A GEOMETRIC APPROACH TO THE LOCAL THEORY OF BI-HAMILTONIAN STRUCTURES

(Bihamiltonian Geometry and Integrable Systems, Summer School, Banach Center, Bedlewo 29 August-2 September 2011)

Francisco-Javier Turiel

Geometría y Topología, Facultad de Ciencias, Campus de Teatinos, 29071 Málaga, Spain

e-mail: turiel@agt.cie.uma.es

Minimal bibliography

Gelfand I.M. et Zakharevich I., *Webs, Veronese Curves and Bihamiltonian Systems*, J. Funct. An., **99** (1991), 150-178.

Magri, F., *A simple model of integrable hamiltonian equations*, J. Math. Phys. **19** (1978) 1156-1162.

Panasyuk A., *Veronese Webs for Bihamiltonian Structures of Higher Corank*, Banach Center Publ. **51** (2000), 251-261.

Rigal, M.-H., *Systèmes bihamiltoniens en dimension impaire*, Ann. Sci. Ec. Norm. Supér., IV. Sér. **31** (1998).

Turiel F.J., *Classification locale d'un couple de formes symplectiques Poisson compatibles*, C.R. Acad. Sci., Paris, Série I **308** (1989) 575-578.

The details of the present lectures can be seen in (by F.J. Turiel)

Classification locale simultanée de deux formes symplectiques compatibles, Manuscripta Math. **82** (1994), 349-362.

On the local theory of Veronese webs, arXiv:1001.3098v1 [math.DG]

The local product theorem for bihamiltonian structures, arXiv:1107.2243v1 [math.SG]

A. PAIRS OF COMPATIBLE SYMPLECTIC FORMS

On a real or complex $2n$-dimensional manifold M consider a pair of closed 2-forms ω, ω_1, symplectic the first one. Let J be the (1,1)-tensor field defined by the relation $\omega_1(X,Y) = \omega(JX,Y)$; then $\omega_k(X,Y) = \omega(J^k X, Y)$ is a 2-form. One will say that ω, ω_1 are *compatible* if ω_2 is closed, which is equivalent to say that $N_J = 0$. When (ω, ω_1) is compatible a local classification can be given on an open dense set of M (the regular open set).

1. Examples. 1) Let Ω be a symplectic form on a complex manifold P of complex dimension $2k$. Set $\Omega = \omega + \imath\omega_1$, where ω, ω_1 are the real and the imaginary part respectively. Then (ω, ω_1) is a compatible pair when P is regarded as a real manifold of dimension $4k$.
2) Take two symplectic manifolds (M_1, β_1), (M_2, β_2), then on $M = M_1 \times M_2$ the pair $\omega = \beta_1 + \beta_2$, $\omega_1 = \beta_1 - \beta_2$ is compatible.
3) Let (G, ω) be a Lie group endowed with a left invariant symplectic form, and let ω_1 be the right invariant 2-form such that $\omega(e) = \omega_1(e)$ where e is the neutral element of G. Then (ω, ω_1) is a compatible pair on G.
4) On a n-manifold N consider a (1,1)-tensor field H, which gives rise to a vector bundle morphism $\varphi_H : T^*N \to T^*N$ by setting $\varphi_H(\lambda) = \lambda \circ H$. Let ω be the symplectic Liouville form of T^*N and let $\omega_1 = (\varphi_H)^*\omega$. Then (ω, ω_1) is a compatible if and only if $N_H = 0$.

2. The algebraic structure.

Let (α, α_1) be a couple of 2-forms on a $2n$-dimensional, real or complex, vector space V the first one symplectic ($\alpha^n \neq 0$). We define $J \in End(V)$ by the relation $\alpha_1(v,w) = \alpha(Jv,w)$ for any $v, w \in V$. Then $\alpha(J\ \ ,\ \ ) = \alpha(\ \ , J\ \ )$ and each $\alpha_k = \alpha(J^k\ \ ,\ \ )$ is a 2-form [k may be negative if J is an isomorphism]. The classification of the couple (α, α_1) is given by that of J and, therefore, by the family of elementary divisors of J. From now on, *the characteristic polynomial, the minimal polynomial and the elementary divisors of* (α, α_1) will be those of J.

For understanding the algebraic classification better consider the following situation. Let H be an endomorphism of a n-dimensional vector space W; then on $W \oplus W^*$ one has the symplectic form $\alpha((v,\lambda),(w,\mu)) = \lambda(w) - \mu(v)$ and the 2-form $\alpha_1((v,\lambda),(w,\mu)) = \lambda(Hw) - \mu(Hv)$, whose relation endomorphism J equals $H \oplus H^*$. *In this way it can be constructed all the models of couples* (α, α_1). Thus the elementary divisors of (α, α_1) occur an even number of times and the characteristic polynomial is the square of another polynomial, which is divided by the minimal polynomial.

3. The regular open set. A first reduction.

Let $\mathbb{K}_P[t]$ be the polynomial algebra in one variable over the ring of differentiable functions on a manifold P. A polynomial $\varphi \in \mathbb{K}_P[t]$ is said *irreducible* if it is irreducible at every point of P. Two polynomials $\varphi, \psi \in \mathbb{K}_P[t]$ are called *relatively prime* if they are at each point. Given a vector bundle E over P of dimension $\widetilde{m}$ and a morphism $H : E \to E$, its characteristic polynomial $\varphi = \sum_{j=0}^{\widetilde{m}} h_j t^j$ belongs to $\mathbb{K}_P[t]$. Set $g_j = trace(H^j)$. Since $h_0, ..., h_{\widetilde{m}-1}$ are, up to sign, the elementary symmetric polynomials of the roots and each g_j the sum of their j-th powers, every function g_j may be expressed as a rational polynomial of $h_0, ..., h_{\widetilde{m}-1}$, and each function h_j like a rational polynomial of $g_1, ..., g_{\widetilde{m}}$. In particular g_j when $j \geq \widetilde{m}+1$ equals a rational polynomial of $g_1, ..., g_{\widetilde{m}}$.

One will say that $H : E \to E$ *has constant algebraic type* if there exist relatively prime irreducible polynomials $\varphi_1, ..., \varphi_s \in \mathbb{K}_P[t]$ and positive integers a_{jk}, $j = 1, ..., r_k$, $k = 1, ..., s$, such that at each point $p \in P$ the family $\{\varphi_k^{a_{jk}}(p)\}$, $j = 1, ..., r_k$, $k = 1, ..., s$, is that of elementary divisors of $H(p)$. Though in general the algebraic type of H is not constant, the set of all points such that around of them H has constant algebraic type is open and dense.

Now suppose that E is a foliation and $N_H = 0$; then $j dg_{j+1} = (j+1) dg_j \circ H$. Therefore $\bigcap_{j=1}^{\widetilde{m}} Ker dg_j(p) = \bigcap_{j=0}^{\widetilde{m}-1} Ker dh_j(p)$ is a H-invariant vector subspace of T_pP because each g_j, $j \geq \widetilde{m}+1$, is a function of $g_1, ..., g_{\widetilde{m}}$.

One will be say that a point $p \in P$ is *regular* if there exists an open neighbourhood B of p such that:

(1) H has constant algebraic type on B,

(2) $\bigcap_{j=1}^{\widetilde{m}} Ker dg_j$, restricted to B, is a vector sub-bundle of E and therefore a foliation,

(3) H restricted to $\bigcap_{j=1}^{\widetilde{m}} Ker dg_j$ has constant algebraic type on B.

The set of all regular points is a dense open set P, called the *regular open set*. On the other hand H, $aH + bI$, $a \in \mathbb{K} - \{0\}, b \in \mathbb{K}$, and H^{-1} if H is invertible, have the same regular open set.

Now consider a compatible pair (ω, ω_1) on a $2n$-manifold M; by definition its regular open set will be that of J. Assume that about some regular point p the characteristic polynomial φ of (ω, ω_1) equals the product $\varphi_1 \cdot \varphi_2$ of two monic polynomials, relatively prime at each point. Then, always around p, (M, ω, ω_1) decomposes into a product $(M', \omega', \omega_1') \times (M'', \omega'', \omega_1'')$ of two compatible pairs in such a way that φ_1 is the characteristic polynomial of (ω', ω_1') [more exactly φ_1 is the pull-back of the characteristic polynomial of (ω', ω_1') by the the first projection], φ_2 that of $(M'', \omega'', \omega_1'')$ and p', p'' are regular points where $p = (p', p'')$.

Reiterating the process reduces the classification problem to the case where φ is a power of an irreducible polynomial, that is to say $\varphi = (t+f)^{2n}$ or $\varphi = (t^2 + ft + g)^n$, with $f^2 - 4g < 0$; this last case only on real manifolds.

3. The local classification.

First one has:

Theorem 1. *Consider a compatible pair (ω, ω_1) whose characteristic polynomial equals $(t+f)^{2n}$. Then about any regular point p there exists a system of coordinates $((x_i^j), y_1, y_2)$, with $p \equiv 0$, such that:*
(a) $i = 1, ..., 2r_j$ and $r_1 \geq ... \geq r_\ell$ including the two limit cases: no coordinates (y_1, y_2) or no coordinates (x_i^j).

(b) $\omega = \sum_{j=1}^{\ell} \sum_{i=1}^{r_j} dx_{2i-1}^j \wedge dx_{2i}^j + dy_1 \wedge dy_2$

$\omega_1 = (y_2 + a)\omega + \tau + \alpha \wedge dy_2$

where a is constant, $\tau = \sum_{j=1}^{\ell} \sum_{i=1}^{r_j - 1} dx_{2i-1}^j \wedge dx_{2i+2}^j$ *and*

$\alpha = dx_2^1 + \sum_{j=1}^{\ell} \sum_{i=1}^{r_j} ([i + \frac{1}{2}] x_{2i}^j dx_{2i-1}^j + [i - \frac{1}{2}] x_{2i-1}^j dx_{2i}^j)$.

Remark. In the foregoing theorem the local model is completely determined by the elementary divisors of (ω, ω_1). When (ω, ω_1) is 0-deformable, that is if there are no coordinates (y_1, y_2), they are $\{(t-a)^{r_j}, (t-a)^{r_j}\}$, $j = 1, ..., \ell$, whereas for the non 0-deformable case one has $(t - (y_2 + a))^{r_1 + 1}, (t - (y_2 + a))^{r_1 + 1}, \{(t - (y_2 + a))^{r_j}, (t - (y_2 + a))^{r_j}\}$, $j = 2, ..., \ell$.

The case where the characteristic polynomial equals $(t^2+ft+g)^n$, $f^2-4g<0$, is dealt with by considering the semi-simple part H of $J_0 = 2(4g-f^2)^{-1/2}J + f(4g-f^2)^{-1/2}Id$, which is a complex structure. If we set $\Omega = \omega + \imath\tilde{\omega}$ and $\Omega_1 = \omega_1 + \imath\tilde{\omega}_1$, where $\tilde{\omega}(X,Y) = -\omega(HX,Y)$ and $\tilde{\omega}_1(X,Y) = -\omega_1(HX,Y)$, then (Ω,Ω_1) is a holomorphic compatible pair whose characteristic polynomial equals $(t+h)^n$, with $h = \frac{1}{2}(f - \imath(4g-f^2)^{-1/2})$. In other words, (ω,ω_1) is the real part of a complex compatible pair with the same regular open set. Thus:

Theorem 2. *Let (ω,ω_1) be a compatible pair. Then its local model around any regular point is a finite product of factors chosen among:*
(a) If the manifold is complex those of theorem 1.
(b) If the manifold is real those of theorem 1 and the real part of the complex models of this theorem.

The elementary divisors of (ω,ω_1) completely determine the local model.

4. Symplectic bihamiltonian structures.

Let (Λ,Λ_1) be a bihamiltonian structure on a $2n$-manifold M. Assume that $rang\Lambda = 2n$ everywhere; let H be the (1,1)-tensor field defined by $\Lambda_1(\alpha,\beta) = \Lambda(H^*\alpha,\beta) = \Lambda(\alpha\circ H,\beta)$. Since the problem is local, one may suppose Λ_1 invertible too (if not take $\Lambda_1 + a\Lambda$ for a suitable scalar a instead of Λ_1). Let (ω,ω_1) be the dual symplectic pair of (Λ,Λ_1). Recall that $H = J^{-1}$ where $\omega_1 = \omega(J,\quad)$; moreover if $\omega(X,\quad) = \alpha$ and $\omega(Y,\quad) = \beta$ then $\Lambda_1(\alpha,\beta) = \omega_1(J^{-1}X, J^{-1}Y) = \omega_{-1}(X,Y)$.

For obtaining the explicit local model of (Λ,Λ_1) at any regular point, it suffices to apply theorem 2 to (ω,ω_{-1}). Indeed, if $(z_1,...,z_{2n})$ are coordinates as in theorems 1 and 2 with respect to (ω,ω_{-1}), then the coefficients of Λ are easily deducted from the expression of ω and those of Λ_1 from ω_{-1} through the rule $\Lambda_1(dz_i,dz_j) = \omega_{-1}(Z_i,Z_j)$, where $dz_k = \omega(Z_k,\quad) = dz_k$, $k=1,...,2n$ (note that each Z_k is just a partial derivative).

B. VERONESE WEBS (LOCAL THEORY)

Veronese webs were introduced as a tool for studying the bihamiltonian structures of constant rank without symplectic factor. As before our structures will be real (at least of class C^∞) or complex (holomorphic). Among the different approaches to this theory, here we will chose that consisting in giving a r-codimensional foliation $\mathcal{F}$ on a n-manifold and a suitable morphism $\ell : \mathcal{F} \to TN$.

1. Algebraic Veronese webs.

Consider a n-dimensional vector space V. A curve $\gamma(t)$, $t \in \mathbb{K}$, in $\Lambda^r V^*$, $r \geq 1$, is called *a Veronese curve* if there exists a basis $\{e_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, of V such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ where each $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1} e^*_{ij}$. Thus γ is a polynomial curve of degree $n - r$; moreover, up to permutation, the family of natural numbers $\{n_1, ..., n_r\}$ only depends on γ and this last one completely determines the Veronese curve. For convenience one will set $\gamma(\infty) = lim \frac{\gamma(t)}{t^{n-r}}$ when $t \to \infty$.

A family $w = \{w(t) \mid t \in \mathbb{K}\}$ of $(n-r)$-planes of V is called *a Veronese web of codimension* r if there exists a Veronese curve γ in $\Lambda^r V^*$ such that $w(t) = Ker\gamma(t)$, $t \in \mathbb{K}$. The curve γ will be named a *representative of* w. If $\tilde{\gamma}$ is another representative of w then $\tilde{\gamma} = a\gamma$, $a \in \mathbb{K} - \{0\}$. This allows us to define $w(\infty) = Ker\gamma(\infty)$, which does not depend on the representative. Moreover if $\{\beta_{ij}\}$, $i = 1, ..., n_j$, $j = 1, ..., r$, is a basis of V^* such that $\gamma(t) = \gamma_1(t) \wedge ... \wedge \gamma_r(t)$ where each $\gamma_j(t) = \sum_{i=1}^{n_j} t^{i-1} \beta_{ij}$, then $w(\infty) = Ker(\beta_{n_1 1} \wedge ... \wedge \beta_{n_r r})$.

Proposition 1. *Consider a n-dimensional vector space V and a natural number $1 \leq r \leq n$.*
(a) Given a r-codimensional vector subspace $W \subset V$ and $J \in End(V)$, if (W', J^*) spans V^* where W' is the annihilator of W in V^* then $\gamma(t) = \varphi(t)((J + tI)^{-1})^*\beta$, where φ is the characteristic polynomial of $-J$ and β a r-form such that $Ker\beta = W$, represents a Veronese web w of codimension r.

Moreover $lim_{t\to\infty} t^{r-n}\gamma(t) = \beta$, $w(\infty) = W$ and $(J + tI)w(\infty) = w(t)$ for any $t \in \mathbb{K}$.
(b) Any Veronese web on V of codimension r may be represented in this way.

(c) Assume that $\gamma(t) = \varphi(t)((J+tI)^{-1})^*\beta$ *and* $\tilde{\gamma}(t) = \tilde{\varphi}(t)((\tilde{J}+tI)^{-1})^*\tilde{\beta}$ *represent two Veronese webs* w *and* $\tilde{w}$ *respectively. Then* $w = \tilde{w}$ *if and only if* $\tilde{\beta} = a\beta$, $a \in \mathbb{K} - \{0\}$, *and* $Ker(\tilde{J} - J) \supset w(\infty) = \tilde{w}(\infty)$.

In this last case $\tilde{\gamma} = \gamma$ *if and only if* $\tilde{\beta} = \beta$.

(d) Up to permutation the family of natural numbers $\{n_1, ..., n_r\}$, *associated to a splitting of a representative of a Veronese web* w, *only depends on* w. *This family characterizes the Veronese web up to isomorphism.*

By definition $n_1, ..., n_r$ *will be called the* the characteristic numbers of w *and their maximum* the height of w.

Remark. It is easily checked that (W', J^*) spans V^* if and only if W does not contain any non-zero J-invariant vector subspace.

By (c) of proposition 1 the restriction of J to $w(\infty)$ gives rise to a morphism $\ell : w(\infty) \to V$ with no ℓ-invariant vector subspace different from zero (this notion is meaningful since : $w(\infty) \subset V$) and which only depends on the Veronese web w. Moreover $(\ell + tI)w(\infty) = w(t)$, $t \in \mathbb{K}$, that is to say $\ell^*\alpha = -t\alpha_{|w(\infty)}$ for any $\alpha \in V^*$ such that $\alpha(w(t)) = 0$ and any $t \in \mathbb{K}$. This last property characterizes ℓ completely because the union of the annihilators of $w(t)$, $t \in \mathbb{K}$, spans V^*.

Conversely given a morphism $\ell : W \to V$ whose only ℓ-invariant vector subspace is zero, we may construct a Veronese web by considering an endomorphism J of V such that $J_{|W} = \ell$ and applying (a) of proposition 1.2 to it. This Veronese web only depends on ℓ. In fact $w(t) = (\ell + tI)W$. Thus:

Giving a Veronese web of codimension $r \geq 1$ *is equivalent to giving a morphism* $\ell : W \to V$, *where* W *is a* r*-codimensional vector subspace, without non-zero* ℓ*-invariant vector subspaces.*

Proposition 2. *Consider a Veronese web* w *of codimension* $r \geq 1$, *a basis* $\{\alpha_1, ..., \alpha_n\}$ *of* V^* *and scalars* $a_1, ..., a_n$. *Assume that* $\alpha_j(w(-a_j)) = 0$, $j = 1, ..., n$. *Then* w *can be constructed through (a) of proposition 1 by means of the endomorphism* J *defined by* $J^*\alpha_j = a_j\alpha_j$, $j = 1, ..., n$.

Before ending this paragraph we recall the classification of pairs of bivectors. Consider, on a finite dimensional vector space W, a pair of bivectors (Λ, Λ_1). One defines *the rank of* (Λ, Λ_1) as the maximum of ranks of $(1-t)\Lambda + t\Lambda_1$,

$t \in \mathbb{K}$. Note that $rank((1-t)\Lambda+t\Lambda_1) = rank(\Lambda, \Lambda_1)$ except for a finite number of scalars t, which is $\leq \frac{dimW}{2}$ (they are given by the polynomial equation $((1-t)\Lambda + t\Lambda_1)^k = 0$ where $rank(\Lambda, \Lambda_1) = 2k$). We will say that (Λ, Λ_1) is *maximal (or of maximal rank)* if $rank(\Lambda) = rank(\Lambda_1) = rank(\Lambda, \Lambda_1)$. Obviously if (Λ, Λ_1) is not maximal one may choose $\Lambda' = (1-a)\Lambda + a\Lambda_1$, $\Lambda'_1 = (1-a_1)\Lambda + a_1\Lambda_1$, with $a \neq a_1$, which is maximal. Consequently it suffices classifying maximal pairs.

Let U be a $(2m-1)$-dimensional vector space. The action of the linear group $GL(U)$ on $(\Lambda^2 U) \times (\Lambda^2 U)$ possesses one dense open orbit; each element of this orbit, all of them isomorphic, will be named *the elementary Kronecker pair in dimension* $2m-1$. By a *Kronecker pair* we mean a pair that is isomorphic to a finite product of Kronecker elementary pairs, while a pair is called *symplectic* if it is the dual of a couple of symplectic forms; in both cases they are maximal. As is well known:

Any maximal pair of bivectors (Λ, Λ_1) on a finite dimensional vector space is isomorphic to the product of a symplectic pair and r Kronecker elementary pairs, where $r = corank(\Lambda, \Lambda_1)$, including the case with no symplectic factor and that where $r = 0$.

Moreover, up to isomorphism and change of order, the factors are unique.

2. Veronese webs on manifolds.

Let N be a real or complex manifold of dimension n. A family $w = \{w(t) \mid t \in \mathbb{K}\}$ of involutive distributions (or foliations) on N of codimension $r \geq 1$ is named a *Veronese web of codimension* r, if for any $p \in N$ there exist an open neighborhood A of this point and a curve $\gamma(t)$ in the module of sections of $\Lambda^r T^* A$ (that is to say $\gamma(t)(q) \in \Lambda^r T^*_q A = \Lambda^r T^*_q N$ for every $q \in A$) such that:

1) $w(t) = Ker\gamma(t)$, $t \in \mathbb{K}$, on A

2) for each $q \in A$, $\gamma(t)(q)$ is a Veronese curve in $\Lambda^r T^*_q N$.

The curve γ is called a *(local) representative of* w.

Although curves $\gamma(t)(q)$ and $\gamma(t)(q')$ could be not isomorphic when $q \neq q'$, $\gamma(t) = \sum_{i=0}^{n-r} t^i \gamma_i$ where $\gamma_0, ...\gamma_{n-r}$ are differentiable r-forms on A. On the other hand $Ker\gamma_{n-r}$ is an involutive distribution of dimension $n-r$ since each $Ker\gamma(t)$ was integrable and $lim t^{r-n}\gamma(t) = \gamma_{n-r}$, $t \to \infty$. This allows us to define $w(\infty) = Ker\gamma_{n-r}$, which does not depend on the representative because if $\tilde{\gamma}$ is another representative then $\tilde{\gamma} = f\gamma$ on the common domain. In particular, there exists a global representative if and only if $w(\infty)$ is transversally orientable. Obviously w as map from $\mathbb{K}\cup\{\infty\} \equiv \mathbb{K}P^1$ to the Grassmann manifold of $(n-r)$-plans of TN is smooth.

Examples. 1) On S^3 regarded as a Lie group consider three left invariant contact forms ρ_1, ρ_2, ρ_3. Suppose that $\rho_1 \wedge \rho_2 \wedge \rho_3 \neq 0$ and set $\gamma(t) = (\rho_1 + t\rho_2) \wedge \rho_3$. Then γ defines a codimension two Veronese web which is not flat because $Ker\rho_3 = w(0) \oplus w(\infty)$ is a contact structure.

2) On $\mathbb{K}^4$ with coordinates (x_1, x_2, y_1, y_2) set $\gamma(t) = (dx_2 \wedge dy_2 + x_2 dx_1 \wedge dx_2) + t(x_2 dx_2 \wedge dy_1 - dx_1 \wedge dy_2) + t^2 dy_1 \wedge dy_2$. Then γ defines a Veronese web of codimension two since $d\gamma(t) = 0$ and $\gamma(t) = (-dx_1 + x_2^{-1} dy_2 + t dy_1) \wedge (-x_2 dx_2 + t dy_2)$ when $x_2 \neq 0$, while $\gamma(t) = (dx_2 - t dx_1 + t^2 dy_1) \wedge dy_2$ if $x_2 = 0$.

Note that $\gamma(t)(q)$ and $\gamma(t)(q')$ are not isomorphic as Veronese curves when $q_2 \neq 0$ and $q'_2 = 0$.

3) Let V be the 3-dimensional Lie algebra spanned by the vectors fields on $\mathbb{K}$: $X_1 = (\partial/\partial t)$, $X_2 = t(\partial/\partial t)$ and $X_3 = t^2(\partial/\partial t)$. Set $\tilde{w}(t) = \{v \in V \mid v(t) = 0\}$. As $\tilde{w}(t) = Ker\{e_1^* + te_2^* + t^2 e_3^*\}$ where $\{e_1^*, e_2^*, e_3^*\}$ is the dual basis of $\{X_1, X_2, X_3\}$, $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$ is an algebraic Veronese web on V. But V is isomorphic to the Lie algebra of $SL(2, \mathbb{K})$ and each $\tilde{w}(t)$ is a subalgebra of V; therefore $\tilde{w}$ gives rise to a Veronese web w of codimension one on any 3-dimensional homogeneous space of $SL(2, \mathbb{K})$.

A local description of Veronese webs is given by the following result:

Theorem 1. *Let N be a n-dimensional real or complex manifold.*
(1) Consider a Veronese web w on N of codimension r and non-equal scalars $a_1, ..., a_{n-k}, a$ where $1 \leq k \leq r$. Then for each $p \in N$ there exist an open set $p \in A$ and a $(1,1)$-tensor field J on A whit characteristic polynomial $\varphi(t) = (\prod_{j=1}^{n-k}(t - a_j))(t - a)^k$, which is flat and diagonalizable, such that:
(I) $(Ker(J^* - a_j I))w(-a_j) = 0$, $j = 1, ..., n-k$, and $(Ker(J^* - aI))w(-a) = 0$.
(II) For any $q \in A$, $(w(\infty)(q)', J^*(q))$ spans $T_q^* A$, that is to say $w(\infty)(q)$ contains no J-invariant vector subspace different from zero.

In particular, if β is a r-form and $Ker\beta = w(\infty)$ then $\gamma(t) = (\prod_{j=1}^{n-k}(t + a_j))(t + a)^k((J + tI)^{-1})^* \beta$ represents w.

Moreover is λ is a closed 1-form such that $Ker\lambda \supset w(\infty)$ then $d(\lambda \circ J)_{|w(\infty)} = 0$.

(2) Conversely, on N consider a foliation $\mathcal{F}$ of codimension $r \geq 1$, a r-form $\bar{\beta}$ such that $Ker\bar{\beta} = \mathcal{F}$ and $(1,1)$-tensor field $\bar{J}$ with characteristic polynomial $\bar{\varphi}(t)$. Suppose that:
(I) $(\mathcal{F}', \bar{J}^*)$ spans T^*N, that is to say $\mathcal{F}$ does not contain any non-zero $\bar{J}$-invariant vector subspace.
(II) $(N_{\bar{J}})_{|\mathcal{F}} = 0$ and $d(\mu \circ \bar{J})_{|\mathcal{F}} = 0$ for each closed 1-form μ such that $Ker\mu \supset \mathcal{F}$ (note that if $\mathcal{F} = Ker(\lambda_1 \wedge ... \wedge \lambda_r)$ where each λ_j is a closed 1-form, this last condition is satisfied if and only if $\lambda_1 \wedge ... \wedge \lambda_r \wedge d(\lambda_j \circ \bar{J}) = 0$, $j = 1, ..., r$).

Then $\bar{\gamma}(t) = (-1)^n \bar{\varphi}(-t)((\bar{J} + tI)^{-1})^* \bar{\beta}$ defines a Veronese web $\bar{w}$ of codimension r for which $\bar{w}(\infty) = \mathcal{F}$. This Veronese web only depends on $\mathcal{F}$ and $\bar{J}$.

Example. On an open set A of $\mathbb{K}^n$ consider a $(1,1)$-tensor field $J = \sum_{j=1}^n f_j(x_j) \frac{\partial}{\partial x_j} \otimes dx_j$ where $f_j(x_j) \neq f_k(x_k)$ whenever $x = (x_1, ..., x_n) \in A$. Set $\beta = \sum_{j=1}^n dx_j$. As $N_J = 0$, (β, J^*) spans T^*A and $d(\beta \circ J) = 0$, by (2) of theorem 1 the curve $\gamma(t) = \prod_{j=1}^n (t + f_j) \beta \circ (J + tI)^{-1} = \sum_{j=1}^n (\prod_{i=1; i \neq j}^n (t + f_i)) dx_j$ defines a Veronese web w on A of codimension one, which generally is not flat.

For obtaining a 2-codimensional Veronese web $\tilde{w}$, one may consider a second 1-form $\beta' = \sum_{j=1}^n g_j(x_j) dx_j$ such that $\beta \wedge \beta'$ never vanishes and set $\tilde{\gamma}(t) = \prod_{j=1}^n (t + f_j)((J + tI)^{-1})^*(\beta \wedge \beta') = \sum_{1 \leq j < k \leq n} (\prod_{i=1; i \neq j,k}^n (t + f_i))(g_k - g_j) dx_j \wedge dx_k$.

Theorem 1 gives a method for constructing all Veronese webs locally. If $r \geq 2$ the scalars $a_1, ..., a_{n-k}, a$ do not determine J which prevent us constructing this tensor globally; on the contrary, when $r = 1$ the tensor J exists on the whole N.

In view of proposition 1, the restriction of J to $w(\infty)$ gives rise to a morphism (of vector bundles) $\ell : w(\infty) \to TN$, which only depends on the Veronese web, without non-zero ℓ-invariant vector subspace at any point of N. Moreover $w(t) = (\ell + tI) w(\infty)$, $t \in \mathbb{K}$.

In some cases the Nijenhuis torsion of a partial (1,1)-tensor field can be defined. More exactly, on a manifold M consider a foliation $\mathcal{F}$ and a morphism (of vector bundles) $G : \mathcal{F} \to TM$. If α is a s-form defined on an open set A of M, then $G^*\alpha$ [we also write $\alpha(G, ..., G)$ or $\alpha \circ G$ instead $G^*\alpha$] is a section on A of $\Lambda^s \mathcal{F}^*$ and can regarded as a s-form on the leaves of $\mathcal{F}$; thus we shall say that is *closed on* $\mathcal{F}$ if it is closed on its leaves. Besides, when $\bar{G} : TM \to TM$ is a prolongation of G, then $d(\bar{G}^*\alpha)_{|\mathcal{F}}$ equals the exterior derivative of $G^*\alpha$ along the leaves of $\mathcal{F}$; thus $G^*\alpha$ is closed on $\mathcal{F}$ if and only if $d(\bar{G}^*\alpha)_{|\mathcal{F}} = 0$. One has:

Lemma 1. *Assume that $G^*\alpha$ is closed on $\mathcal{F}$ for every closed 1-form α such that $Ker\alpha \supset \mathcal{F}$. Then the restriction of $N_{\bar{G}}$ to $\mathcal{F}$, which will be named* the Nijenhuis torsion of G *and denoted by N_G, does not depend on the prolongation $\bar{G}$.*

Note that the Nijenhuis torsion of $\ell : w(\infty) \to TN$ vanishes and $\ell^*\alpha$ is closed on $w(\infty)$ for every closed 1-form α such that $Ker\alpha \supset w(\infty)$ since J, its local prolongation given by (1) of theorem 1, has zero Nijenhuis torsion and $d(\alpha \circ J)_{|w(\infty)} = 0$.

Conversely, given a foliation $\mathcal{F}$ on N of codimension $1 \leq r \leq n$ and a morphism $\ell : \mathcal{F} \to TN$ with the algebraic and differentiable properties stated before, then $w(t) = (\ell + tI)\mathcal{F}$, $t \in \mathbb{K}$, defines a Veronese web of codimension r for which $w(\infty) = \mathcal{F}$. Indeed apply (2) of theorem 1 to a prolongation $\bar{J}$ of ℓ. Thus:

Giving a Veronese web on N of codimension $r \geq 1$ is equivalent to giving a morphism $\ell : \mathcal{F} \to TN$, where $\mathcal{F}$ is a r-codimensional foliation without non-vanishing ℓ-invariant vector subspace at any point such that:
1) whenever α is a closed 1-form whose kernel contains $\mathcal{F}$, restricted to the domain of α, then $\ell^*\alpha$ is closed on $\mathcal{F}$,
2) $N_\ell = 0$.

Note that if $\mathcal{F} = Ker(\alpha_1 \wedge ... \wedge \alpha_r)$, where $d\alpha_1 = ... = d\alpha_r = 0$, then $\ell^*\alpha$ is closed on $\mathcal{F}$ for any 1-form α such that $d\alpha = 0$ and $Ker\alpha \supset \mathcal{F}$, if and only if $\ell^*\alpha_1, ..., \ell^*\alpha_r$ are closed on $\mathcal{F}$.

Example. On an open set A of $\mathbb{K}^{2m}$, endowed with coordinates $(x, y) = (x_1, ..., x_m, y_1, ..., y_m)$, consider the foliation $\mathcal{F}$ defined by $dy_1 = ... = dy_m = 0$ and the morphism $\ell : \mathcal{F} \to TA$ given by $\ell(\frac{\partial}{\partial x_j}) = \sum_{k=1}^m f_{jk} \frac{\partial}{\partial y_k}$, $j = 1, ...m$. Assume $| \; f_{jk} \; | \neq 0$ everywhere, which implies that $\ell : \mathcal{F} \to TA$ defines a m-codimensional Veronese distribution w on A with characteristic numbers $n_1 = ... = n_m = 2$. Then w is a Veronese web if and only if $d(\sum_{j=1}^m f_{jk} dx_j)_{|\mathcal{F}} = 0$, $k = 1, ..., m$, and $\left[\sum_{k=1}^m f_{jk} \frac{\partial}{\partial y_k}, \sum_{\tilde{k}=1}^m f_{\tilde{j}\tilde{k}} \frac{\partial}{\partial y_{\tilde{k}}}\right] = 0$, $1 \leq j < \tilde{j} \leq m$ (indeed consider the prolongation J of ℓ given by $J(\frac{\partial}{\partial y_k}) = 0$, $k = 1, ..., m$).

When $m = 1$ there are no conditions at all. If $m = 2$ one has a partial differential system of order one with four equations and four functions; for $m \geq 3$ the system is over-determined.

More generally when $n = 2m$, the m-dimensional Veronese webs on N, with characteristic numbers $n_1 = ... = n_m = 2$, are given by a morphism $\ell : \mathcal{F} \to TN$ such that $dim\mathcal{F} = m$ and $TN = \mathcal{F} \oplus Im\ell$. As ℓ is determined by its image and its graph, which may be identified to $w(1) = (\ell + I)\mathcal{F}$, from the algebraic viewpoint giving a Veronese web w with all its characteristic number equal to 2 is like giving the 3-web $\{\mathcal{F} = w(\infty), w(0), w(1)\}$. Conversely, for any 3-web $D = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ on N there exists just one Veronese distribution w_D such that $w_D(\infty) = \mathcal{D}_1$, $w_D(0) = \mathcal{D}_2$ and $w_D(1) = \mathcal{D}_3$. It is easily seen that w_D is a Veronese web if and only if the torsion of the Chern connection of D vanishes (the Chern connection of D is the only connection making $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ parallel such that $T(\mathcal{D}_1, \mathcal{D}_2) = 0$).

3. Kronecker bihamiltonian structures.

A bihamiltonian structure (Λ, Λ_1) on a m-dimensional manifold M is called *Kronecker* when there exists $r \in \mathbb{N} - \{0\}$ such that each $(\Lambda(p), \Lambda_1(p))$, $p \in M$, is the product of r Kronecker elementary pairs. In this case from the algebraic model at each point follows that $m - r = rank(\Lambda, \Lambda_1) = rank(\Lambda) = rank(\Lambda_1) = rank(\Lambda + t\Lambda_1)$ for any $t \in \mathbb{K}$; moreover $\mathcal{D} = \bigcap Im(\Lambda + t\Lambda_1)$, $t \in \mathbb{K}$, is a foliation of dimension $\frac{m-r}{2}$ lagrangian for both Λ and Λ_1, and $\mathcal{D} \subset Im\Lambda_1$. This foliation will be named the *axis of* (Λ, Λ_1).

Let N be the local quotient of M by the foliation $\mathcal{D}$, which is a manifold of dimension $n = \frac{m+r}{2}$, and let $\pi : M \to N$ be the canonical projection. Then $w = \{w(t) = \pi_*(Im(\Lambda + t\Lambda_1)) \mid t \in \mathbb{K}\}$ is a Veronese web of codimension r, whose limit when $t \to \infty$ equals $\pi_*(Im\Lambda_1)$. Thus a Veronese web of codimension r is locally associated to any Kronecker bihamiltonian structure with r factors, and:

Theorem 2. *From the local viewpoint the Veronese web completely determines the Kronecker bihamiltonian structure, at least, in the following four cases: complex manifold, real analytic category, C^∞ category when $r = 1$, and* *flat Veronese web.*

Besides:

Proposition 3. *Consider a Veronese web w of codimension $r \geq 1$ defined on a n-manifold N. Let $T^*w(0)$ be the cotangent bundle of the foliation $w(0)$, which is a vector bundle over N of dimension $n-r$ (so as manifold $dimT^*w(0) = 2n-r$). Then, on $T^*w(0)$, there exists a Kronecker bihamiltonian structure (Λ, Λ_1) of corank r such that:*
(1) The axis $\mathcal{D}$ is given by the fibres of the fiber bundle $T^*w(0) \to N$; therefore $\frac{T^*w(0)}{\mathcal{D}} = N$.
(2) w is the Veronese web associated to (Λ, Λ_1).

4. Versal models; local classification of codimension one Veronese webs.

On a real or complex manifold N of dimension n consider a Veronese web w of codimension $r \geq 1$. Given non-equal scalars $a_1, ..., a_{n-r}, a$ and any point $p \in N$, let J be a $(1,1)$ tensor field like in part (1) of theorem 1 and let $(x_1, ..., x_{n-r}, y_1, ..., y_r)$ be a system of coordinates, around p, such that $dx_j \circ J = a_j dx_j$, $j = 1, ..., n-r$, and $Ker(dy_1 \wedge ... \wedge dy_r) = w(\infty)$. Then $dy_k \circ J = a dy_k + \tilde{\alpha}_k$, $k = 1, ..., r$, where each $\tilde{\alpha}_k = \sum_{j=1}^{n-r} f_{kj} dx_j$. As $(w(\infty)', J^*)$ spans the cotangent bundle around p, by linearly recombining functions $y_1, ..., y_r$ and considering $b_j x_j$ instead x_j for a suitable $b_j \in \mathbb{K} - \{0\}$, we assume that each $f_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number.

On the other hand $d(dy_k \circ J) \wedge dy_1 \wedge ... \wedge dy_r = 0$ and $N_J = 0$; a calculation shows that these last two conditions are equivalent to system

$$(*) \qquad \begin{cases} d\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \\ \\ \left(d(\tilde{\alpha}_k \circ J_0) - \displaystyle\sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \end{cases}$$

where $J_0 = \sum_{j=1}^{n-r} a_j \frac{\partial}{\partial x_j} \otimes dx_j + \sum_{\ell=1}^{r} a \frac{\partial}{\partial y_\ell} \otimes dy_\ell$.

Moreover $\gamma(t) = (\prod_{j=1}^{n-r} (t + a_j))(t+a)^r ((J + tI)^{-1})^* (dy_1 \wedge ... \wedge dy_r)$ represents w.

Therefore, in view of (2) of theorem 1, locally Veronese webs correspond to those solutions of system $(*)$ such that $f_{11}(p), ..., f_{1n-r}(p) \in \mathbb{R}^+$ (this last assumption implies that $(dy_1, ..., dy_r, J^*)$ spans the cotangent bundle near p).

Hereafter *the standard case* will mean that the structured considered are complex, real analytic, or C^∞ with $r = 1$ in this last case. Let S be the submanifold defined by $x_j - x_{n-r} = x_j(p) - x_{n-r}(p)$, $j = 1, ..., n-r-1$ [$S = M$ if $n = r, r+1$].

The next theorem gives us all solutions of (*) suitable for our purposes.

Theorem 3. *In the standard case, given a germ at p of maps $\varphi_{kj} : S \to \mathbb{K}$, $k = 1, ..., r$, $j = 1, ..., n-r$, such that every $\varphi_{1j}(p)$, $j = 1, ..., n-r$, is a positive real number, then there exists one and only one germ at p on M of 1-forms $\tilde{\alpha}_1 = \sum_{j=1}^{n-r} f_{1j} dx_j, ..., \tilde{\alpha}_r = \sum_{j=1}^{n-r} f_{rj} dx_j$ such that*

$$(*) \qquad \begin{cases} d\tilde{\alpha}_k \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \\ \\ \left(d(\tilde{\alpha}_k \circ J_0) - \displaystyle\sum_{\ell=1}^{r} \tilde{\alpha}_\ell \wedge \frac{\partial \tilde{\alpha}_k}{\partial y_\ell} \right) \wedge dy_1 \wedge ... \wedge dy_r = 0, \quad k = 1, ..., r \end{cases}$$

and that $f_{kj|S} = \varphi_{kj}$, $k = 1, ..., r$, $j = 1, ..., n-r$.

When $r \geq 2$ the tensor field J is not unique and consequently we may associate more than one model to a same Veronese web; *thus our model of every Veronese web is versal.*

To remark that a classification in codimension≥ 2 seems rather difficult as the following example shows. Consider a field of 2-planes and a local basis of it $\{X, Y\}$. Let $\tilde{w}(t)$, $t \in \mathbb{K}$, be the 1-foliation defined by $X + tY$. Then to classify the 1-dimensional (local) Veronese web $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$, roughly speaking, is like locally classifying the fields of 2-planes in any dimension; but it is well known the difficult of this problem (first dealt with by Élie Cartan in "Les systèmes de Pfaff à cinq variables" and later on by several authors).

Now let us examine the remainder case. *Assume $r = 1$ until the end of this section.* Then $a_1, ..., a_{n-1}, a$ completely determines J since $Ker(J^* - a_j I)$, $j = 1, ..., n-1$, is the annihilator of $w(-a_j)$ and $Ker(J^* - aI)$ that of $w(-a)$. The next step will be to construct an intrinsic surface S. By technical reasons *one will suppose that $a_1, ..., a_{n-1}, a$ are non-equal real numbers.*

The polynomial $\sum_{j=1}^{n-1}\prod_{k=1;k\neq j}^{n-1}(t+a_k)$ has $n-2$ different roots $b_1,...,b_{n-2}$ since it is the derivative of $\prod_{k=1}^{n-1}(t+a_k)$, whose roots are $-a_1,...,-a_{n-1}$; moreover $b_\ell \neq -a_j$, $\ell=1,...,n-2$, $j=1,...,n-1$.

Let R be the germ at p of the leaf of the 1-foliation $w(b_1)\cap ...\cap w(b_{n-2})\cap w(\infty)$ passing through this point, and let S_0 be the germ at p of the surface containing R and to which the 1-foliation $w(-a_1)\cap ...\cap w(-a_{n-1})$ is tangent. By construction S_0 is intrinsic.

Since R is transverse to every $w(-a_j)$, $j=1,...,n-1$, one may take coordinates $(x_1,...,x_{n-1},y)$ constructed before, with two additional properties: R is defined by the equations $x_1=...=x_{n-1}, y=0$, and $x_1(p)=...=x_{n-1}(p)=y(p)=0$; of course we write y and $\tilde{\alpha}=\sum_{j=1}^{n-r} f_j dx_j$ instead y_1 and $\tilde{\alpha}_1=\sum_{j=1}^{n-r} f_{1j}dx_j$. In these coordinates S_0 is defined by the equations $x_1=...=x_{n-1}$. Moreover

$$\gamma(t)=-\sum_{j=1}^{n-1}\left(\prod_{k=1;k\neq j}^{n-1}(t+a_k)f_j\right)dx_j+\prod_{k=1}^{n-1}(t+a_k)dy.$$

On the other hand $\gamma(b_\ell)(q)((\partial/\partial x_1)+...+(\partial/\partial x_{n-1}))=0$, $\ell=1,...,n-2$, for every $q\in R$ because $(\partial/\partial x_1)+...+(\partial/\partial x_{n-1})$ is tangent to R and $T_qR=(w(b_1)\cap ...\cap w(b_{n-2})\cap w(\infty))(q)$. Therefore $b_1,...,b_{n-2}$ are the roots of $\sum_{j=1}^{n-1}\prod_{k=1;k\neq j}^{n-1}(t+a_k)f_j(q)$ when $q\in R$; so $f_1=...=f_{n-1}$ on R since $b_1,...,b_{n-2}$ are the roots of $\sum_{j=1}^{n-1}\prod_{k=1;k\neq j}^{n-1}(t+a_k)$ too, which implies that both polynomials are equal up to multiplicative factor (conversely, if $f_1=...=f_{n-1}$ on R then $(\partial/\partial x_1)+...+(\partial/\partial x_{n-1})$ is tangent to this curve and R is defined by $x_1=...=x_{n-1}, y=0$).

The change of coordinates between two of such system can be regarded as a diffeomorphism $(x_1,...,x_{n-1},y)\to G(x_1,...,x_{n-1},y)$. But G has to preserve R, S_0, the foliations of dimension $n-1$ defined by dx_1,..., dx_{n-1} and dy respectively (that is to say $w(-a_1)$,..., $w(-a_{n-1})$ and $w(\infty)$), and the origin. Therefore $G(x_1,...,x_{n-1},y)=(h_1(x_1),...,h_1(x_{n-1}),h_2(y))$ where h_1,h_2 are one variable functions such that $h_1(0)=h_2(0)=0$ and $h_1'(0)\neq 0$, $h_2'(0)\neq 0$.

Denote by J' the pull-back of J by the diffeomorphism G. Then $dx_j\circ J'=a_jdx_j$, $j=1,...,n-1$, and $dy\circ J'=ady+\tilde{\alpha}'$ where

$$\tilde{\alpha}' = \sum_{j=1}^{n-1} h_1'(x_j)(h_2'(y))^{-1} f_j(h_1(x_1), ..., h_1(x_{n-1}), h_2(y))dx_j.$$

Now we may take h_1, h_2 in such a way that

$$h_1'(x_1)(h_2'(y))^{-1} f_1(h_1(x_1), ..., h_1(x_{n-1}), h_2(y)) = 1$$

on the curves $x_1 = ... = x_{n-1}$, $y = 0$, and $x_1 = ... = x_{n-1} = 0$.

In other words, there exist coordinates $(x_1, ..., x_{n-1}, y)$ as before with a third additional property: $f_1 = ... = f_{n-1} = 1$ on the curve $x_1 = ... = x_{n-1}$, $y = 0$, and $f_1 = 1$ on the curve $x_1 = ... = x_{n-1} = 0$.

In turn, a change of coordinates between two system with this last property is given by two functions h_1, h_2 such that $h_1'(x_1)(h_2'(y))^{-1} = 1$ on the curves $x_1 = ... = x_{n-1}$, $y = 0$, and $x_1 = ... = x_{n-1} = 0$. Therefore h_1', h_2' are constant. In short, the only possible change of coordinates is a homothety of ratio $b \in \mathbb{K} - \{0\}$, and $\tilde{\alpha}'(x_1, ..., x_{n-1}, y) = \tilde{\alpha}(bx_1, ..., bx_{n-1}, by)$.

A germ at the origin of a map $\phi = (\varphi_1, ..., \varphi_{n-1})$ from S_0 to $\mathbb{K}^{n-1}$ will be called *admissible* if $\varphi_1 = ... = \varphi_{n-1} = 1$ on the curve $x_1 = ... = x_{n-1}$, $y = 0$, and $\varphi_1 = 1$ on the curve $x_1 = ... = x_{n-1} = 0$. Two admissible germs ϕ and $\bar{\phi}$ will be named *equivalent* if there exists $b \in \mathbb{K} - \{0\}$ such that $\bar{\phi}(x_1, ..., x_{n-1}, y) = \phi(bx_1, ..., bx_{n-1}, by)$.

From theorem 1, theorem 3 and system (*), applied to the last kind of coordinates system, follows (remark that in this last step the number a does not play any role, which is due to the fact that a Veronese web is determined by $w(\infty)$ and $J_{|w(\infty)}$):

Theorem 4. *Consider non-equal real numbers $a_1, ..., a_{n-1}$. One has:*
(1) Given a Veronese web of codimension 1 on a real or complex n-manifold N and any point $p \in N$, there exist coordinates $(x_1, ..., x_{n-1}, y)$ around p such that $x_1(p) = ... = x_{n-1}(p) = y(p) = 0$ and the Veronese web is represented by

$$\gamma(t) = -\sum_{j=1}^{n-1} \left(\prod_{k=1; k\neq j}^{n-1} (t + a_k) f_j \right) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy,$$

where $\tilde{\alpha} = \sum_{j=1}^{n-r} f_j dx_j$ satisfies to the system

$$\begin{cases} d\tilde{\alpha} \wedge dy = 0 \\ \\ \left(d \left(\sum_{j=1}^{n-1} a_j f_j dx_j \right) - \tilde{\alpha} \wedge \dfrac{\partial \tilde{\alpha}}{\partial y} \right) \wedge dy = 0, \end{cases}$$

$f_1 = ... = f_{n-1} = 1$ on the curve $x_1 = ... = x_{n-1}$, $y = 0$, and $f_1 = 1$ on the curve $x_1 = ... = x_{n-1} = 0$.

(2) Let S_0 be the surface of equation $x_1 = ... = x_{n-1}$ and let $\phi = (\varphi_1, ..., \varphi_{n-1})$ be a germ at the origin of a map from S_0 to $\mathbb{K}^{n-1}$. Assume ϕ admissible. Then there exists one and only one germ at the origin of 1-form $\tilde{\alpha} = \sum_{j=1}^{n-r} f_j dx_j$, which satisfies to the system of part (1) and such that ${f_j}_{|S_0} = \varphi_j$, $j = 1, ..., n-1$.

Moreover

$$\gamma(t) = -\sum_{j=1}^{n-1} \left(\prod_{k=1;k\neq j}^{n-1} (t + a_k) f_j \right) dx_j + \prod_{k=1}^{n-1} (t + a_k) dy,$$

defines a Veronese web of codimension 1 around the origin.

(3) Finally given two admissible germs at the origin ϕ and $\bar{\phi}$ of maps from S_0 to $\mathbb{K}^{n-1}$, the germs of 1-codimensional Veronese webs associated to them by virtue of part (2) are equivalent, by diffeomorphism, if and only if ϕ and $\bar{\phi}$ are equivalent as admissible germs.

C. THE LOCAL PRODUCT THEOREM FOR BIHAMILTONIAN STRUCTURES

One will shows that, around each point of a dense open set (regular points), a real analytic or holomorphic bihamiltonian structure decomposes into a product of a Kronecker bihamiltonian structure and a symplectic one if a necessary condition on the characteristic polynomial of the symplectic factor holds. Moreover we will give an example of bihamiltonian structure for showing that this result does not extend to the C^∞- category.

As main tool for this purpose, to any bihamiltonian structure we associate a new object called a Veronese flag, which generalizes the notion of Veronese web introduced by Gelfand and Zakharevich (codimension one) and later on by others authors (higher codimension). After that, and roughly speaking, the crucial point is to show that, about each regular point, a Veronese flag is the product of a Veronese web and a pair of compatible symplectic forms.

1. Veronese flags.

On a manifold P consider a foliation $\mathcal{F}$ of positive codimension and a morphism of vector bundles $\ell : \mathcal{F} \to TP$. Let $\mathcal{A}(p)$, $p \in P$, be the largest ℓ-invariant vector subspace of $\mathcal{F}(p)$. We will say that the pair $(\mathcal{F}, \ell)$ is a *weak Veronese flag* if the following three conditions hold:

1) $\ell^*\alpha$ is closed on $\mathcal{F}$ for every closed 1-form α such that $Ker\alpha \supset \mathcal{F}$,
2) $N_\ell = 0$,
3) $dim\mathcal{A}(p)$ does not depend on p.

It is easily seen that the distribution $\mathcal{A} = \bigcup_{p\in P} \mathcal{A}(p)$ is a foliation when $(\mathcal{F}, \ell)$ is a weak Veronese flag.

Lemma 1 *Consider a weak Veronese flag $(\mathcal{F}, \ell)$ and for every integer $k \geq 0$ set $g_k = trace((\ell_{|\mathcal{A}})^k)$. Then $kdg_{k+1} = (k+1)dg_k \circ \ell$ on $\mathcal{F}$.*

Now let ω, ω_1 be a couple of 2-forms defined on $\mathcal{A}$. One will say that $(\mathcal{F}, \ell, \omega, \omega_1)$ is a *Veronese flag* on P if:

1) $(\mathcal{F}, \ell)$ is a weak Veronese flag.
2) ω is symplectic on $\mathcal{A}$, ω_1 closed and $\omega_1 = \omega(\ell, \quad)$ [that is $\omega_1(X,Y) = \omega(\ell X, Y)$].

3) Whenever f is a function on an open set of P such that $\ell^* df$ is closed on $\mathcal{F}$, then $L_{X_f}\ell = 0$ where X_f is the ω-hamiltonian of f along $\mathcal{A}$.

When $\mathcal{A} = 0$, Veronese web and Veronese flag are equivalent notions.

Example. Consider a Veronese web w on a manifold M_1 and a compatible symplectic pair (ω, ω_1) on a manifold M_2. Associated to w one has a morphism $\ell_1 : \mathcal{F}_1 \to TM_1$, where $\mathcal{F}_1 = w(\infty)$; let ℓ_2 be the $(1,1)$-tensor field on M_2 defined by $\omega_1 = \omega(\ell_2, \quad)$. With the obvious identifications, on $M_1 \times M_2$ we may consider the foliation $\mathcal{F} = \mathcal{F}_1 \oplus TM_2$, the morphism $\ell = \ell_1 \oplus \ell_2$ and the forms ω, ω_1 along $\{0\} \times TM_2$. Then $(\mathcal{F}, \ell)$ is a weak Veronese flag, for which $\mathcal{A} = \{0\} \times TM_2$, and $(\mathcal{F}, \ell, \omega, \omega_1)$ a Veronese flag.

Coming back to the general problem, on a real or complex m-manifold M consider a bihamiltonian structure (Λ, Λ_1) such that:
1) (Λ, Λ_1) is maximal, that is every $(\Lambda(p), \Lambda_1(p))$, $p \in M$, is maximal,
2) the rank of (Λ, Λ_1) and the dimension of the the symplectic factor at each point are constant.

Set $r = corank(\Lambda, \Lambda_1)$ and let $2m'$ be the dimension of the symplectic factor. Since r is the number of Kronecker elementary factors, $m + r$ is even and one may set $m = 2m' + 2n - r$. Note that, at every point, $2n - r$ equals the sum of the dimensions of the Kronecker elementary factors (warning these last dimensions could depend on the point).

Our next aim is locally to associate a Veronese flag in dimension $2m' + n$ to (Λ, Λ_1). For each $p \in M$ let $\mathcal{A}_1(p)$ be the intersection of all vector subspaces $Im(\Lambda + t\Lambda_1)(p)$, $t \in \mathbb{K}$, such that $rank(\Lambda + t\Lambda_1)(p) = m - r$. From the algebraic model follows that $dim\mathcal{A}_1(p) = m - n = 2m' + n - r$. It is not hard to see that $\mathcal{A}_1$ is a foliation, which be called *the (primary) axis of* (Λ, Λ_1).

Moreover, $\mathcal{A}_1 \subset Im\Lambda_1$ and $dim(Im(\Lambda + t\Lambda_1) + \mathcal{A}_1) = m - r$, $t \in \mathbb{K}$. Set $\tilde{w}(t) = Im(\Lambda + t\Lambda_1) + \mathcal{A}_1$, $t \in \mathbb{K}$; then $\tilde{w} = \{\tilde{w}(t) \mid t \in \mathbb{K}\}$ is a family of foliations of codimension r whose limit at each point, when $t \to \infty$, is $Im\Lambda_1$.

Let N be the local quotient of M by $\mathcal{A}_1$, which is a n-dimensional manifold, and $\pi_N : M \to N$ the canonical projection. Then $\bar{w} = \{\bar{w}(t) = (\pi_N)_* \tilde{w}(t) \mid t \in \mathbb{K}\}$ is a Veronese web on N of codimension r.

The Poisson structure Λ is given by a symplectic form $\tilde{\omega}$ defined on $Im\Lambda$

while Λ_1 is given by a symplectic form form $\tilde{\omega}_1$ on $Im\Lambda_1$. Therefore the restricted 2-forms $\tilde{\omega}_{|\mathcal{A}_1}$ and $\tilde{\omega}_{1|\mathcal{A}_1}$ are closed; besides $Ker(\tilde{\omega}_{|\mathcal{A}_1}) = Ker(\tilde{\omega}_{1|\mathcal{A}_1}) = \Lambda(\mathcal{A}'_1, \quad) = \Lambda_1(\mathcal{A}'_1, \quad)$ where $\mathcal{A}'_1$ is the annihilator of $\mathcal{A}_1$ and $dim(Ker(\tilde{\omega}_{|\mathcal{A}_1})) = n - r$. Thus $\mathcal{A}_2 = Ker(\tilde{\omega}_{|\mathcal{A}_1})$ is a foliation of dimension $n - r$, which will be called *the secondary axis of* (Λ, Λ_1), and $\mathcal{A}_2 \subset \mathcal{A}_1$.

Let P be the local quotient of M by $\mathcal{A}_2$ and $\pi_P : M \to P$ the canonical projection; then $dimP = 2m' + n$, $\mathcal{A}_1$ projects into a $2m'$-dimensional foliation $\mathcal{A}$ and $\tilde{\omega}_{|\mathcal{A}_1}$, $\tilde{\omega}_{1|\mathcal{A}_1}$ in two symplectic forms ω, ω_1 on $\mathcal{A}$. Moreover Λ projects in the Poisson structure defined by $\mathcal{A}$ and ω, whereas Λ_1 does in the Poisson structure defined by $\mathcal{A}$ and ω_1. Let $\mathcal{F}$ be the r-codimensional foliation on P projection of $Im\Lambda_1$. Obviously the local quotient of P by $\mathcal{A}$ is identified in a natural way to N and $\pi \circ \pi_P = \pi_N$ where $\pi : P \to N$ is the canonical projection. In short, we have three of the four elements of a Veronese flag on P. Let us construct the fourth one.

As $\Lambda(\mathcal{A}'_1, \quad) = \Lambda_1(\mathcal{A}'_1, \quad) = \mathcal{A}_2$ and $\mathcal{A}'_1$ contains $Ker\Lambda$ and $Ker\Lambda_1$, the Poisson structures Λ, Λ_1 give rise to two isomorphisms $\tilde{\lambda}$, $\tilde{\lambda}_1$ from $\frac{T^*M}{\mathcal{A}'_1}$ to $\frac{Im\Lambda}{\mathcal{A}_2}$ and $\frac{Im\Lambda_1}{\mathcal{A}_2}$ respectively, by setting $\tilde{\lambda}([\alpha]) = [\Lambda(\alpha, \quad)]$ and $\tilde{\lambda}_1([\alpha]) = [\Lambda_1(\alpha, \quad)]$. Thus $\tilde{\ell} = \tilde{\lambda} \circ \tilde{\lambda}_1^{-1}$ is a monomorphism from $\frac{Im\Lambda_1}{\mathcal{A}_2}$ to $\frac{TM}{\mathcal{A}_2}$ whose image equals $\frac{Im\Lambda}{\mathcal{A}_2}$. By construction $\tilde{\ell}$ is an invariant of (Λ, Λ_1); moreover $\tilde{\ell}$ projects into a morphism $\ell : \mathcal{F} \to TP$. In turn, a non-elementary calculation shows that $(\mathcal{F}, \ell, \omega, \omega_1)$ *is a Veronese flag.*

2. The local product theorem

Consider a bihamiltonian structure (Λ, Λ_1) on a real or complex manifold M of dimension m. The set of all $p \in M$ such that $rank(\Lambda, \Lambda_1)$ is constant about p is open and dense. For simplicity sake, suppose $r = corank(\Lambda, \Lambda_1)$ locally constant by the moment. Since our problem is local, by considering $(1-b)\Lambda + b\Lambda_1$ and $(1 - b')\Lambda + b'\Lambda_1$ for suitable scalars b, b' instead of Λ, Λ_1, we may assume maximal (Λ, Λ_1), that is $r = corank\Lambda = corank\Lambda_1 = corank(\Lambda, \Lambda_1)$, without loss of generality.

In turn, it is easily seen that the dimension of $\mathcal{A}_1$ and that of the symplectic factor are locally constant on a dense open set.

Observe that if (Λ, Λ_1) decomposes into a product near p, then the dimension of the symplectic factor has to be constant close to p.

In short, suppose that on an open set $M' \subset M$ the bihamiltonian structure is maximal and its rank and the dimension of the symplectic factor are constant. Then, following section 1, set $m = 2m' + 2n - r$ where $2m'$ is the dimension of the symplectic factor and consider the Veronese flag $(\mathcal{F}, \ell, \omega, \omega_1)$ on the local quotient P of M' by the secondary axis $\mathcal{A}_2$.

Let $\widetilde{\varphi} = t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j t^j$ be the characteristic polynomial of the symplectic factor of (Λ, Λ_1) on M', that is $t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j(p) t^j$, for each $p \in M'$, is the characteristic polynomial of the symplectic factor of $(\Lambda(p), \Lambda_1(p))$ when regarded as a couple of (linear) symplectic forms.

On the other hand let $\varphi = t^{2m'} + \sum_{j=0}^{2m'-1} h_j t^j$ be the characteristic polynomial of $\ell_{|\mathcal{A}}$. By means of the algebraic model of $(\Lambda(p), \Lambda_1(p))$ it is not hard to see that the symplectic factor of $(\Lambda(p), \Lambda_1(p))$ is isomorphic to $(\omega(\pi_P(p)), \omega_1(\pi_P(p)))$. Thus the characteristic polynomial of $(\ell_{|\mathcal{A}})(\pi_P(p))$ equals $\widetilde{\varphi}(p)$, that is locally $\widetilde{h}_j = h_j \circ \pi_P$, $j = 0, ..., 2m'-1$, which in particular shows the differentiability of $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$.

Proposition 1. *The functions $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ are in involution both for Λ and Λ_1. Moreover $\{\Lambda(d\widetilde{h}_j, \quad)(p)\}_{j=0,...,2m'-1}$ and $\{\Lambda_1(d\widetilde{h}_j, \quad)(p)\}_{j=0,...,2m'-1}$ span the same vector subspace of T_pM' for any $p \in M'$.*

Now assume that (M', Λ, Λ_1) is diffeomorphic to a product of a Kronecker bihamiltonian structure and a symplectic one $(M_1, \Lambda', \Lambda'_1) \times (M_2, \Lambda'', \Lambda''_1)$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the foliations given by the first and second factor respectively. Then $\mathcal{A}_1 \supset \mathcal{B}_2$ and $\widetilde{h}_0, ..., \widetilde{h}_{2m'-1}$ are $\mathcal{B}_1$-foliate functions; therefore the dimension of the vector subspace of T_q^*M' spanned by $d\widetilde{h}_0(q), ..., d\widetilde{h}_{2m'-1}(q)$ equals the dimension of the vector subspace of $\mathcal{A}_1^*(q)$ spanned by $d\widetilde{h}_{0|\mathcal{A}_1(q)}, ..., d\widetilde{h}_{2m'-1|\mathcal{A}_1(q)}$ whenever $q \in M'$. Thus *the foregoing property is necessary* for the existence of a local decomposition into a product of a Kronecker bihamiltonian structure and a symplectic one.

A point p of M is called *regular* for (Λ, Λ_1) if the three following conditions hold:

1) The rank (Λ, Λ_1) is constant on an open neighbourhood M' of this point.

Observe that this first condition allows assuming maximal (Λ, Λ_1) by replacing (Λ, Λ_1) by $(1-b)\Lambda + b\Lambda_1$ and $(1-b')\Lambda + b'\Lambda_1$, for suitable scalars b, b', and shrinking M'. Then:

2) The dimension of the symplectic factor is constant near p, that is on M' by shrinking this neighbourhood again if necessary.

3) The point $\pi_P(p)$ is regular for $\ell_{|\mathcal{A}}$.

Obviously there are many choices of scalars b, b' such that $((1-b)\Lambda+b\Lambda_1, (1-b')\Lambda+b'\Lambda_1)$ is maximal around p, but it is easily checked that conditions 2) and 3) do not depend on them.

Since the set of regular points of $\ell_{|\mathcal{A}}$ is open and dense and the projection π_P is a submersion, the set of regular points of (Λ, Λ_1) is dense and open on M; it will be named *the regular open set*.

Theorem 1. *Consider a real analytic or holomorphic bihamiltonian structure (Λ, Λ_1) on M and a regular point p. Let $\widetilde{\varphi} = t^{2m'} + \sum_{j=0}^{2m'-1} \widetilde{h}_j t^j$ be the characteristic polynomial of the symplectic factor of (Λ, Λ_1) near p. Assume that when q is close to p the vector subspace spanned by $d\widetilde{h}_0(q), ..., d\widetilde{h}_{2m'-1}(q)$ and that spanned by $d\widetilde{h}_{0|\mathcal{A}_1(q)}, ..., d\widetilde{h}_{2m'-1|\mathcal{A}_1(q)}$ have the same dimension. Then, around p, (Λ, Λ_1) decomposes into a product of a Kronecker bihamiltonian structure and a symplectic one.*

Moreover, if $\tilde{\varphi}(p)$ only has real roots then in the C^∞ category (Λ, Λ_1) locally decomposes into a product Kronecker-symplectic.

The way for proving this theorem is to show that the Veronese flag $(\mathcal{F}, \ell, \omega, \omega_1)$, associated to (Λ, Λ_1) on the local quotient by the secondary axis, locally decomposes into a product. That is to say, showing the existence around any point of a $(1,1)$-tensor field G that extends ℓ and coordinates $(x, z) = (x_1, ..., x_n, z_1, ..., z_{2m'})$ such that $dx_1 = ... = dx_n = 0$ defines $\mathcal{A}$, $G = \sum_{j,k=1}^{n} g_{jk}(x)(\partial/\partial x_j) \otimes dx_k + \sum_{j,k=1}^{2m'} h_{jk}(z)(\partial/\partial z_j) \otimes dz_k$ and the coefficient functions of ω, ω_1 with respect to $dz_{1|\mathcal{A}}, ..., dz_{2m'|\mathcal{A}}$ only depend on z. Then, it suffices to consider, on an open set of M, the foliation $Ker(\pi_P^*(dz_1 \wedge ... \wedge dz_{2m'}))$ and that spanned by the Λ-hamiltonians of $z_1 \circ \pi_P, ..., z_{2m'} \circ \pi_P$, which gives us the required decomposition into a product.

Note that if a bihamiltonian structure decomposes into a product Kronecker-symplectic, then its associated Veronese flag does into a product as well.

3. The bihamiltonian structure over a $(1,1)$-tensor field and a

foliation.

Let N be a n-manifold. Recall that on $\Lambda^r T^*N$ it is defined a r-form R, called the Liouville r-form, as follows: if $v_1, ..., v_r \in T_\mu(\Lambda^r T^*N)$ then $R(v_1, ..., v_r) = \mu(\pi_* v_1, ..., \pi_* v_r)$ where $\pi : \Lambda^r T^*N \to N$ is the canonical projection. In turn $\Omega = dR$ will be named the Liouville $(r+1)$-form of $\Lambda^r T^*N$. When $r = 1$, that is on the cotangent bundle, the Liouville forms will be denoted ρ and ω respectively.

Given a $(1,1)$-tensor field H on N, let $\varphi_H : T^*N \to T^*N$ be the morphism of vector bundles defined by $\varphi_H(\tau) = \tau \circ H$, that is $\varphi_H(\tau)(v) = \tau(Hv)$. Set $\omega_1 = \varphi_H^* \omega$. Then the formula $\omega_1(X, Y) = \omega(H^* X, Y)$ gives rise to a $(1,1)$-tensor field H^* on T^*N as manifold, which will be called *the prolongation of* H (to the cotangent bundle).

Now suppose that H is an inversible $(1,1)$-tensor field and $\mathcal{G}$ a r-codimensional foliation both of them defined on N. Assume that:
1) $\alpha \circ H$ is closed on $\mathcal{G}$ whichever α is a closed 1-form such that $Ker\alpha \supset \mathcal{G}$,
2) the restriction of N_H to $\mathcal{G}$ vanishes.

Let $\mathcal{G}_0$ be the ω-orthogonal of the foliation $\pi_*^{-1}(H\mathcal{G}) = \{v \in T(T^*N) \mid \pi_* v \in H\mathcal{G}\}$, which equals the ω_1-orthogonal of the foliation $\pi_*^{-1}(\mathcal{G}) = \{v \in T(T^*N) \mid \pi_* v \in \mathcal{G}\}$ because $\omega_1 = \omega(H^*, \quad)$ and H^* projects in H. Note that $\mathcal{G}_0$ is a symplecticly complete foliation (also called a Libermann foliation) for ω and ω_1. On the other hand the quotient M of T^*N by $\mathcal{G}_0$ is globally defined and there is a projection $\pi' : M \to N$ such that $\pi' \circ \tilde{\pi} = \pi$, where $\tilde{\pi} : T^*N \to M$ is the canonical projection. In fact, M can be regarded as the quotient of T^*N by a vector sub-bundle and $\pi' : M \to N$ as its quotient vector bundle.

Since $\mathcal{G}_0$ is both ω and ω_1 symplecticly complete, the Poisson structures Λ_ω and Λ_{ω_1}, respectively associated to ω and ω_1, project in two Poisson structures Λ and Λ_1 on M.

Proposition 2. *The pair* (Λ, Λ_1) *is a bihamiltonian structure.*

Examples. 1) On $N = \mathbb{K}^n$, $n \geq 1$, consider the foliation given by the closed 1-form $\alpha = \sum_{j=1}^n dx_j$ and the $(1,1)$-tensor field $H = \sum_{j=1}^n h_j(x_j)(\partial/\partial x_j) \otimes dx_j$ where the functions $h_1, ..., h_n$ never vanish. Then the associated bihamiltonian structure (Λ, Λ_1), defined on $M = T^*(\mathbb{K}^n)/\mathcal{G}_0$, has a symplectic factor of positive dimension at $p \in M$ if and only if $\tilde{h}(\pi'(p)) = 0$ where $\tilde{h} = \prod_{1\leq j<k\leq n}(h_j - h_k)$. In other words (Λ, Λ_1) is Kronecker just on the open set $(\tilde{h}\circ\pi')^{-1}(\mathbb{K} - \{0\})$.

2) Now on $N = \mathbb{R}^n - \{0\}$, $n \geq 1$, consider the foliation $\mathcal{G}$ defined by $\alpha = \sum_{j=1}^{n} x_j^{a_j} dx_j$, where $a_1, ..., a_r$ are positive natural numbers, and the $(1,1)$-tensor field $H = \sum_{j=1}^{n} j(\partial/\partial x_j) \otimes dx_j$. Then the associated bihamiltonian structure (Λ, Λ_1), defined on $M = T^*(\mathbb{R}^n - \{0\})/\mathcal{G}_0$, has non-trivial symplectic factor on the closed set $(h\circ\pi')^{-1}(0)$, where $h = x_1 \cdots x_n$, and is Kronecker on the open set $(h\circ\pi')^{-1}(\mathbb{R} - \{0\})$.

Let ϕ_t be the flow of the vector field $\xi = \sum_{j=1}^{n}(a_j+1)^{-1}x_j\partial/\partial x_j$. As $L_\xi\alpha = \alpha$ and $L_\xi H = 0$, the foliation $\mathcal{G}$ and the $(1,1)$-tensor field H project in a foliation $\tilde{\mathcal{G}}$ and a $(1,1)$-tensor field $\tilde{H}$ respectively, defined on the quotient manifold $\tilde{N} = (\mathbb{R}^n - \{0\})/G$ where $G = \{\phi_k \mid k \in \mathbb{Z}\}$. Obviously $\tilde{\mathcal{G}}$ and $\tilde{H}$ satisfy 1) and 2), which gives rise to a bihamiltonian structure on $\tilde{M} = (T^*\tilde{N})/\tilde{\mathcal{G}}_0$. Moreover $\tilde{N}$ is diffeomorphic to $S^1 \times S^{n-1}$.

4. A counter-example.

Here one will apply the construction of the foregoing section to $N = \mathbb{R}^7$ endowed with coordinates $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3, y_4)$, the foliation $\mathcal{G} = Ker(\alpha_1 \wedge \alpha_2)$ where $\alpha_1 = dx_1 - dx_2$ and $\alpha_2 = x_2 dx_2 - dx_3$, and the $(1,1)$-tensor field

$$
\begin{aligned}
H = \textstyle\sum_{j=1}^{3} a_j(\partial/\partial x_j)\otimes dx_j + \sum_{j=1}^{2}[(\partial/\partial y_{2j})\otimes dy_{2j-1} - (\partial/\partial y_{2j-1})\otimes dy_{2j}]\\
+(\partial/\partial y_1)\otimes dy_3 + (\partial/\partial y_2)\otimes dy_4\\
+[(y_3 g_1 - y_4 g_2)(\partial/\partial y_1) + (y_3 g_2 + y_4 g_1)(\partial/\partial y_2)]\otimes dx_1
\end{aligned}
$$

where a_1, a_2, a_3 are non-equal and non-vanishing real numbers and functions g_1, g_2 only depend on x.

Now $dim\mathcal{G}_0 = 2$, $dimM = 12$, the secondary axis $\mathcal{A}_2$ of (Λ, Λ_1) has dimension one and the quotient $P = M/\mathcal{A}_2$, which is globally defined, is a vector bundle over $\mathbb{R}^7$.

Moreover, (Λ, Λ_1) defines a G-structure and the characteristic polynomial of its symplectic factor equals $(t^2+1)^4$; therefore any point of M is regular and the hypothesis of theorem 1 on the coefficients of this polynomial automatically holds.

A calculation shows that if the Veronese flag on P associated to (Λ, Λ_1) decomposes into a product around some point, then there exists a function φ on some non-empty open set such that

$$(*) \qquad (JX - \imath X) \cdot \varphi + g_1 + \imath g_2 = 0$$

where $X = \partial/\partial x_1 + \partial/\partial x_2 + x_2 \partial/\partial x_3$ and $JX = a_1 \partial/\partial x_1 + a_2 \partial/\partial x_2 + a_3 x_2 \partial/\partial x_3$.

But this equation is equivalent to the Lewy's example, which allows us to chose two C^∞ functions $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$ for which (*) has no solution in any neighbourhood of any point of $\mathbb{R}^3$.

Thus a counter-example to theorem 1 is constructed in the C^∞ category.