Artifacts for Stamping Symmetric Designs

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Abstract. It is well known that there are 17 crystallographic groups that determine the possible tessellations of the Euclidean plane. We approach them from an unusual point of view. Corresponding to each crystallographic group there is an orbifold. We show how to think of the orbifolds as artifacts that serve to create tessellations.

1. TESSELLATIONS. The history of human civilization gives an enormous number of examples of tessellations or, equivalently, tilings or mosaics. A tessellation is a design made up of one or more types of tiles (tessellas) that are repeated several times and are placed in such a way that they cover completely a surface without gaps or overlaps.

Since humans started to build walls, floors, and ceilings, the placement of stones or bricks (as tiles) has naturally given rise to tessellations. The search for beautiful designs, the selection of shapes and colors of the tiles, and the systematic repetition of motifs produced symmetric patterns as examples of tesellations.

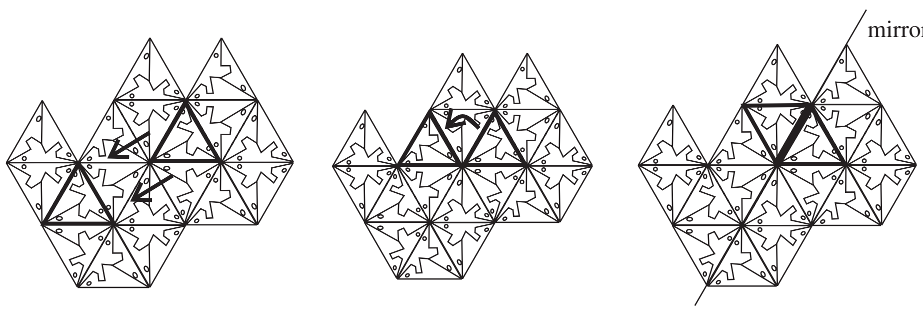

Figure 1. Some rigid moves on a plane.

Even though we have infinitely many ways to make symmetric designs, basically there are 17 “methods” which suffice to construct all symmetric tessellations on a plane. In fact, the crystallographer E. S. Fedorov proved in 1891 that, up to a natural equivalence, there are only 17 plane crystallographic groups. Having a tile on a plane, it is possible to translate it, to rotate it about a point, and to reflect it about an axis. These moves are rigid, i.e., they neither deform nor change the size of the tile but change the place of the tile on the plane; see Figure 1. Any other rigid move can be obtained using a combination of translations, reflections, or rotations. Taking these rigid moves as generators, we generate 17 different algebraic groups. They are called the *two-dimensional* or *plane crystallographic groups*. Each one of the 17 plane crystallographic groups gives one possible way of making a symmetric tessellation. See [**8**].

To study symmetric tessellations there are two points of view. On the one hand, we can use algebraic language, and in this sense we speak in terms of crystallographic

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groups; on the other hand, we are able to use geometric or topological language, and in this sense we speak in terms of orbifolds. Historically, symmetry was first understood from the point of view of algebra. Following the ideas of W. Thurston (see [**11**]) it is possible to understand the symmetry of designs by means of the concept of orbifold. This concept belongs to the world of geometry and topology. Both languages are equivalent. The advantage of the geometric language is that orbifolds can be visualized as artifacts that stamp the corresponding symmetric design. For each Euclidean crystallographic group there is basically one Euclidean orbifold.

In the interest of rigor, at this point we would like to define some terms with more precision. A plane crystallographic group is a subgroup of the symmetry group of a tessellation of the Euclidean plane such that its translation subgroup is generated by translations in two independent directions. Two such groups are said to be equivalent if they are conjugate in the larger group of affine transformations of the Euclidean plane. In two dimensions these groups turn out to be equivalent if and only if they are isomorphic as groups. The idea of this equivalence is that mere changes of scale and shears should not give rise to inequivalent groups. The reader is referred to the website `http://en.wikipedia.org/wiki/Wallpaper_group` where the classification into 17 groups is explained in great detail.

Given such a group G, a fundamental domain for G is a subset D of the plane (in this paper always a convex polygon) such that for any point x in the plane there is an element g of G with $g(x)$ in D and such that for any two distinct points x and y in the interior of D there is no element in G with $g(x) = y$. The tiles in this paper are fundamental domains.

As the group G acts on the Euclidean plane there is an equivalence relation on the points of the plane. Point x is equivalent to point y if there is an element in G with $g(x) = y$. The quotient space of this action, the set of equivalence classes of points in the plane, turns out to be a two-dimensional Euclidean orbifold and can be constructed from a fundamental domain by pasting edges.

A two-dimensional Euclidean orbifold is a compact surface with or without boundary. Points in the interior that are not singular have neighborhoods that are isometric to Euclidean discs. Singular interior points are called cone points and have neighborhoods isometric to Euclidean cones with cone angle an integral fraction of 360°, that is to say, spaces constructed by starting with a radial disc segment of angle $360°/n$ and gluing the two linear sides.

Boundary points that are not singular have neighborhoods isometric to Euclidean half discs. Singular boundary points have neighborhoods isometric to radial disc segments of angle $360°/n$.

Alternatively, two-dimensional Euclidean orbifolds can be thought of as surfaces that are locally isometric to the quotient spaces arising from the actions of rotation groups or dihedral groups on a Euclidean disk.

We can illustrate these concepts with a couple of examples. Consider the tessellation of the plane by squares with integer coordinate vertices. Let G_1 be the group generated by reflections in the lines $x =$ an integer and $y =$ an integer. It can be shown that the translation subgroup T of G_1 is generated by translations by 2 in the x and y directions and that a fundamental domain for G_1 is the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$. No two points in the square are equivalent, so the quotient orbifold is the square itself. The vertices of the square are singular points. Later, in Section 3.1, you will see that this orbifold is $D_{\overline{2222}}$; see Figure 6.

Let G_2 be the orientation preserving subgroup of G_1. It can be shown that G_1 is generated by 180-degree rotations about points with integer coordinates and that a fundamental domain for G_2 is the rectangle $\{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq 1\}$. There are equivalences on the boundary of this rectangle. The point $(-1, y)$ is equivalent to $(1, y)$, $(-x, 0)$ to $(x, 0)$, and $(-x, 1)$ to $(x, 1)$. Gluing points to their equivalent points we obtain a topological sphere. Thus the quotient orbifold is topologically the 2-sphere. There are four singular points, all with cone angle 180°. Later, in Section 4.3, you will see that this orbifold is S_{2222}. We invite the reader to see [**5**] and [**6**].

Table 1 contains the list of the 17 plane crystallographic groups. On it we find both the notation used in the literature of algebra and the notation used in the literature of the geometry of orbifolds. See [**2**] and [**9**].

Table 1. The 17 crystallographic groups.

Crystallograpic Groups

Orbifolds

$$
\begin{tikzcd}[row sep=small]
p6 \arrow[r, leftrightarrow] & S_{236} \arrow[r] & D_{\overline{2}\,\overline{3}\,\overline{6}} \arrow[r, leftrightarrow] & p6m \\
p4 \arrow[r, leftrightarrow] & S_{244} \arrow[r] \arrow[dr] & D_{\overline{2}\,\overline{4}\,\overline{4}} \arrow[r, leftrightarrow] & p4m \\
 & & D_{\overline{2}4} \arrow[r, leftrightarrow] & p4g \\
p3 \arrow[r, leftrightarrow] & S_{333} \arrow[r] \arrow[dr] & D_{\overline{3}\,\overline{3}\,\overline{3}} \arrow[r, leftrightarrow] & p3m1 \\
 & & D_{\overline{3}3} \arrow[r, leftrightarrow] & p31m \\
 & & P_{22} \arrow[r, leftrightarrow] & pgg \\
p2 \arrow[r, leftrightarrow] & S_{2222} \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & D_{22} \arrow[r, leftrightarrow] & pmg \\
 & & D_{\overline{2}\,\overline{2}\,\overline{2}\,\overline{2}} \arrow[r, leftrightarrow] & pmm \\
 & & D_{\overline{2}\,\overline{2}2} \arrow[r, leftrightarrow] & cmm \\
 & & A \arrow[r, leftrightarrow] & pm \\
p1 \arrow[r, leftrightarrow] & T \arrow[ur] \arrow[r] \arrow[dr] & M \arrow[r, leftrightarrow] & cm \\
 & & K \arrow[r, leftrightarrow] & pg
\end{tikzcd}
$$

Admittedly, this notation is confusing. In the course of the paper we will explain the notation, at least in the case of orbifolds. For the moment the reader ought simply to observe that there are 17 plane crystallographic groups, 17 entries in columns two and three of Table 1 and 17 entries in columns one and four of Table 1, and that the letters S, T, D, P, A, M, and K are the first letters of sphere, torus, disk, projective plane, annulus, Möbius band, and Klein bottle, respectively. See [**4**] and [**8**].

2. KALEIDOSCOPES AND THE FIRST EXAMPLES OF ORBIFOLDS. Real world kaleidoscopes, those that one could buy in a museum shop, consist of three or four rectangular mirrors fastened along pairs of opposite edges in such a way that the unfastened edges form triangles or rectangles at both ends. A triangle or a rectangle is decorated and glued to one of the ends. We look through the other end and we see a tessellation of the plane in which the decorated triangle or rectangle lies, and we think that this plane is covered completely with a design formed by the repetition of the decorated triangle. Given, for example, a fixed triangular kaleidoscope we can obtain using it infinitely many symmetric designs by changing the decoration of the given triangle. In this way we can think of a kaleidoscope as an artifact for producing symmetric designs. Examples of these kinds of tessellations are found in the mosaics of the Alhambra in Granada, Spain (see the institutional DVD [**7**] and [**8**]), and also in the beautiful well-known Escher designs. For example, Figure 2 is a sketch of the Escher design known as the Lizard, Fish, and Bat. This design can be found online

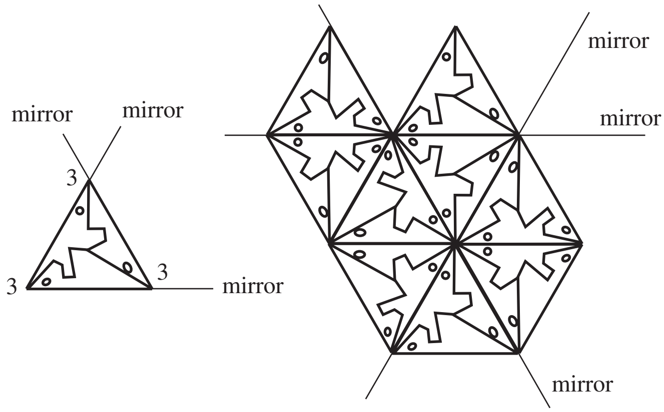

Figure 2. Tile and tessellation of $D_{\overline{333}}$.

at http://britton.disted.camosun.bc.ca/jbescher2.htm, see [**3**]. Figures 3 and 4 are sketches of Alhambra mosaics. Figure 5 is inspired by Escher (see the design "1950 4 motifs" in the gallery of http://www.tessellations.org).

Not all triangles produce tessellations. There are only three that work: an equilateral triangle, which we denote by $D_{\overline{333}}$, an isosceles right triangle with two acute angles of 45°, which we denote by $D_{\overline{244}}$,and a right triangle with acute angles of 30° and 60°, which we denote by $D_{\overline{236}}$. There is one more (not so common) kaleidoscope constructed with 4 mirrors placed so as to form a rectangle. This one is denoted by $D_{\overline{2222}}$. The letter D in the notation refers to the fact that we have a topological disk with a border, and the index $\overline{n}$ means that we have an angle of $180°/n$ formed by two consecutive mirrors. See Figures 2 through 5. The patterns on the tiles in Figures 2 through 5, and in others figures throughout this paper, have been chosen arbitrarily and have no mathematical significance.

In each of these four cases the triangle or rectangle corresponds to the quotient space of the action of a plane crystallographic group. Therefore $D_{\overline{333}}$, $D_{\overline{244}}$, $D_{\overline{236}}$, and $D_{\overline{2222}}$ are Euclidean orbifolds. We will refer to them as "kaleidoscopic" orbifolds. The crystallographic groups corresponding to these kaleidoscopic orbifolds are, respectively, $p3m1$, generated by three reflections in the sides of an equilateral triangle; $p4m$, generated by three reflections in the sides of an isosceles right triangle with two acute

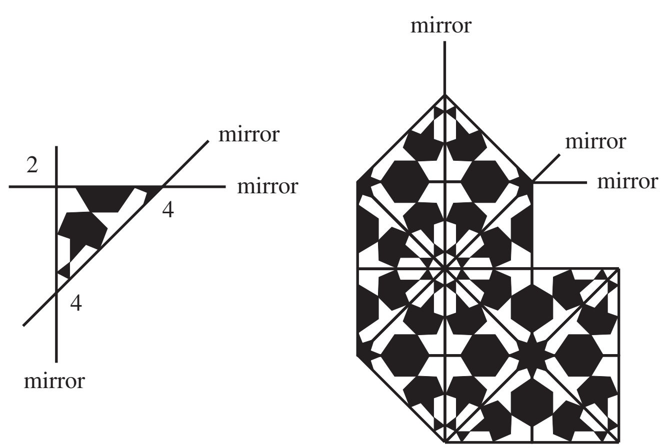

Figure 3. Tile and tessellation of $D_{\overline{244}}$.

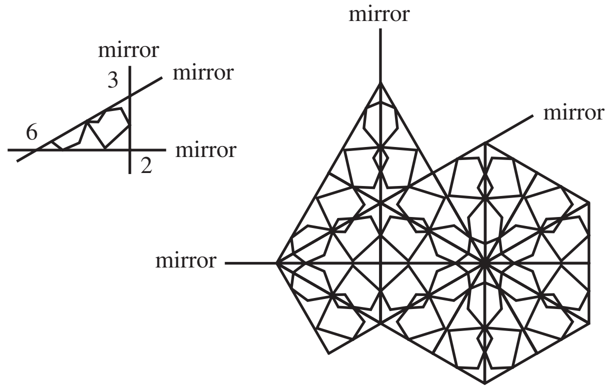

Figure 4. Tile and tesellation of $D_{\overline{236}}$.

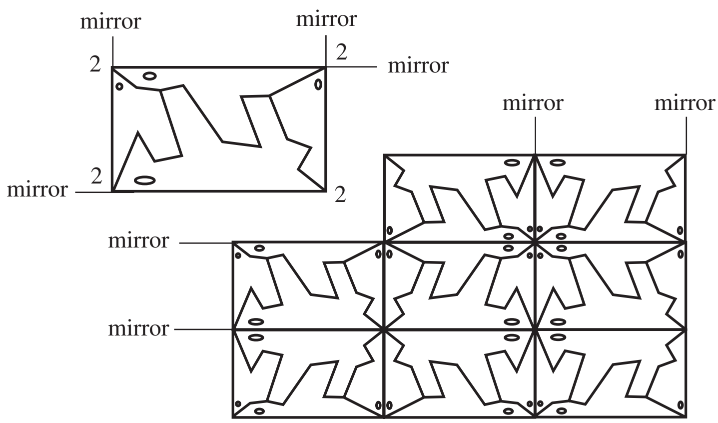

Figure 5. Tile and tesellation of $D_{\overline{2222}}$.

angles of 45°; the group $p6m$, generated by three reflections in the sides of a right triangle with acute angles 30° and 60°; and, finally, the group pmm, generated by four reflections in the sides of a rectangle. This notation is the usual notation in crystallography. The tile associated to a crystallographic group is an example of a fundamental domain. Figures 2 to 5 illustrate tessellations corresponding to these four kaleidoscopic orbifolds.

The vertices of the triangles delineated by the three mirrors in Figures 2, 3, and 4, and of the rectangle delineated by the four mirrors in Figure 5, are centers of rotation. If the rotation is $360°/n$ it is called an *n-fold center*. In Figures 2 to 5 an n-fold center is labeled by n. It is known (Barlow's theorem; although we do not prove it here) that for any n-fold center, n must be equal to 2, 3, 4, or 6. (See [**2**] and [**9**].)

3. ARTIFACTS FOR RIGID MOVES. We have already said that rigid moves can be produced from reflections, translations, and rotations. Let us see how to produce these basic moves with some simple artifacts.

3.1. Reflections. Think of a very thin, porous tile with a design on it (specially shaped with straight borders), such that when it is impregnated with ink on one side, the other side also gets inked. Roll it over a plane stamping alternately with one side and then the other. The manner of stamping each side is similar to the technique used in silk-screening. When we say "roll it," we mean that we do not want to separate the tile from the surface while we are turning it, i.e., the border of the tile has to be in contact with the surface while we are turning the tile. If this special tile is a rectangle we have in our hands an artifact that mimics the effect of the kaleidoscope corresponding to the kaleidoscopic orbifold $D_{\overline{2222}}$. So we do not need mirrors anymore. See Figure 6.

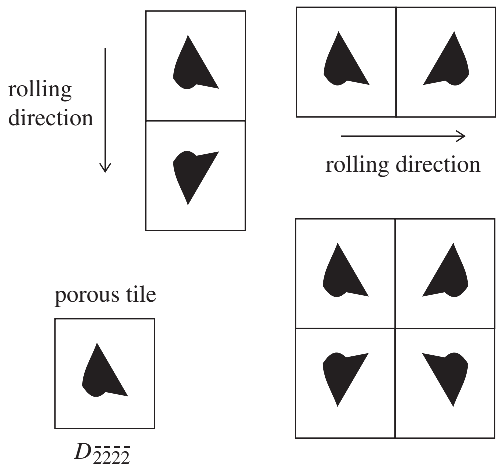

Figure 6. Tile and tessellation of $D_{\overline{2222}}$.

Now, if the shape of the tile is not rectangular but an equilateral triangle like the tile in Figure 2, the above procedure of stamping with this artifact will produce exactly the tessellation corresponding to the kaleidoscopic orbifold $D_{\overline{333}}$. Similarly, using shapes like the tiles of Figures 3 and 4 there are two more artifacts that mimic the other two kaleidoscopic orbifolds.

3.2. Translations. Ancient cultures used seals to decorate their fabrics. A common seal was shaped like a cylinder that when impregnated with ink and rolled over a cloth produced a pattern that is translated in one fixed direction, in this case, perpendicular to the axis of the cylinder. Gluing two opposite sides of a rectangle we are able to construct two different kinds of cylinders giving different patterns along a ribbon. We illustrate both situations in Figure 7.

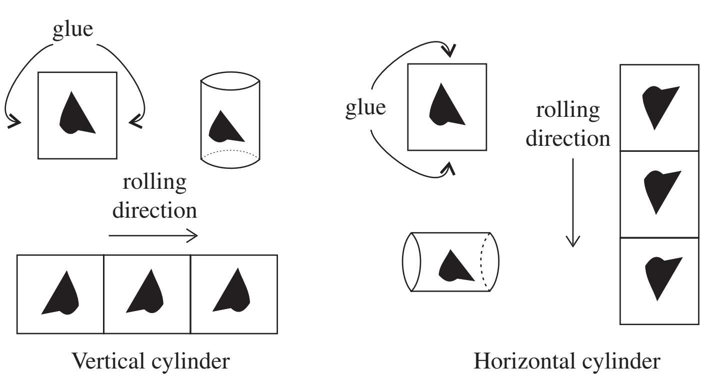

Figure 7. Horizontal and vertical translations.

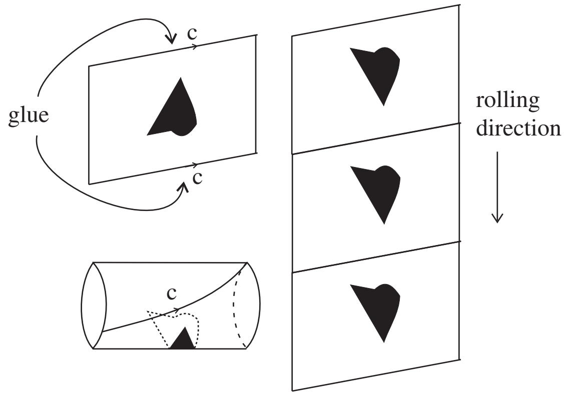

Figure 8. Translations corresponding to a parallelogram.

Gluing opposite sides of a parallelogram we obtain cylinders that give translations in two directions that are not orthogonal. As an exercise, we invite the reader to make a cylinder from a parallelogram of paper. See Figure 8.

In this way, the effect of translations can be produced by artifacts modelled as cylinders.

3.3. Glide reflection. When we compose a reflection followed by a translation we obtain an interesting rigid move that can be observed in many symmetric designs of Escher or in the Alhambra. It is called a *glide reflection*.

What artifact can produce a glide reflection? We need an artifact similar to those that produce reflections. Think, for example of a thin, flexible, and porous parallelogram and construct a Möbius band with it, i.e., choose a pair of opposite sides of the parallelogram, make a twist in the parallelogram, and glue the chosen opposite sides. Rolling a Möbius band is difficult to visualize. We invite the reader to actually construct one and to roll it. Impregnating the band with ink and rolling it over a plane we obtain a rigid move consisting of a reflection followed by a translation, i.e., a glide reflection. We illustrate the case when the parallelogram is a rectangle in Figure 9.

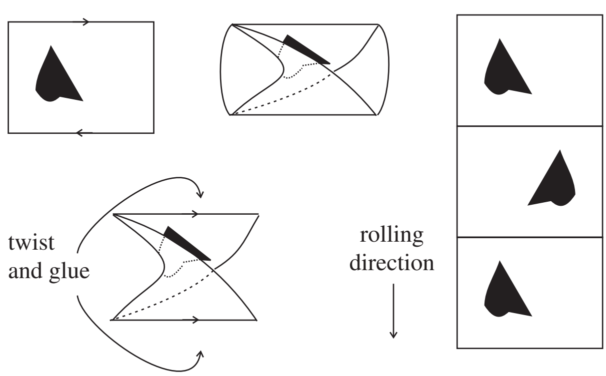

Figure 9. Band and stamped ribbon with glide reflection.

3.4. Rotations. What about a rotation? A seal shaped like a cone (with special angle) rolling on a plane produces the same effect as a rotation. Figure 10 illustrates what happens with a cone with an angle of 90°. The center of the circle is a 4-fold center.

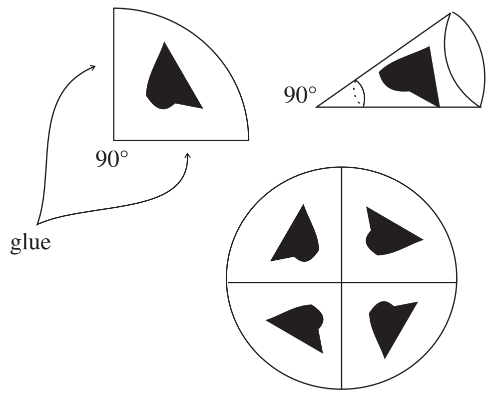

Figure 10. Stamping with a 90° cone.

Similarly, to obtain an n-fold center the cone must have an angle of $360°/n$. We note that n has to be a positive integer in order to guarantee no overlapping.

4. SEVENTEEN ARTIFACTS.

4.1. The groups $p3m1$, $p4m$, $p6m$, and pmm. In the last section we have seen that different artifacts mimic rigid moves. Moreover we already have found *four* artifacts that produce the same symmetric designs we can get with the four kaleidoscopic orbifolds $D_{\overline{333}}$, $D_{\overline{244}}$, $D_{\overline{236}}$, and $D_{\overline{2222}}$ or, respectively, with the crystallographic groups $p3m1$, $p4m$, $p6m$, and pmm. See Table 1.

In the sequel we are going to explain how to obtain thirteen more artifacts, one for each of the remaining crystallographic groups.

4.2. The group $p1$. Gluing a pair of opposite sides of a rectangle we construct a cylindrical seal that mimics translations in one determined direction. Now if we want translations in two simultaneous perpendicular directions we need to glue simultaneously both pairs of opposite sides. See Figure 11. We get a *torus*, which we can ideally

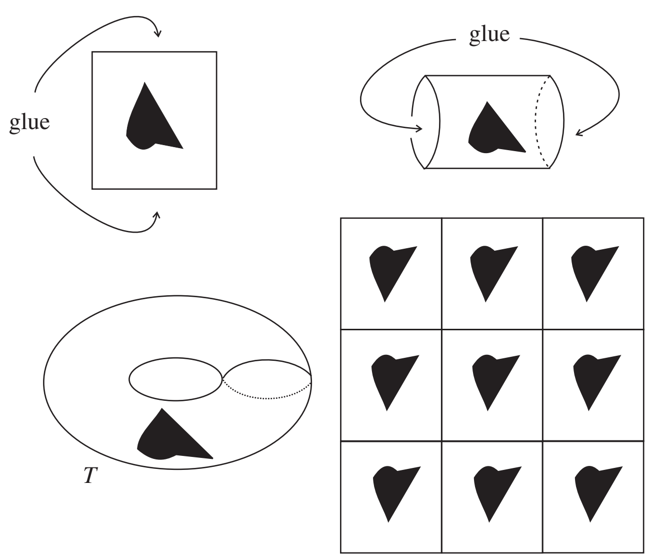

Figure 11. Stamped design with a torus.

think of as an artifact such that rolling it along a plane produces translations in two perpendicular directions. If we want to produce nonperpendicular translations we start with a nonrectangular parallelogram to construct the torus. See, for example, Escher's Pegasus in the gallery of `http://www.tessellations.org`.

This artifact, shaped like a torus, and denoted by T, produces symmetric designs that are the same as those produced by the crystallographic group $p1$, which is generated by two nonparallel translations.

In the real world it is not easy to obtain a material so flexible that it makes a torus that neatly stamps the designs. There is another possibility for this artifact. Let us describe it. Start with a parallelogram with zippers on its sides as shown in Figure 12. We have to operate both zippers alternately. For example, first we close the zipper labeled with A, obtaining a vertical cylinder that we roll stamping a ribbon. Next we open the zipper A and close the zipper B, getting a horizontal cylinder that we roll a complete turn, producing a translation perpendicular to the first stamped ribbon. Next we open the zipper B, close the zipper A, again roll the cylinder, and so on. In this manner it is possible to completely stamp the plane.

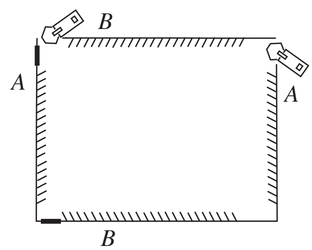

Figure 12. Zippered torus.

4.3. The group $p2$. Now, let us construct another artifact. Glue a pair of opposite sides of a parallelogram to construct a cylinder. Now, look for the midpoint of each of the two free sides and glue half of each side with the other half of the same side as illustrated in Figures 13 and 14. Note that if the parallelogram is a rectangle we obtain a pillow (see Figure 13). Each of the four vertices is the vertex of a cone with an angle

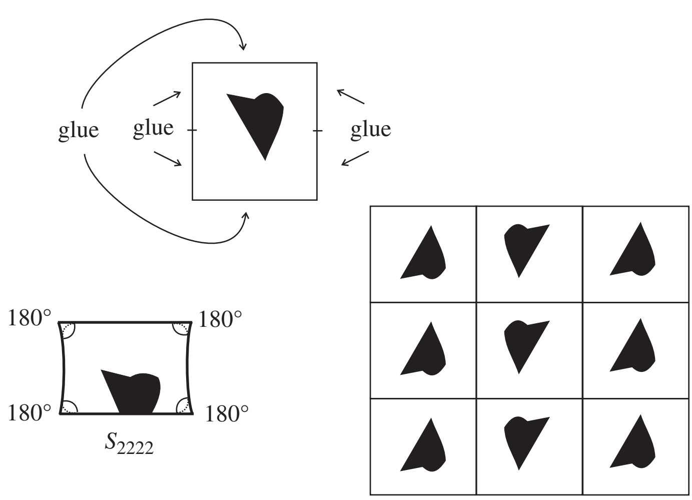

Figure 13. Pillow S_{2222} constructed from a rectangle and its stamped design.

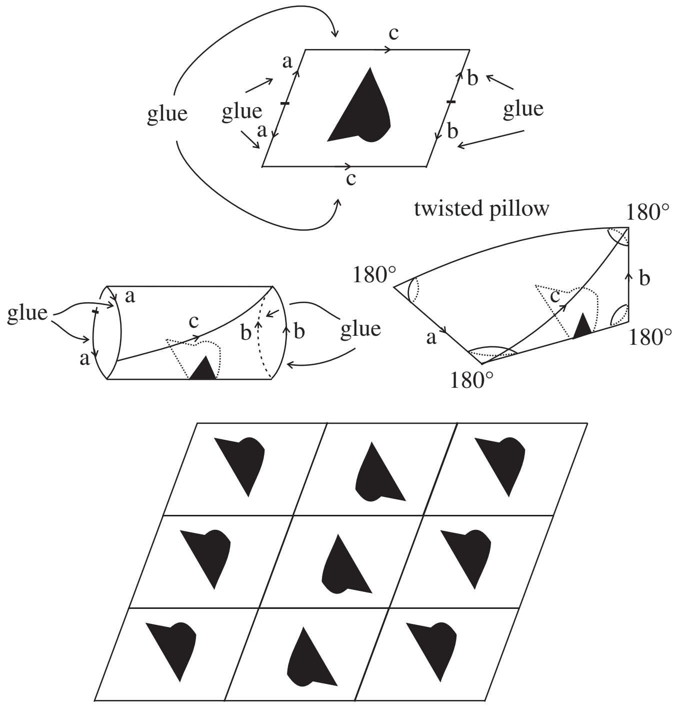

Figure 14. Translations corresponding to a twisted pillow.

of 180°. If the parallelogram is nonrectangular we obtain a twisted pillow, as in Figure 14, that can be thought of as being tetrahedral, with four vertices that are vertices of cones with 180° angles. Again, we invite the reader to do the exercise using paper parallelograms. See Escher's Squirrels (1936), Frogs (1942), Pattern (1967), or Flower Pattern (1967) in the gallery of http://www.tessellations.org.

Any of these pillows is an artifact that impregnated with ink stamps symmetric designs when it rolls on a plane as a seal. It is denoted by S_{2222} and corresponds to the crystallographic group $p2$, which is generated by four (actually, three is enough) rotations of 180°.

The letter S, in S_{2222}, means that we have a topological sphere, and the subindex n means that there is a vertex that is an n-fold center, i.e., we have a rotation of $360°/n$; in other words, each subindex n means that the pillow has a cone point of angle $360°/n$.

There is another way to construct the twisted pillow. Start from an acute triangle. Draw the three lines that join consecutive midpoints of its sides. Fold along these lines and you get a tetrahedron with vertices that are cones with angle 180°. (See Figure 15.) Observe that the four triangles in the tetrahedron pillow correspond to the four triangles in a parallelogram on the right side of Figure 15 when the pillow is rolled.

4.4. The groups *pm* and *cm*. If we make a cylinder of very flexible porous material we are able to turn the cylinder inside out and to perform reflections using it. Rolling the cylinder we stamp a ribbon with motifs that are translated along it, and turning the cylinder inside out we stamp another parallel ribbon that is the reflection of the first one. As a cylinder is topologically an annulus, this new artifact is denoted by A and is called a *ring*. It corresponds to the crystallographic group pm generated by two parallel reflections and a translation.

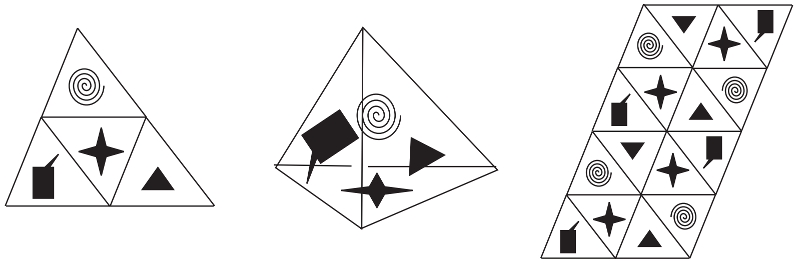

Figure 15. Acute triangle, tetrahedron, and its stamped design.

We have already explained that a Möbius band permits us to stamp a ribbon where a motif is glide reflected and translated along the ribbon. In fact, using the same artifact, we are able to fill a plane with a symmetric design. Let us explain: First stamp a ribbon rolling a porous Möbius band (as in Figure 9), and next turn the band "inside out" and stamp another parallel ribbon that reflects the first one. Continue in this way to fill out the plane. This artifact is denoted by M and is called a *Möbius band*. It corresponds to the crystallographic group cm, which is generated by a reflection and a glide reflection, where both axes of reflection are parallel. See Escher's Scarabs (1953) in the gallery of `http://www.tessellations.org`. This motif represents an A tessellation if the colors are considered but it represents an M tessellation if we do not consider the colors.

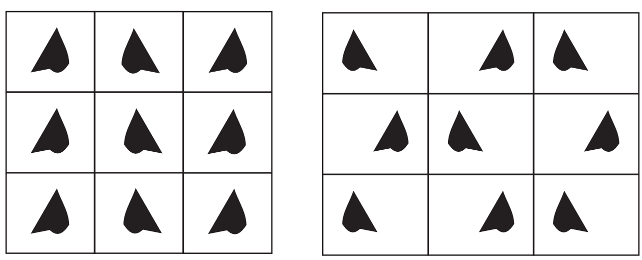

Figure 16. A and M symmetric designs.

4.5. The groups $p3$, $p4$, and $p6$. We have already studied the kaleidoscopic orbifolds $D_{\overline{333}}$, $D_{\overline{244}}$, $D_{\overline{236}}$, and $D_{\overline{2222}}$. Comparing $D_{\overline{2222}}$ with the pillow S_{2222}, we see that they are very similar except that S_{2222} is thick and has two sides with different designs, while $D_{\overline{2222}}$ is very thin and has two sides but the porosity of the material effects the transfer of the design on one side to the other side. We say that S_{2222} is the double of $D_{\overline{2222}}$. Recall that S_{2222} has four singularities that are vertices of four cones with angles of 180°.

Similarly, the doubles of $D_{\overline{333}}$, $D_{\overline{244}}$, and $D_{\overline{236}}$ exist. These doubles are like "air cushions" and are denoted by S_{333}, S_{244}, and S_{236}, respectively. Each of these "cushion" orbifolds is a topological sphere and has three cone-point singularities with cone angles of $360°/n$, where n is one subindex in the notation. We illustrate these three cushions in Figure 17.

The crystallographic group corresponding to S_{333} is the group $p3$, which is generated by two rotations of 120° with different centers. The crystallographic group corresponding to S_{244} is the group $p4$, which is generated by a rotation of 180° and a rotation of 90°. And finally, the crystallographic group corresponding to S_{236} is the group $p6$,

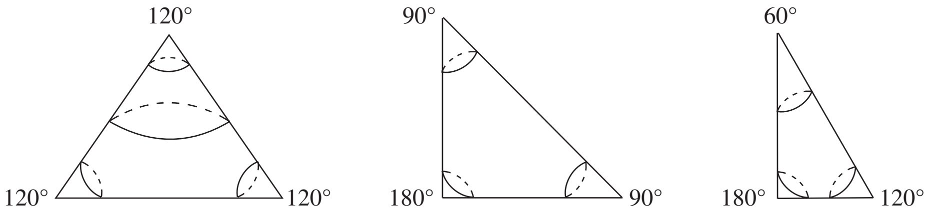

Figure 17. S_{333}, S_{244}, and S_{236}.

which is generated by rotations of 180° and 120°. For example, the Escher design Fish (1964) illustrated in http://www.tessellations.org/-eschergallery26.htm can be stamped by rolling the cushion S_{244} along a plane.

4.6. The groups *cmm* and *pmg*. To construct another artifact, let us start with the following observation. Consider an untwisted pillow S_{2222} with no design on it. Think of it as an air pillow, i.e., a surface with nothing inside. Now, let us cut the pillow along the plane of symmetry that passes through the four singularities. Then we get two rectangular pieces of the same size. If, moreover, they are made of porous material, then each piece could be thought of as the artifact $D_{\overline{2222}}$ studied before. So from S_{2222} we have obtained two copies of $D_{\overline{2222}}$. This is the reason we call S_{2222} the double of $D_{\overline{2222}}$.

Analogously, for each plane of symmetry of an untwisted pillow S_{2222} we could cut the pillow into two pieces (with the same shape and size) and each of them will furnish another artifact that, if made of porous material, will stamp a different symmetric design. There exist three types of planes of symmetry for S_{2222}. The first type is the plane P_1 that passes simultaneously through the four singularities A, B, C, and D (this is the plane mentioned in the paragraph above). The second type is represented by a plane P_2 that passes through the midpoints of two opposite sides of the rectangle $ABCD$. And the third type is represented by a plane P_3 that passes simultaneously through two opposite singularities, for example, through A and C (in this case the rectangle has to be a square, and since we have four singularities, two planes of this kind are possible). In Figure 18, we illustrate the three types of planes of symmetry of S_{2222}.

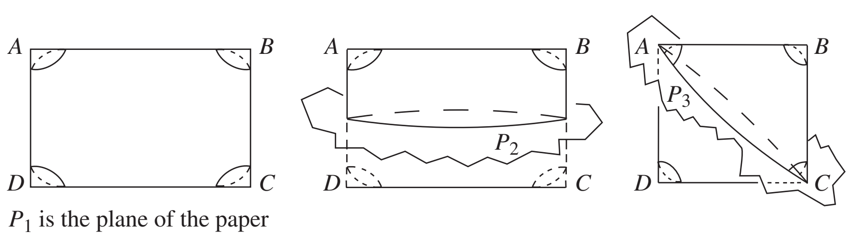

Figure 18. Symmetry planes for S_{2222}.

So, cutting the pillow along any of these planes we obtain three different artifacts. We have already seen that cutting along the plane P_1 we obtain the kaleidoscopic orbifold $D_{\overline{2222}}$. Cutting along the plane P_2 we obtain the artifact denoted by D_{22} (also a topological disc). It resembles a pillowcase with two singularities of 180° and a flat border. The flat border permits us to perform reflections with it when we roll it turning inside to outside. We think of it as being made with a porous material. The

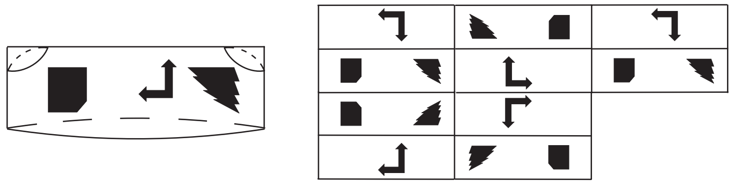

Figure 19. D_{22} and its symmetric design.

crystallographic group corresponding to D_{22} is the group pmg, which is generated by a reflection and two rotations of 180° with different centers. See Figure 19.

Now, cutting along the plane P_3 we obtain the artifact denoted by $D_{\overline{22}2}$ (also a topological disc). It resembles a party hat with a singularity of 180° and also with two 90° angles on the border. The border of this strange hat makes it possible to perform reflections with it. If it is made with a porous material we could stamp a special design by rolling the hat and alternately using outside and inside. This artifact corresponds to the crystallographic group cmm, which is generated by two perpendicular reflections and one 180° rotation. See Figure 20. The artifact $D_{\overline{22}2}$ can also be obtained from a twisted pillow S_{2222} by cutting along a plane of symmetry (if it exists) that passes simultaneously through two singularities of 180° and the middle point of the edge that joins the other two singularities.

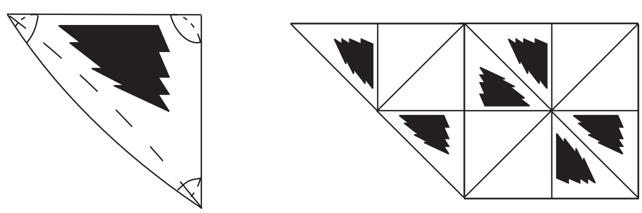

Figure 20. $D_{\overline{22}2}$ and its symmetric design.

4.7. The groups *p*4*g* and *p*31*m*. Now let us construct two more artifacts. Like $D_{\overline{22}2}$, both of them resemble party hats and are topological discs.

The first one, denoted by $D_{\overline{2}4}$, corresponds to the crystallographic group $p4g$, generated by a 90° rotation and a reflection. It has a border with one 90° angle and also it has one 90° singularity. To produce this artifact we start from the cushion orbifold S_{244} and cut it along the plane of symmetry that passes through the 180° singularity and the midpoint between the other two singularities. Once more, we think of this artifact as being made with a porous flexible material. See Figure 21. See Escher's Strong Men (1936) in the gallery of `http://www.tessellations.org`.

The second artifact is analogous to $D_{\overline{2}4}$. It is denoted by $D_{\overline{3}3}$ (Figure 22) and it is constructed from S_{333} by cutting along the plane of symmetry that passes through one of the singularities and also through the midpoint between the other two singularities. $D_{\overline{3}3}$ corresponds to the crystallographic group $p31m$, which is generated by a rotation of 120° and a reflection. See Escher's China Boy (1936) in the gallery of `http://www.tessellations.org`.

5. THE GROUPS *pg* AND *pgg*. To finish we are going to describe the last two artifacts. The first one corresponds to the group pg, generated by one glide reflection and one translation. It is denoted by K, and it is called a *Klein bottle*. Theoretically,

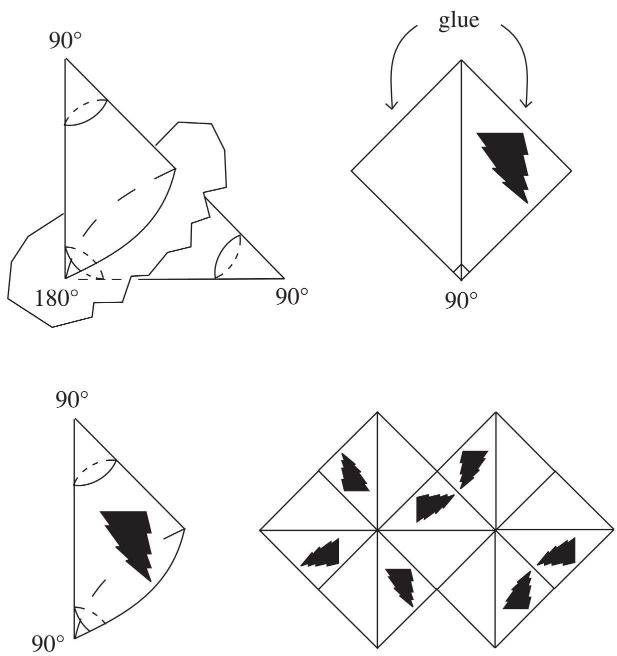

Figure 21. $D_{\bar{2}4}$ and its symmetric design.

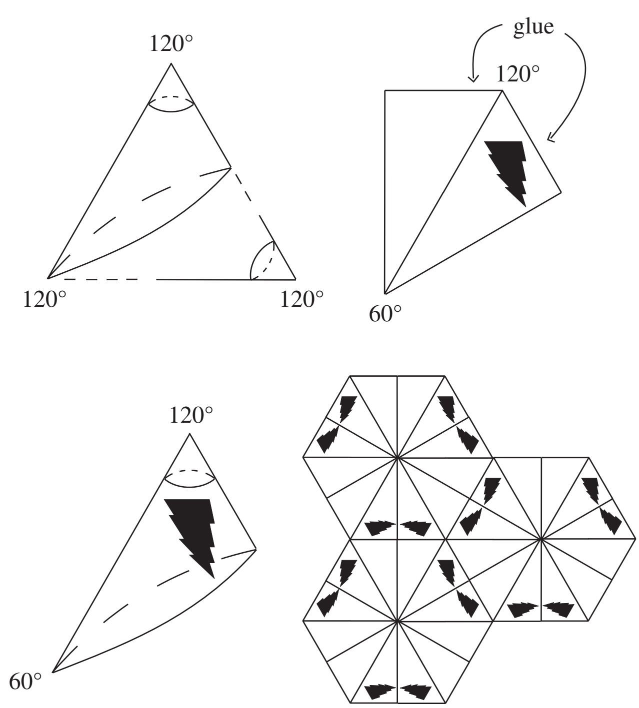

Figure 22. $D_{\bar{3}3}$ and its symmetric design.

a Klein bottle could be constructed from a quadrilateral by gluing its sides as illustrated in Figure 23. It is impossible to construct a Klein bottle in our real world. It is necessary to have at least 4 dimensions, but fortunately we can imagine an artifact that could stamp a symmetric design that would correspond to a design stamped by an ideal Klein bottle.

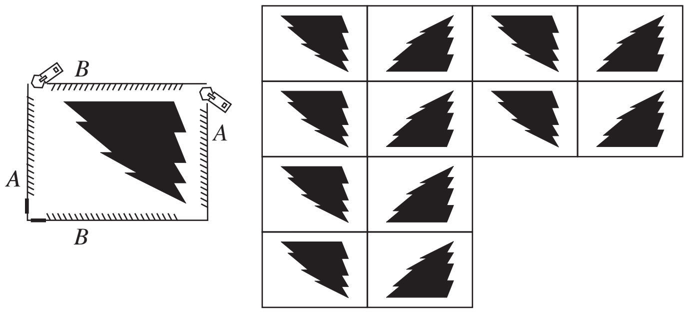

Figure 23. Zippered Klein bottle and stamped design.

In this case the artifact is made from one rectangular piece that has zippers on its sides as shown in Figure 23. Compare the direction of the zippers in Figures 12 and 24. We have to operate both zippers alternately. For example, we close the zipper labeled with A to obtain a Möbius band that when rolling stamps a ribbon. Next we open the zipper A and close the zipper B, obtaining a cylinder that when rolled one complete turn produces a translation perpendicular to the first stamped ribbon. Next we open the zipper B, close the zipper A, and again roll the Möbius band, and so on. See Escher's Swallows and Dragonflies (1938) in the gallery of http://www.tessellations.org.

Finally, the last artifact corresponds to the group pgg, generated by two perpendicular glide reflections. This artifact is denoted by P_{22} and it is called a *projective plane*. The P in the notation recalls the projective plane and the subindices mean that we have two singularities of 180°. As in the case of the Klein bottle a projective plane can ideally be constructed from a rectangle by gluing its sides as shown in Figure 24. Also, in our real world it is impossible to construct a projective plane because we would again need at least 4 dimensions, but fortunately we also can think of an artifact that could stamp the same symmetric design as an ideal projective plane. As in the last case we have a zippered artifact as shown in Figure 24. In the same way, we operate alternately

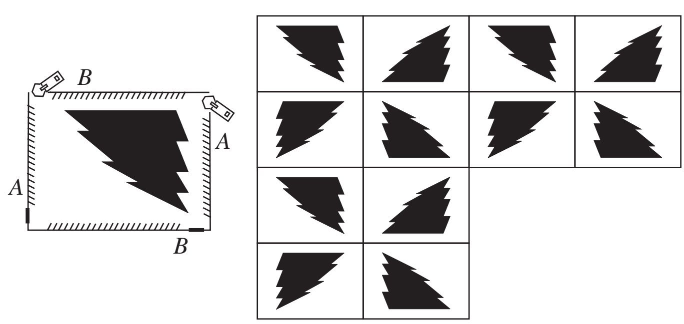

Figure 24. Zippered projective plane and stamped design.

both zippers. The difference in this case is that closing any zipper we obtain a Möbius band. To operate this artifact we start, for example, by closing the zipper labeled A, obtaining a Möbius band that we roll stamping a ribbon. Next we open the zipper A and close the zipper B, getting a new Möbius band that we roll perpendicularly to the first stamped ribbon, and so on. We notice the centers of rotation of 180° placed on the vertices of the rectangle. See Escher's Lions (1937) or Dogs (1938) in the gallery of http://www.tessellations.org.

6. CONCLUSION. The concept of orbifold introduced by Thurston in the seventies gave rise to a new way of looking at manifolds. This plays an important role in geometry and topology today, such as in Perelman's proofs of the Poincaré conjecture and geometrization conjecture.

In this paper we have tried to avoid technical language and to approach orbifolds from an intuitive point of view. Thus we have described seventeen artifacts which give interpretations of the seventeen crystallographic groups or the seventeen two-dimensional orbifolds arising from them. Readers are referred to the Associação Atractor–Matemática Interactiva (Atractor) website http://www.atractor.pt/mat/orbifolds/info-en.htm, where they will find some applets and videos that illustrate some of these artifacts. The Atractor project also has a beautiful interactive DVD, *Symmetry–dynamical presentation*.

In higher dimensions orbifold theory becomes more complicated. For example, in dimension three knot theory comes into play. The figure eight knot and the Borromean rings can be considered as the singular sets of both Euclidean and hyperbolic three-dimensional orbifolds. See Figure 25. In three dimensions there are exactly 219 pairwise nonisomorphic crystallographic Euclidean groups. The interested reader should read the beautiful paper of Bonahon and Siebenmann [**1**]. We strongly recommend as an introduction to tessellations the nice and elementary book of Seymour and Britton [**10**].

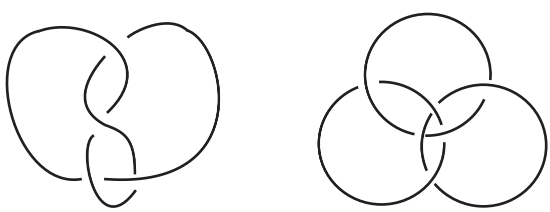

Figure 25. The figure eight knot and Borromean rings.

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