *some* groups for which the above implication is true, but we want a result that matches the generality of Theorem 1. The reader is invited to consider the group $G := C_p \times C_p$, the direct product of two cyclic groups of odd prime order p. This group has trivial Frattini subgroup, while $\Phi(\mathrm{Aut}(G)) \neq 1$.

By now it has become clear that we need to look at some special $X(G)$'s. Here is a list of three properties that a generic subgroup $X(G)$ might have. The first two are properties of $Z(G)$.

(a) $Z(G) \leq X(G)$,

(b) $H \cap X(G) \leq X(H)$, whenever $H \trianglelefteq G$, and

(c) $X(H) \leq X(G)$ whenever $H \trianglelefteq G$.

For convenience, we will call a generic subgroup having both properties (a) and (b) an *ab-subgroup* and one having property (c) a *c-subgroup*. According to this definition, $Z(G)$, $R_2(G)$, $F(G)$, and $S(G)$ are ab-subgroups, but G' and $\Phi(G)$ are not. However, G' and $\Phi(G)$ are c-subgroups.

Now comes the surprise: as it happens, all we need to extend Theorem 1 is just a generic ab-subgroup G. We have the following result.

Theorem 2. *Let G be a group and let $X(G)$ be a generic ab-subgroup of G. If $X(G) = 1$, then $X(\mathrm{Aut}(G)) = 1$.*

Proof. Since $Z(G) \leq X(G)$ by property (a), the hypothesis $X(G) = 1$ forces $Z(G) = 1$ and consequently $G \cong \mathrm{Inn}(G)$. But then $X(\mathrm{Inn}(G)) = 1$, and since $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ we have $\mathrm{Inn}(G) \cap X(\mathrm{Aut}(G)) \leq X(\mathrm{Inn}(G)) = 1$ by property (b). Thus $\mathrm{Inn}(G) \cap X(\mathrm{Aut}(G)) = 1$.

Since $X(\mathrm{Aut}(G)) \trianglelefteq \mathrm{Aut}(G)$, we have $[\mathrm{Inn}(G), X(\mathrm{Aut}(G))] \leq \mathrm{Inn}(G) \cap X(\mathrm{Aut}(G)) = 1$, so $X(\mathrm{Aut}(G)) \leq \mathrm{Aut}_c(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$. Finally, as we have seen in the proof of Theorem 1, $Z(G) = 1$ implies $\mathrm{Aut}_c(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = 1$, whence $X(\mathrm{Aut}(G)) = 1$, as asserted. ■

The examples given above show that Theorem 2 can be useless when dealing with generic subgroups $X(G)$ not containing $Z(G)$, like for example G' and $\Phi(G)$. But there is a nice family of generic groups to which this theorem applies: the centralizer in G of a generic c-subgroup of G is a generic ab-subgroup. Indeed, let $Y(G)$ be a generic c-subgroup of G and consider $X(G) = C_G(Y(G))$. Then $X(G)$ is a generic subgroup of G and, moreover, it is also an ab-subgroup. The inclusion $Z(G) \leq X(G)$ is trivial, and if $H \trianglelefteq G$, then $H \cap X(G) = H \cap C_G(Y(G)) = C_H(Y(G)) \leq C_H(Y(H)) = X(H)$. The last containment follows from the fact that $Y(H) \leq Y(G)$, which is just the c-property enjoyed by $Y(G)$.

We thus have the following result.

Corollary 3. *Let G be a group and let $Y(G)$ be a generic c-subgroup of G. If $C_G(Y(G)) = 1$, then $C_{\mathrm{Aut}(G)}(Y(\mathrm{Aut}(G))) = 1$.*

After all is said and done, a new question comes to mind: Is Theorem 2 the best possible result in this direction? But that is a question for another time.

REFERENCES

1. M. Deaconescu and G. L. Walls, Right 2-Engel elements and commuting automorphisms, *J. Algebra* **238** (2001) 479–484. doi:10.1006/jabr.2000.8655
2. H. Wielandt, Eine Verallgemeinerung der invarianten Untergruppen, *Math. Z.* **45** (1939) 209–244. doi:10.1007/BF01580283
3. H. Zassenhaus, *The Theory of Groups*, Chelsea, New York, 1956.

Department of Mathematics, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
mdeaconescu@yahoo.com

Department of Mathematics, Southeastern Louisiana University, Box 60787, Hammond, LA 70402, USA
gary.walls@selu.edu

The Unreasonable Slightness of E_2 over Imaginary Quadratic Rings

Bogdan Nica

Abstract. It is almost always the case that the elementary matrices generate the special linear group SL_n over a ring of integers in a number field. The only exceptions to this rule occur for SL_2 over rings of integers in imaginary quadratic fields. The surprise is compounded by the fact that, in these cases when elementary generation fails, it actually fails rather badly: the group E_2 generated by the elementary 2-by-2 matrices turns out to be an infinite-index, non-normal subgroup of SL_2.

We give an elementary proof of this strong failure of elementary generation for SL_2 over imaginary quadratic rings.

1. INTRODUCTION. For $n \geq 2$, the special linear group $\mathrm{SL}_n(\mathbb{Z})$ is generated by the elementary matrices. That is, every $n \times n$ matrix with integer entries and determinant 1 can be written as a product of matrices which have 1 on the diagonal and at most one nonzero off-diagonal entry. The proof rests on the fact that $\mathbb{Z}$ is a Euclidean domain: row- and column-operations dictated by division with remainder will reduce any matrix in $\mathrm{SL}_n(\mathbb{Z})$ to the identity matrix.

What happens if we replace $\mathbb{Z}$ by another ring of integers? To make this question precise, we first pass from $\mathbb{Q}$ to a number field $\mathbb{K}$—that is, a finite field extension of $\mathbb{Q}$. Next, we take the ring of integers of $\mathbb{K}$—i.e., the ring consisting of all elements from $\mathbb{K}$ which are roots of monic polynomials with coefficients in $\mathbb{Z}$—and we denote it by $\mathcal{O}_{\mathbb{K}}$. Our vague query crystallizes into the following problem: is $\mathrm{SL}_n(\mathcal{O}_{\mathbb{K}})$ generated by the elementary matrices?

We owe to Cohn [**3**] the first result in this direction: when $n = 2$, the answer is negative for all but five imaginary quadratic fields.

doi:10.4169/amer.math.monthly.118.05.455