Singularities of integrable Hamiltonian systems: a criterion for non-degeneracy, with an application to the Manakov top

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Liouville integrable systems: generalities

A *Liouville integrable Hamiltonian system (IHS)* $(M, \omega, h_1, \dots, h_n)$ is a symplectic $2n$-manifold (M, ω) with commuting functions $h_1, \dots, h_n : M \to \mathbb{R}$ whose differentials are almost everywhere independent. The *momentum map* $\mathcal{F} : M \to \mathbb{R}^n$ is given by $\mathcal{F}(x) := (h_1(x), \dots, h_n(x))$. A point $x \in M$ is called a *singular (critical) point of rank* r, $0 \leq r < n$, if $\mathrm{rk}\ \mathrm{d}\mathcal{F}(x) = r$.

Definition. Let $(M, \omega, h_1, \dots, h_n)$ be an IHS and $P \in M$ be a zero-rank singular point, i.e. $\mathrm{d}h_i(P) = 0$ for each i. The point $P \in M$ is called **non-degenerate** if operators $\omega^{-1}\mathrm{d}^2 h_1, \dots, \omega^{-1}\mathrm{d}^2 h_n$ generate a Cartan subalgebra of $sp(2n, \mathbb{R})$.

Among other reasons which make this definition important there is a theorem (Eliasson, Ito, Vey) stating that integrable systems have simple *normal forms* in a neighborhood of a non-degenerate singularity.

We present a criterion for non-degeneracy of zero-rank singularities.

A criterion for non-degeneracy

Theorem 1.
Consider an analytic IHS $(M, \omega, h_1, \ldots, h_n)$. Let $\mathcal{F} : M \to \mathbb{R}^n$ be the momentum map and $P \in M$ be a zero-rank singular point. Denote by K the set of all singular points of rank 1 in a neighborhood of P. If the following conditions hold, then P is non-degenerate:
(a) $\bigcap_{i=1}^n \ker \mathrm{d}^2 h_i(P) = \{0\}$.
(b) The image $\mathcal{F}(K \cup \{P\})$ contains n smooth curves $\gamma_1, \ldots, \gamma_n$, each curve having P as its end point or its inner point. The vectors tangent to $\gamma_1, \ldots, \gamma_n$ at $\mathcal{F}(P)$ are independent in $\mathbb{R}^n$. (Figures 1,2,3 show examples for $n = 2$.)

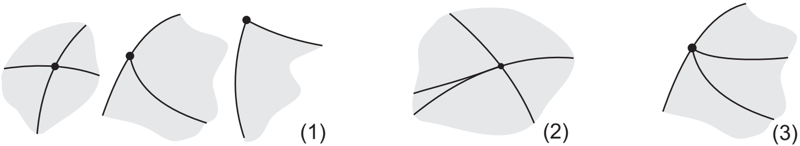

Figure: Images $\mathcal{F}(K \cup \{P\})$ diagrams satisfying condition (b). The diagram (2) appears in the non-analytic case and (3) only when the zero-rank point is degenerate (in this situation condition (a) fails). The image of the momentum map is shaded gray.

Informal explanation of Theorem 1

If P is a non-degenerate zero-rank singular point, then the bifurcation diagram around $\mathcal{F}(P)$ looks (up to a diffeomorphism) in one of the several standard ways, depending on the type of the singularity.

In physical examples, the converse is usually true: if the bifurcation diagram around $\mathcal{F}(P)$ looks in a standard way (Condition (b)), then P is non-degenerate. The converse is not true in general, as illustrated by the following artificial example: $M = \mathbb{R}^4(p_1, p_2, q_2, q_2)$, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, $h_1 = p_1^4 + q_1^4$, $h_2 = p_2^4 + q_2^4$.

Theorem 1 says that P will be non-degenerate if we add to Condition (b) the simple Condition (a) involving $\mathrm{d}^2 h_i(P)$ but not the symplectic structure.

Application to the Manakov top, I

We omit the definition of the 2DoF Manakov top system (aka the 4-dimensional rigid body) here. Its *bifurcation diagram* (i.e. the $\mathcal{F}$-image of all singular points) was found by A. Oshemkov [3]. The system depends on several parameters (moments of inertia and the values of Casimir functions of the $so(4)$-bracket).
We deduce the following natural result from Theorem 1.

Proposition 2. Consider the Manakov top system with a given set of parameters S. If for any sufficiently small change of parameters the bifurcation diagram of the system remains the same up to a diffeomorphism on $\mathbb{R}^2$, all zero-rank singular points of the system with parameters S are non-degenerate.

Application to the Manakov top, II

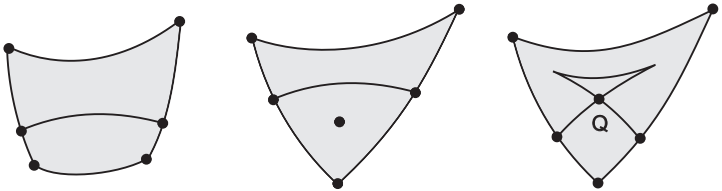

Figure: Three types of generic bifurcation diagrams of the Manakov top. Point Q is the image of two saddle-saddle singularities; it will be studied further.

Using Theorem 1, Proposition 2 is proven easily, with only minor computation. Alternatively, it can be proven using the bi-Hamiltonian structure of the system. On the other hand, in preprint [1], a result essentially equivalent to Proposition 2 is obtained by rather hard direct computation. Thus Theorem 1 does simplify computation.

Saddle-saddle singularity of the Manakov top

Proposition 2 describes when non-degenerate singularities appear in the Manakov top. Now we explicitly describe the singular Liouville foliation on a neighborhood of the fiber containing non-degenerate *saddle-saddle* singularities of the Manakov top.

Proposition 3. Let $Q \in \mathbb{R}^2$ be the point on the bifurcation diagram of the Manakov top shown above and $V \subset \mathbb{R}^2$ be its neighborhood. The preimage $\mathcal{F}^{-1}(V)$ is fiberwise homeomorphic to the quotient $(\widetilde{C_2} \times \widetilde{C_2})/(\alpha, \alpha)$ where $\widetilde{C_2}$ is the fibered 2-manifold with boundary shown on the figure below, and $\alpha : \widetilde{C_2} \to \widetilde{C_2}$ is rotation by π.

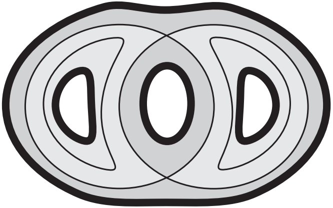

Figure: The fibered 2-manifold $\widetilde{C_2}$

An application to the quantum Manakov top, I

Proposition 3 yields the following result.

Proposition 4. Suppose the parameters of the Manakov top system are taken such that a saddle-saddle point P is present, and the group of $\mathcal{F}$-fiberwise symplectomorphisms of the system acts transitively on the set of connected components of each Liouville fiber. Let $V \subset \mathbb{R}^2$ be a neighborhood of $Q := \mathcal{F}(P)$ and $U := \mathcal{F}^{-1}(V)$. There is a 1-form θ on U such that $d\theta = \omega|_U$.

For $h > 0$, let L_h be the union of all Liouville tori in U satisfying the following condition: the values of all action functions (with respect to the 1-form θ) on the torus belong to $2\pi h\mathbb{Z}$.

Let $D^2 \subset \mathbb{R}^2(x, y)$ be the unit disc. There is a homeomorphism ψ of the plane which takes the bifurcation diagram to some transversal curves and for each $h \in \mathbb{R}_+$ takes $\mathcal{F}(L_h)$ to the sublattice of the straight $2\pi h\mathbb{Z} \oplus 2\pi h\mathbb{Z}$-lattice shown on the next frame.

An application to the quantum Manakov top, II

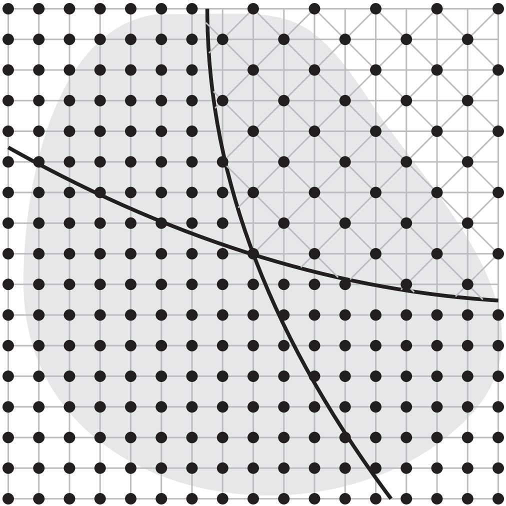

Figure: The lattice $\psi\mathcal{F}(L_h)$ from Proposition 4

An application to the quantum Manakov top, III

Propositions 3 and 4 are an easy application of Fomenko's theory [2] and Zung's theorem. Proposition 3 is interesting in the context of quantization of the Manakov top.

Suppose $\hat{h}_1, \hat{h}_2$ are operators obtained by some quantization procedure applied to the classical integrals of Manakov top. Their joint spectrum can usually be approximated by the lattice $\mathcal{F}(L_h)$ defined above (after a possible Maslov-type correction). For example, this is true for Toeplitz quantization on Kähler manifolds, which works for the Manakov top.

So Proposition 3 predicts the qualitative view of the joint spectrum lattice of the quantum Manakov top.

It completely agrees with the picture of the joint spectrum obtained through numerical computation by Sinitsyn and Zhilinskii [4] which we reproduce on the final frame.

The actual spectrum lattice of the quantum Manakov top

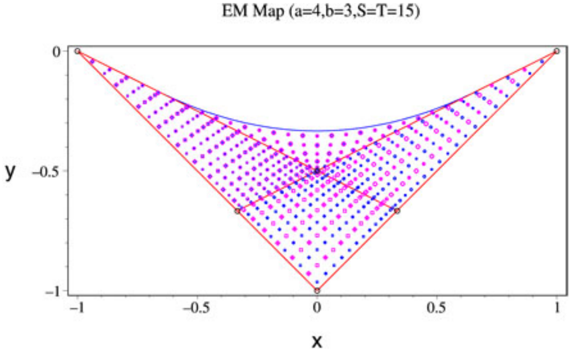

Figure: The joint spectrum of the two operators of the quantum Manakov top, taken from [4]. It agrees with Proposition 4.

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