# Introduction to Poisson and bihamiltonian geometry

Andriy Panasyuk
Dep. of Mathematics and Computer Science
University of Warmia and Mazury at Olsztyn

*Lectures presented at Summer school "Bihamiltonian Geometry and Integrable Systems", Będlewo 28.08-2.09.2011*

## Lecture I

*References:* [Arn73]

**Basic objects:** $M$ a manifold (smooth, analytic etc.), *for example* $M = \mathbb{R}^n$; the tangent bundle $\tau : TM \to M$ , *for example* $\mathrm{pr}_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$; the cotangent bundle $\pi : T^*M \to M$, *for example* $\mathrm{pr}_1 : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}^n$, $\mathcal{E}(M)$ the set of functions on $M$ (smooth, analytic, etc.)

**A vector field $v \in \Gamma(TM)$:** a section of the tangent bundle, i.e. a map (smooth, analytic etc.) $v : M \to TM$ such that $\tau \circ v = \mathrm{Id}_M$. Local description $v(x) = (\sum_i)v^i(x)\frac{\partial}{\partial x^i}$ (we will skip the sum sign - "Einstein convention"), here $x \in M$, $(x^1, \ldots, x^n)$ local coordinates on $U \subset M$, $\{\frac{\partial}{\partial x^i}\}$ the corresponding basis of the fiber of the tangent bundle, $v^i \in \mathcal{E}(U)$. *For example* $v(x) = (x, v^1(x) \ldots, v^n(x))$ (a vector function).

Another point of view: $v$ is a *differentiation* of the algebra $\mathcal{E}(M)$, i.e. an $\mathbb{R}$-linear map $v : \mathcal{E}(M) \to \mathcal{E}(M)$ with $v(fg) = v(f)g + fv(g)$.

**Commutator of vector fields:** $[,] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), [v, w]^i(x) = v^j(x)\frac{\partial w^i(x)}{\partial x^j} - w^j(x)\frac{\partial v^i(x)}{\partial x^j}$.

Another point of view: $[v, w]f = (vw - wv)f$ (commutator of differentiations). *Exercise:* check that the commutator of differentiations is a differentiation.

**A Lie algebra on a vector space $V$:** a bilinear skew-symmetric operation $[,] : V \times V \to V$ satisfying the Jacobi Identity:

1. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\ \forall x, y, z \in V$, or, equivalently,
2. $\mathrm{ad}_x[y, z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z]\ \forall x, y, z \in V$, where $\mathrm{ad}_x y := [x, y]$, or, equivalently,
3. $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]\ \forall x, y \in V$, where the bracket in the RHS denotes the commutator of the operators.

The second condition means that $\mathrm{ad}_x$ is a differentiation of the bracket $[,]$. The third one has the following interpretation. A pair $(V, [,])$, where $V$ is a vector space and $[,] : V \times V \to V$ is a bilinear operation, is called an *algebra*. Given algebras $(V_1, [,]_1)$ and $(V_2, [,]_2)$, we say that a linear map $L : V_1 \to V_2$ is a *homomorphism* of algebras, if $L[x, y]_1 = [Lx, Ly]_2 \ \forall x, y \in V_1$.

So the third condition means that the map $x \mapsto \mathrm{ad}_x : V \to \mathrm{End}(V)$ a *homomorphism* of algebras $(V, [,])$ and $(\mathrm{End}(V), [,])$. Note that the last algebra is in fact a Lie algebra. A homomorphism of Lie algebras $(V, [,]) \to (\mathrm{End}(W), [,])$ is called a *representation* of the Lie algebra $(V, [,])$ in the vector space $W$ (so $x \mapsto \mathrm{ad}_x$ is a representation of $(V, [,])$ in $V$).

EXAMPLES:

1. $V = \mathrm{End}(W)$ with commutator, in other words $V = Mat_{n\times n}(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}), [A, B] := AB - BA$.
2. $V = \mathfrak{sl}(n, \mathbb{R})$ (traceless matrices), $V = \mathfrak{so}(n, \mathbb{R})$ (skew symmetric matrices), etc.
3. $V = \Gamma(TM)$ with commutator of vector fields.

**Ordinary differential equation on a manifold:**

$$\frac{dc}{dt}(x) = v(x), (\text{or } \dot{x} = v(x) \text{ for short})$$

here $v \in \Gamma(TM)$ is given, $c$ is unknown. A *solution* of this equation (or a *trajectory* of $v$) with an initial condition $x_0 \in M$ is a curve $c : \mathbb{R} \to M$ such that $c(0) = x_0$ and the vector $v(x)$ is tangent to $c$ at any $x \in c(\mathbb{R})$.

A solution always exists *locally* and is unique: in local coordinates $(x^1, \ldots, x^n)$ we have $v = v^i(x)\frac{\partial}{\partial x^i}$ and the equation is equivalent to the system of ODE

$$\frac{dc^i(t)}{dt} = v^i(c^1(t), \ldots, c^n(t)), i = 1, \ldots, n$$

with the initial condition $c^i(0) = x_0^i, i = 1, \ldots, n$, and we can use the corresponding existence-uniqueness theorem.

*Globally*, if $\operatorname{supp} v := \overline{\{x \in M \mid v(x) \neq 0\}}$ is compact (eg. $M$ is compact itself) one can extend any local solution to a global (in time and space) solution.

EXAMPLE 1: "NONEXTENDABILITY IN TIME": $M := ]0, 1[, \dot{x} = 1$.

EXAMPLE 2: "NONEXTENDABILITY IN SPACE": $M := \mathbb{R}, \dot{x} = x^2$.

EXAMPLE 3: "WINDING LINE ON A TORUS": $M := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the vector field $v_{a,b} := a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2}$, where $a, b \in ]0, \infty[$ are fixed, can be projected onto the vector field $\tilde{v}_{a,b}$ on $\mathbb{T}^2$. Its trajectories are the projections $t \to P(x^1 + at, x^2 + bt)$ of the lines $t \to (x^1 + at, x^2 + bt)$.

*Rational case:* $b/a$ is a rational number, $b = m\lambda, a = n\lambda$ for some $\lambda \in \mathbb{R}$. Then for $t := 1/\lambda$ we have $(x^1 + at, x^2 + bt) = (x^1 + m, x^2 + n)$ and $P(x^1 + at, x^2 + bt) = P(x^1, x^2)$ (the trajectory is closed, i.e. periodic).

*Irrational case:* $b/a$ is an irrational number (any trajectory is dense in $M$).

**A submanifold $S$ of $M$ of codimension $r$:** A subset $N \subset M$ such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}, \psi_\alpha = (\psi_\alpha^1, \dots \psi_\alpha^n) : U_\alpha \to \mathbb{R}^n$, on $M$ with $N \cap U_\alpha = \{x \in U_\alpha \mid \psi_\alpha^1(x) = 0, \dots \psi_\alpha^r(x) = 0\}$ for those $\alpha \in A$ for which $N \cap U_\alpha \neq \emptyset$.

**Smooth maps and submanifolds:** A smooth map $F : M_1 \to M_2$ is called an *immersion* if $T_m F : T_m M_1 \to T_{F(m)} M_2$ is injective for any $m \in M_1$. The image of an injective immersion is called an *immersed submanifold*. An injective immersion $F$ is an *embedding* if $F$ is a homeomorphism onto $F(M_1)$, where $F(M_1)$ is endowed with the topology induced from $M_2$.

*Remarks:* 1. The image $N := F(M_1)$ of an embedding is a submanifold in $M_2$ and, vice versa, given a submanifold $N \subset M$, the inclusion $N \hookrightarrow M$ is an embedding. 2. If $N \subset M$ is an immersed submanifold, then for any $x \in N$ there exists an open neighbourhood $U$ of $x$ in $M$ such that the connected component of $N \cap U$ containing $x$ is a submanifold in $U$. Vice versa, ...

**Example of an immersed submanifold, which is not a submanifold:** "The irrational torus winding" $\mathbb{R} \to \mathbb{T}^2$.

**A foliation $\mathcal{F}$ of codimension $r$ on $M$ :** A collection $\mathcal{F} = \{F_\beta\}_{\beta \in B}$ of path-connected sets on $M$ such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on $M$ with the following properties:

1. $M = \bigcup_{\beta \in B} F_\beta$;
2. $F_\beta \cap F_\gamma = \emptyset$ for any $\beta, \gamma, \beta \neq \gamma$;
3. for any $\alpha \in A$ and any $(c_1, \dots, c_r) \in \psi_\alpha(U_\alpha)$ there exists $\beta \in B$ such that the set $\{x \in U_\alpha \mid \psi^1(x) = c_1, \dots \psi^r(x) = c_r\}$ coincides with one of the path-connected components of the set $U_\alpha \cap \mathcal{F}_\beta$ if it is nonempty.

By the remark above the sets $F_\beta$ are immersed submanifolds.

EXAMPLE: Collection of the trajectories of the vector field $\tilde{v}_{a,b}$ on $\mathbb{T}^2$.

**A distribution $\mathcal{D}$ on $M$ of codimension $r$:** A subbundle of the tangent bundle $TM$ with the $r$-codimensional fiber, or in other words a collection of subspaces $D_x \subset T_x M$ smoothly (analytically) depending on $x \in M$. Such a distribution is locally spanned by $n - r$ linearly independent (at each point) vector fields.

EXAMPLE : The distribution tangent to a foliation: $\mathcal{D} = T\mathcal{F} := \{v \in TM \mid v \text{ is tangent to } \mathcal{F}\}$.

**Integrable distribution:** A distribution which is tangent to some foliation.

EXAMPLE: Take a *nonvanishing* vector field $v \in \Gamma(TM)$ (if it exists) and put $D_x = \mathbb{R}v(x)$. This is an integrable 1-dimensional distribution tangent to the trajectories of the vector field $v$.

**Involutive distribution:** A distribution $\mathcal{D}$ such that for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$ (equivalently, locally there exist $v_1, \dots, v_m, v_i \in \Gamma(TM)$, and functions $f_{ij}^k$ such that $\mathrm{Span}\{v_1, \dots, v_m\} = \mathcal{D}$ and $[v_i, v_j] = f_{ij}^k v_k$; *Exercise: prove the equivalence*).

**The Frobenius theorem (standard):** A distribution $\mathcal{D}$ is integrable if and only if it is involutive.

**Example of nonintegrable distribution:** $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}$.

**A generalized distribution $\mathcal{D}$ on $M$ of codimension $r$:** A collection of subspaces $D_x \subset T_xM$ locally spanned by $n-r$ vector fields linearly independent at least at one point (but not necessarily linearly independent at other points).

**A generalized foliation $\mathcal{F}$ on $M$:** ...

**Example of a generalized foliation which is not a foliation:** The trajectories of a vector field $x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$.

**The generalized Frobenius theorem (Nagano [Nag66]):** An analytic generalized distribution $\mathcal{D}$ is integrable if and only if it is *involutive*, i.e. for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$.

**An example of smooth involutive nonintegrable distribution:** Let $\varphi(x)$ be a smooth function on $\mathbb{R}$ such that $\varphi(x) \equiv 0$ for $x \leqslant 0$ and $\varphi(x) > 0$ for $x > 0$. Take $X = \frac{\partial}{\partial x}, Y = \varphi\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then $[X, Y] := \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}$ can be expressed as a linear combination of $X, Y$. But it is nonintegrable: look at its "leaves".

# Lecture II

*References:* [dSW99, Arn89]

**A bivector field on $M$:** A section (smooth, analytic) $\eta$ of the second exterior power of the tangent bundle $\bigwedge^2 TM$. Locally $\eta = \eta^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, $\eta^{ij}(x)$ being a skew-symmetric matrix depending on $x \in M$.

**A covector field on $M$ (differential 1-form):** A section $\gamma$ of the bundle $T^*M$. Locally $\gamma = \gamma_i(x)dx^i$.

**A differential 2-form on $M$:** A section $\omega$ of the second exterior power of the cotangent bundle $\bigwedge^2 T^*M$. Locally $\omega = \omega_{ij}(x)dx^i \wedge dx^j$.

**Bivector fields and 2-forms as morphisms:** Let $\eta \in \Gamma(\bigwedge^2 TM)$ and $\gamma \in \Gamma(T^*M)$. The *contraction* $\gamma \lrcorner \eta =: \eta(\gamma)$ (in the first index) is a vector field defined by $v = v^j(x)\frac{\partial}{\partial x^j}, v^j(x) := \gamma_i(x)\eta^{ij}(x)$. Since this operation is pointwise it defines a morphism of bundles $\eta^\sharp : T^*M \to TM$, i.e a smooth map such that the following diagram is commutative

$$\begin{array}{ccc} T^*M & \xrightarrow{\eta^\sharp} & TM \\ \tau\big\downarrow & & \big\downarrow\pi \\ M & = & M \end{array}$$

and the induced mappings $\eta^\sharp_x : T^*_xM \to T_xM$ are linear for any $x \in M$. Note that it is skew-symmetric, i.e. $(\eta^\sharp)^* = -\eta^\sharp$. Conversely, given such a morphism, we can construct a bivector field.

Analogously, a differential 2-form $\omega$ defines a skew-symmetric morphism $\omega^\flat : TM \to T^*M$.

**A symplectic form on $M$:** A differential 2-form (2-form for short) $\omega$ on $M$ such that

1. $\omega$ is nondegenerate, i.e. $\omega^\flat$ is an isomorphism of bundles, or, equivalently, $\omega_{ij}(x)$ is a nondegenerate matrix for any $x$ in some (consequently in any) local coordinate system;

2. $d\omega = 0$.

**A nondegenerate Poisson structure on $M$:** A bivector field (bivector for short) $\eta$ such that $\eta^\sharp : T^*M \to TM$ is inverse to $\omega^\flat : TM \to T^*M$ for some symplectic form $\omega$.

**The Poisson bracket on $\mathcal{E}(M)$:** Given a bivector field $\eta : T^*M \to TM$ (not necessarily Poisson), put $\{f, g\} := \eta(df)g, f, g \in \mathcal{E}(M)$. (From now on we will often skip $\sharp$ and $\flat$ indices.) Then $\{,\}$ is a bilinear skew-symmetric operation on $\mathcal{E}(M)$. We say that $\eta(f) := \eta(df)$ is a *hamiltonian* vector field corresponding to the function $f$.

PROPOSITION. Let $\eta$ be a nondegenerate bivector. Then it is Poisson if and only if $\{,\}$ satisfies the Jacobi identity, $\sum_{c.p.\ f,g,h}\{\{f, g\}, h\} = 0$. $\square$

*Proof* Put $\omega := \eta^{-1}$, i.e. $\omega(\eta(\alpha), v) = \alpha(v)$ for any vector field $v$ and 1-form $\alpha$. Then $\eta(f)\omega(\eta(g), \eta(h)) = \eta(f)(dg(\eta(h))) = \eta(f)(\eta(h)g) = \eta(f)\{h, g\} = \{f, \{h, g\}\} = -\{f, \{g, h\}\}$ and $\omega([\eta(f), \eta(g)], \eta(h)) = -\omega(\eta(h), [\eta(f), \eta(g)]) = -dh([\eta(f), \eta(g)]) = -[\eta(f), \eta(g)]h = -\eta(f)\eta(g)h + \eta(g)\eta(f)h = -\eta(f)\{g, h\} + \eta(g)\{f, h\} = -\{f, \{g, h\}\} + \{g, \{f, h\}\}$. Thus $d\omega(\eta(f), \eta(g), \eta(h)) = \sum_{c.p.\ f,g,h} \eta(f)\omega(\eta(g), \eta(h)) - \omega([\eta(f), \eta(g)], \eta(h)) = -\sum_{c.p.\ f,g,h}\{g, \{f, h\}\}$ (we use the Cartan formula $(d\omega)(X, Y, Z) = \sum_{c.p. X,Y,Z} X\omega(Y, Z) - \omega([X, Y], Z)$). So, if $d\omega = 0$, then $\{,\}$ satisfies the Jacobi identity.

Conversely, if the JI holds, $d\omega$ vanishes on all hamiltonian vector fields. To finish the proof it remains to note that the hamiltonian vector fields span $T_xM$ at any $x \in M$. Indeed, it is enough to take $\eta(x^i)$, where $(x^i)$ are local coordinates.

**Example: the canonical symplectic structure on the cotangent bundle $T^*Q$:** Let $\pi_Q : T^*Q \to Q$ be a cotangent bundle to a manifold $Q$. There is a canonical differential 1-form $\lambda \in \Gamma(T^*M), M := T^*Q$ determined uniquely by the following condition: for any $\alpha \in \Gamma(T^*Q)$, the following equality holds $\alpha^*\lambda = \alpha$, here $\alpha$ in the LHS is regarded as a map $\alpha : Q \to T^*Q$. We call $\lambda$ the *Liouville* 1-form. If $(U, q^1, \ldots, q^n)$ is a local chart on $Q$, the 1-forms $dq^1, \ldots, dq^n$ form a basis of the vector space $T^*_xQ, x \in U$, and define the chart $(\pi_Q^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$. In these coordinates $\lambda = p_i dq^i$. Indeed, $\alpha : (q^1, \ldots, q^n) \mapsto (q^1, \ldots, q^n, \alpha_1(q), \ldots, \alpha_n(q))$, where $\alpha = \alpha_i(q)dq^i$. Thus $\alpha^*\lambda = \alpha_i(q)dq^i = \alpha$.

The canonical symplectic form $\omega$ on $M$ is given by $\omega := d\lambda$, or, locally, $\omega = dp_i \wedge dq^i$.

**Hamiltonian differential equation on a symplectic manifold $(M, \omega)$:** The ODE related to a hamiltonian vector field $\eta(f), f \in \mathcal{E}(M)$, here $\eta = \omega^{-1}$. In the context of the example above (in the canonical coordinates $(q, p)$): $\eta = -\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, \eta(H) = \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$, the corresponding equations read:

$$\dot{q}^i = -\frac{\partial H(q, p)}{\partial p^i}, \dot{p}_i = \frac{\partial H(q, p)}{\partial q_i}.$$

**A Poisson structure on $M$:** A bivector $\eta : T^*M \to TM$ (not necessarily nondegenerate) such that the corresponding bracket $\{,\}$ on $\mathcal{E}(M)$ satisfies the Jacobi identity (JI for short).

Consider the Lie algebra $(\mathcal{E}(M), \{,\})$ on a Poisson manifold. The corresponding $\mathrm{ad}_f$-operator, $f \in \mathcal{E}(M)$, coincides with $\eta(f) : \mathcal{E}(M) \to \mathcal{E}(M)$.

**The characteristic (generalized) distribution of a Poisson structure $\eta : T^*M \to TM$:** $\mathcal{D}_\eta := \operatorname{im} \eta$ (locally generated by the hamiltonian vector fields $\eta(x^1), \ldots, \eta(x^n)$, where $(x^1, \ldots, x^n)$ are some local coordinates).

By the third form of the JI the map $f \mapsto \eta(f), (\mathcal{E}(M), \{,\}) \to (\Gamma(TM), [,])$ is a homomorphism of Lie algebras, here $[,]$ is the commutator of vector fields. This implies involutivity of $\mathcal{D}_\eta$: $[\eta(x^i), \eta(x^j)] = \eta(\{x^i, x^j\}) = \eta(\eta^{ij}(x))$, where $\eta = \eta^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. On the other hand, $\eta(f) = \eta^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^i}\eta(x^i)$ for any $f$. In particular, $[\eta(x^i), \eta(x^j)]$ is a linear combination (with smooth coefficients) of $\eta(x^1), \ldots, \eta(x^n)$.

**Theorem:** The characteristic distribution $\mathcal{D}_\eta$ is integrable (we call the corresponding foliation *characteristic* or *symplectic*).

*Proof* In analytic category this follows from the involutivity of $\mathcal{D}$ by the generalized Frobenius theorem. In the smooth case this is also true, but the proof is more complicated, so we skip it. □

**Digression on linear algebra of bivectors:** Let $V$ be a vector space and $e$ a bivector on $V$. Then $e$ can be treated as: 1) an element $e \in \bigwedge^2 V$; 2) a linear skew-symmetric map $e^\sharp : V^* \to V$; 3) a bilinear form $\tilde{e}$ on $V^*$.

Proposition. Let $W := \operatorname{im} e^\sharp \subset V$. Then there exists a correctly defined bivector $e|_W \in \bigwedge^2 W$, called the *restriction* of $e$ to $W$. Moreover, the restriction $e|_W$ is nondegenerate, i.e. $e|_W^\sharp : W^* \to W$ is an isomorphism.

*Proof I.* A theorem from linear algebra says that there exists a basis $v_1, \ldots, v_n$ of $V$ such that $e = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$ (the number $2k$ is equal to $\dim W$ and is called the *rank* of $e$). It is easy to see that $v_1, \ldots, v_{2k}$ span $W$. □

*Proof II.* $e$ is skew-symmetric, i.e. $(e^\sharp)^* = -e^\sharp$. This implies $\ker e^\sharp = (\operatorname{im} e^\sharp)^\perp$, where $(\cdot)^\perp$ stands for the annihilator of $(\cdot)$. So the natural isomorphism $\hat{e} : V^*/\ker e^\sharp \to \operatorname{im} e^\sharp = W$ induced by $e^\sharp$ can regarded as a map from $W^* \cong V^*/(W^\perp)$ to $W \subset V$. The map $\hat{e}$ being skew-symmetric induces the element of $\bigwedge^2 W$, which we denote by $e|_W$. □

*Proof III.* Let $\omega$ be a skew-symmetric bilinear form on a vector space $L$. Put $\ker \omega := \{x \in L \mid \omega(x, y) = 0 \ \forall y \in L\}$. The form is called *nondegenerate* if $\ker \omega = \{0\}$.

Any $\omega$ induces a nondegenerate skew-symmetric bilinear form on the vector space $L/\ker \omega$.

Treating $e$ as a skew-symmetric bilinear form $\tilde{e}$ on $V^*$ we have $\ker \tilde{e} = \ker e^\sharp$. The restriction $e|_W$ treated as a skew-symmetric bilinear form on $W^* \cong V^*/\ker \tilde{e}$ is the above mentioned nondegenerate form induced from $\tilde{e}$. □

**Symplectic leaves of a Poisson structure $\eta$ on $M$:** These are the leaves of the characteristic foliation $\mathcal{D}_\eta$. Since $D_{\eta,x} = \operatorname{im} \eta_x^\sharp$ for any $x \in M$, the bivector $\eta$ admits a restriction $\eta|_S$ to any symplectic leaf $S \subset M$, which is a nondegenerate bivector on $S$. Moreover, since any hamiltonian vector field $\eta(f)$ is tangent to $S$ at points of $S$, the value $\{f, g\}(x) = (\eta(f)g)(x), x \in S$, depends only of $g|_S$ and by the skew-symmetry the same is true with respect to $f$. In other words, $\{f|_S, g|_S\}_{\eta|_S} =$

$(\{f,g\}_\eta)|_S$ for any $f,g\in\mathcal{E}(M)$ and the operation $\{,\}_{\eta|S}$ satisfies the JI, hence $\eta|_S$ is a nondegenerate Poisson structure on $S$. This explains the term "symplectic leaf" ($(\eta|_S)^{-1}$ is a symplectic form).

**Example 1:** Let $M:=\mathbb{R}^2, \eta=x^1\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$. On the open set $U:=\{x^1\neq 0\}$ the form $(\eta|_U)^{-1}=-(1/x^1)dx^1\wedge dx^2$ is symplectic. Thus the JI holds for $\{,\}_\eta$ on $U$ and by continuity it holds also on the whole $M$. The symplectic leaves are $U$ and all the points on the line $\{x^1=0\}$.

**Example 2:** Let $M:=\mathbb{R}^3, \eta=\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$. On each plane $P_c:=\{x^3=c\}$ the form $(\eta|_{P_c})^{-1}=-dx^1\wedge dx^2$ is symplectic. The JI holds for $\{,\}_\eta$ on $P_c$ for any $c\in\mathbb{R}$. Since $P_c$ sweep the whole space $M$ as $c$ runs through $\mathbb{R}$, the JI holds for $\{,\}_\eta$ globally. The symplectic leaves are the planes $P_c$.

**Example 3:** Let $M:=\mathbb{R}^3, \eta=x^1\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}+x^2\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^1}+x^3\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$ (we will prove that this is a Poisson bivector later). The symplectic leaves are ...

**Example 4:** Let $M=\mathbb{T}^2\times\mathbb{R}$, let $y$ be a coordinate on the second component. Put $\eta=\tilde{v}_{a,b}\wedge\frac{\partial}{\partial y}$, where $\tilde{v}_{a,b}$ is the generator of winding line. $\eta$ is Poisson because locally it looks like the bivector from Example 2. If $b/a$ is irrational, the symplectic leaves (which are two-dimensional) are dense in $M$.

**Casimir functions of a Poisson structure $\eta$ on $M$:** Let $U\subset M$ be an open set. We say that $f\in\mathcal{E}(U)$ is a *Casimir function* if $\eta(f)\equiv 0$ on $U$. In particular, since $\{f,g\}=\eta(f)g$ on $U$ the Casimir functions constitute the centre of the Lie algebra $(\mathcal{E}(U),\{,\}|_U)$. The space of Casimir functions over $U$ will be denoted by $\mathcal{C}_\eta(U)$.

PROPOSITION. The Casimir functions are constant on the leaves of the symplectic foliation.

*Proof* We have $\eta(f)g=-\eta(g)f=0$ for any $f\in\mathcal{C}_\eta(U), g\in\mathcal{E}(U)$. So, since $\eta(g)$ span the characteristic distribution, $f$ is constant along its leaves. □

**Example 1':** $\mathcal{C}_\eta(M)=\mathbb{R}$, the space of constant functions.

**Example 2':** $\mathcal{C}_\eta(M)=\mathcal{F}un(x^3)$, the space of functions functionally generated by $x^3$.

**Example 3':** $\mathcal{C}_\eta(M)=\mathcal{F}un((x^1)^2+(x^2)^2+(x^3)^2)$. Hence the symplectic leaves are the concentric spheres and the point $\{(0,0,0)\}$.

**Example 4':** If $b/a$ is irrational $\mathcal{C}_\eta(M)=\mathbb{R}$. However, for sufficiently small $U$ the space $\mathcal{C}_\eta(U)$ will be functionally generated by one nonconstant function. So "local Casimirs" are not obtained as the restriction of the "global Casimirs".

**Lie–Poisson structures, Definition I:** Let $(\mathfrak{g},[,])$ be a finite-dimensional Lie algebra, $\mathfrak{g}^*$ its dual space (space of linear functionals on $\mathfrak{g}$). Given $f,g\in\mathcal{E}(\mathfrak{g}^*)$ define $\{f,g\}_\mathfrak{g}(x):=\langle x,[df|_x,dg|_x]\rangle, x\in\mathfrak{g}^*$. Here we identify $T_x^*\mathfrak{g}^*$ with $\mathfrak{g}$, $\langle,\rangle$ stands for the canonical pairing between vectors and covectors.

**Lie–Poisson structures, Definition II:** Let $(\mathfrak{g},[,])$ be a finite-dimensional Lie algebra, $e_1,\ldots,e_n\in\mathfrak{g}$ its basis, $x_1=e_1,\ldots,x_n=e_n$ these vectors regarded as linear functions on $\mathfrak{g}^*$ (in particular $x_1,\ldots,x_n$ are linear coordinates on $\mathfrak{g}^*$). Let $[e_i,e_j]=c_{ij}^ke_k$ ($c_{ij}^k$ are called the *structure constants* corresponding to the basis $e_1,\ldots,e_n$). Put $\eta_\mathfrak{g}:=c_{ij}^kx_k\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$.

PROPOSITION. The bivector corresponding to the bracket $\{,\}_\mathfrak{g}$ coincides with $\eta_\mathfrak{g}$.

*Proof Exercise:* Prove that, given $\{,\}: \mathcal{E}(M)\times\mathcal{E}(M)\to\mathcal{E}(M)$, a bilinear skew-symmetric operation being a differentiation with respect to each argument, there exists a bivector $\eta\in\Gamma(\bigotimes^2 TM)$ such that $\{f,g\}=\eta(df,dg)$.

Let $\eta=\eta^{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ be the bivector corresponding to $\{,\}_{\mathfrak{g}}$. Take $f:=x_i, g:=x_j$, then $\{f,g\}(x)=\eta^{ij}(x)$. On the other hand, by Definition I, $\{f,g\}(x)=\langle x,[x_i,x_j]\rangle=c_{ij}^k x_k$. $\square$

**Exercise:** 1) Let $\eta\in\Gamma(\bigwedge^2 TM)$, in local coordinates $\eta=\eta^{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$. Show that the JI for $\{,\},\{f,g\}=\eta^{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ holds if and only if the expression

$$[\eta,\eta]_S^{ijk}:=\sum_{c.p.\,i,j,k}\eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

vanishes for all $i,j,k\in\{1,\ldots,n\}$. 2) Show that, given $\eta,\zeta\in\Gamma(\bigwedge^2 TM),\eta=\eta^{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\zeta=\zeta^{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$, the expression

$$[\eta,\zeta]_S^{ijk}:=\frac{1}{2}\sum_{c.p.\,i,j,k}\eta^{ir}(x)\frac{\partial}{\partial x^r}\zeta^{jk}(x)+\zeta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

is a local representation of a trivector on $M$ (called the *Schouten bracket* of $\eta$ and $\zeta$).

**Proof of the Jacobi identity for the Lie–Poisson structure:** $[\eta_{\mathfrak{g}},\eta_{\mathfrak{g}}]_S^{ijk}=\sum_{c.p.\,i,j,k}c_{ir}^l x_l c_{jk}^r$. The last expression vanishes for all $i,j,k$ if and only if $\sum_{c.p.\,i,j,k}c_{ir}^l c_{jk}^r=0$ for all $l,i,j,k$, which is equivalent to the JI for $[,]$. $\square$

**An action of a Lie algebra $\mathfrak{g}$ on a manifold:** A homomorphism of Lie algebras $\rho:(\mathfrak{g},[,])\to(\Gamma(TM),[,])$ (in the target space $[,]$ stands for the commutator of vector fields) is called a *(right) action* of $\mathfrak{g}$ on $M$ (a left action corresponds to an antihomomorphism, i.e. a map $\rho:(\mathfrak{g},[,])\to(\Gamma(TM),[,])$ such that $\rho([v,w])=-[\rho(v),\rho(w)], v,w\in\mathfrak{g}$).

**Orbits of an action $\rho:(\mathfrak{g},[,])\to(\Gamma(TM),[,])$:** Put $D_x:=\{\rho(v)|_x\mid v\in\mathfrak{g}\}, x\in M$.

PROPOSITION. Let $\mathfrak{g}$ be finite-dimensional. Then the generalized distribution $\mathcal{D}:=\{D_x\}_{x\in M}$ is integrable.

*Proof* The distribution $\mathcal{D}$ is involutive: $[\rho(v),\rho(w)]=\rho([v,w])$. Thus in the analytic category the proof follows from the generalized Frobenius theorem. We skip the proof in the smooth case (roughly it consists in integrating the action of the Lie algebra to a local action of the corresponding Lie group). $\square$

The leaves of the corresponding generalized foliation are called the *orbits* of the action $\rho$. If the Lie algebra $\mathfrak{g}$ is finite-dimensional, the action can be "integrated" to a local action of a Lie group $G$ such that $\mathfrak{g}$ is its Lie algebra. Then the orbits of the Lie algebra action and of the Lie group action coincide.

**Linear representations and actions:** Let $V$ be a vector space and $A\in\mathrm{End}(V)$ a linear operator. It induces a uniquely defined vector field $\tilde{A}$ on $V$ given by $x\mapsto(x,Ax):V\to V\times V\cong TV$. If

$e_1, \dots, e_n$ is a basis of $V$, $x^1, \dots, x^n$ the dual basis of $V^*$ (i.e. the coordinates on $V$) and $Ae_i = A_{ji}e_j$, we have $\tilde{A} = A_{ji}x^i\frac{\partial}{\partial x^j}$.

*Exercise:* The map $A \mapsto \tilde{A} : \mathrm{End}(V) \to \Gamma(TV)$ is an antihomorphism of Lie algebras, i.e. a left action of the Lie algebra $\mathrm{End}(V)$ on $V$.

Let $L : (\mathfrak{g}, [,]) \to (\mathrm{End}(V), [,])$ be a representation of a Lie algebra $\mathfrak{g}$ in a vector space $V$. Then the map $\tilde{L} : \mathfrak{g} \to \Gamma(TM), \tilde{L}(x) := \widetilde{L(x)}$ is a left action of $\mathfrak{g}$ on the manifold $V$. Note, that the dual representation $L^* : \mathfrak{g} \to \mathrm{End}(V^*)$ given by $L^*(v) := (L(v))^*$ is an antihomomorphism, hence the map $\tilde{L^*} : \mathfrak{g} \to \Gamma(TM), \tilde{L^*}(v) := \widetilde{(L(v))^*}$ is a right action of $\mathfrak{g}$ on $V$.

**The adjoint and coadjoint actions:** Let $(\mathfrak{g}, [,])$ be a Lie algebra. The homomorphism $v \mapsto \mathrm{ad}_v : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, where $\mathrm{ad}_v w := [v, w]$, gives the *adjoint* representation (of $\mathfrak{g}$ on $\mathfrak{g}$). The corresponding (left) action $v \mapsto \widetilde{\mathrm{ad}_v} : \mathfrak{g} \to \Gamma(T\mathfrak{g})$ is also called *adjoint*. The homomorphism $v \mapsto \mathrm{ad}^*_v : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, where $\mathrm{ad}^*_v$ is the transposed operator to $\mathrm{ad}_v$, and the corresponding (right) action $v \mapsto \widetilde{\mathrm{ad}^*_v} : \mathfrak{g} \to \Gamma(T\mathfrak{g}^*)$ are called the *coadjoint* (anti)representation and action, respectively.

**The symplectic leaves of the Lie-Poisson structure $\eta_\mathfrak{g}$ on $\mathfrak{g}^*$ coincide with the orbits of the coadjoint action :** We claim that $\widetilde{\mathrm{ad}^*_v} = \eta_\mathfrak{g}(v')$, where $v'$ denotes the linear function on $\mathfrak{g}^*$ defined by an element $v \in \mathfrak{g}$. Indeed, let $v = v^j e_j$, then $v' = v^j x_j$. Here $x_1, \dots, x_n$ are the elements $e_1, \dots, e_n$ regarded as linear functions on $\mathfrak{g}^*$. Then $\mathrm{ad}_v e_i = v^j c^k_{ji} e_k, \mathrm{ad}^*_v x^i = v^j c^i_{jk} x^k$, hence $\widetilde{\mathrm{ad}^*_v} = v^j c^i_{jk} x_i \frac{\partial}{\partial x_k}$. The last expression obviously coincides with $\eta_\mathfrak{g}(v')$. $\square$

**An invariant symmetric bilinear form on $(\mathfrak{g}, [,])$:** A symmetric bilinear form $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying the equality $(\mathrm{ad}_x y, z) = -(y, \mathrm{ad}_x z)$ for any $x, y, z \in \mathfrak{g}$.

PROPOSITION. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the map $v \mapsto (v, \cdot)$. Then the adjoint orbits become coadjoint ones under this identification.

*Proof* Indeed, if $A : \mathfrak{g} \to \mathfrak{g}$ is a linear operator the transposed operator $A^* : \mathfrak{g}^* \to \mathfrak{g}^*$ becomes the adjoint one under this identification: $(A^* y, z) = (y, Az)$ for any $y, z \in \mathfrak{g}$. Thus $\mathrm{ad}^*_x$ becomes $-\mathrm{ad}_x$. $\square$

**Notations (for the Lie algebras):** $\mathfrak{gl}(n, \mathbb{R}) := \{n \times n - \text{matrices with real entries}\}, \mathfrak{sl}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{Tr}(x) = 0\}, \mathfrak{so}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid x = -x^T\}$

The sets above are Lie algebras with respect to the commutator of matrices.

**Notations (for the Lie Groups):** $GL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X \neq 0\}, SL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X = 1\}, SO(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid XX^T = I_n\}$. All these sets are groups with respect to the matrix multiplication. It is easy to see that if $x \in \mathfrak{g}$, where $\mathfrak{g}$ is one of the Lie algebras above, then $\exp(x) \in G$, where $G$ is the corresponding Lie group. Also $\mathfrak{g} = T_I G$.

The Lie algebras from Examples 1-4, below, have an invariant nondegenerate symmetric form $(x, y) = \mathrm{Tr}(xy)$ by means of which we can make an identification $\mathfrak{g} \cong \mathfrak{g}^*$. The coadjoint orbits are identified with the adjoint ones, which can be described as the orbits of the corresponding Lie group with respect to the conjugation of matrices: $\{XxX^{-1} \mid X \in G\}, x \in \mathfrak{g}$.

**Example 1:** $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{R}), \mathcal{C}_{\eta_\mathfrak{g}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x), \mathrm{Tr}(x^2), \ldots, \mathrm{Tr}(x^n))$.

**Example 2:** $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R}), \mathcal{C}_{\eta_\mathfrak{g}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \ldots, \mathrm{Tr}(x^n))$. In particular, for $n = 2$ we have a basis $e_1 := e_{11} - e_{22}, e_2 := e_{12}, e_2 := e_{21}$ and the commutation relations $[e_1, e_2] = 2e_2, [e_1, e_3] = -2e_3, [e_2, e_3] = e_1$. Hence $\eta_\mathfrak{g} = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - 2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}$. The Casimir function $\mathrm{Tr}(x^2)$ reads as $x_1^2/2 + 2x_2x_3$. The symplectic leaves are the 1-sheet hyperboloids, sheets of 2-sheet hyperboloids, two sheets of the cone (without zero) and the point 0.

**Example 3:** $\mathfrak{g} := \mathfrak{so}(2n, \mathbb{R}), \mathcal{C}_{\eta_\mathfrak{g}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2n-2}), \mathrm{Pf}(x))$.

**Example 4:** $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{R}), \mathcal{C}_{\eta_\mathfrak{g}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2n}))$.

**Example 5 (the Heisenberg algebra):** $\mathfrak{g} := \mathbb{R}^3, [e_1, e_2] = e_3$, here $e_1, e_2, e_3$ is the standard basis of $\mathbb{R}^3$. We have $\eta_\mathfrak{g} = x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \mathcal{C}_{\eta_\mathfrak{g}}(\mathfrak{g}^*) = \mathcal{F}un(x_3)$, so the coadjoint orbits consist of the planes $\{x_3 = c\}, c \neq 0$ and of the points of the plane $\{x_3 = 0\}$. The adjoint orbits are generated by the vector fields $c_{ij}^k x^i \frac{\partial}{\partial x^k}$, where $\{x^i\}$ is the basis dual to $\{x_i\}$, i. e. by $x^1 \frac{\partial}{\partial x^3}, x^2 \frac{\partial}{\partial x^3}$, so they are the lines parallel to the $x_3$-axis and the points of this axis.

**The Arnold–Liouville theorem:** Let $(M, \omega)$ be symplectic, $\dim M = 2n$. Assume a hamiltonian vector field $v(H)$ admits $n$ functionally independent integrals $g_1 = H, g_2, \ldots, g_n$ in involution. Then

1. if the common level sets $M_c := \{x \in M \mid g_i = c_i, i = 1, \ldots, n\}$ of these integrals are compact and connected, they are diffeomorphic to ($n$-dimensional) tori $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mathrm{mod} 2\pi\}$;

2. the restriction of the initial hamiltonian equation to $\mathbb{T}^n$ gives an almost periodic motion on $\mathbb{T}^n$, i.e. in the "angle coordinates" $\varphi$ the equation has the form

$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a},$$

   here $\overrightarrow{a} = (a_1, \ldots, a_n)$ is a constant vector depending only on the level;

3. the initial equation can be integrated in "quadratures", i.e. the solutions can be obtained by means of a finite number of algebraic operations and operations of taking integral.

The proof of this theorem essentially breaks into two parts. The first shows that a compact $n$-dimensional manifold with $n$ commuting nonvanishing vector fields $v_1, \ldots, v_n$ (in our case $v_i = \eta(g_i)$) is diffeomorphic to $\mathbb{T}^n$.

The second builds special coordinates on $M$, the "action-angle" coordinates. The "angles" $\varphi_1, \ldots, \varphi_n$ are defined in the fiest part of the proof for a fixed level set $M_c$, but it turns out that they smoothly depend on $c$. The "action" coordinates $I^1, \ldots, I^n$ depend only on $g_1, \ldots, g_n$ and satisfy $\omega = dI^i \wedge d\varphi_i$ (i.e. $(\varphi, I)$ are canonical or Darboux coordinates). The initial equations in these coordinates are of the form

$$\frac{d\overrightarrow{I}}{dt} = 0, \frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(I).$$

Due to the fact that $(I, \varphi)$-coordinates are canonical, we get $\overrightarrow{a}(I) = -\frac{\partial H}{\partial \overrightarrow{I}}, \frac{\partial H}{\partial \overrightarrow{\varphi}} = 0$. Thus, knowing the "action-angle" coordinates, we can easily calculate the vector of "frequences" $\overrightarrow{a}$ and the solutions: $\overrightarrow{\varphi} = \overrightarrow{\varphi_0} + t\,\overrightarrow{a}$.

**Example (harmonic oscillator I):** Let $M = \mathbb{R}^2, H = (1/2)(p^2+q^2), \omega = dp\wedge dq$. Then in the polar coordinates $q = r\cos\varphi, p = r\sin\varphi$ we have $dp \wedge dq = -\sin\varphi dr \wedge r\sin\varphi d\varphi + \cos\varphi r d\varphi \wedge \cos\varphi dr = -rdr \wedge d\varphi = d(-r^2/2) \wedge d\varphi$. Hence $I = -H, a_1 = 1$, the solution is $\varphi(t) = \varphi(0) + t$, i.e.

$$t \mapsto (R\cos(\varphi(0) + t), R\sin(\varphi(0) + t)).$$

**Example (harmonic oscillator II):** Let $M = \mathbb{R}^2, H = (1/2)(a^2p^2 + b^2q^2), \omega = dp \wedge dq$. The hamiltonian vector field is $\eta(H) = -a^2p\frac{\partial}{\partial q} + b^2q\frac{\partial}{\partial p}$, here $\eta = \omega^{-1} = -\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$. The level sets $M_c = \{(q,p) \mid H(q,p) = c\}$ are ellipses $\{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) = 1\}$ with the semiaxes $\sqrt{2c}/b, \sqrt{2c}/a$. Note that the standard parametrization of the ellipse, $\varphi \mapsto (\sqrt{2c}/b\cos\varphi, \sqrt{2c}/a\sin\varphi)$ is not a trajectory of $\eta(H)$

The recipe gives $I(c) = \frac{1}{2\pi}\int_{M_c} pdq = \frac{1}{2\pi}\int_{\overline{M}_c} \omega = -\frac{c}{ab}$, which up to $-\frac{1}{2\pi}$ is the area of the figure $\overline{M}_c := \{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) \leqslant 1\}$ bounded by the ellipse. From this we conclude that $H = -abI$ and that the solution of the hamiltonian system

$$\dot{q} = -a^2p, \dot{p} = b^2q$$

is given by $H = c, \varphi(t) = \varphi(0) - t\frac{\partial H}{\partial I} = \varphi(0) + tab$ or, in other words, by

$$t \mapsto ((\sqrt{2c}/b)\cos(t_0 + tab), (\sqrt{2c}/a)\sin(t_0 + tab)).$$

**Example (harmonic oscillator III):** Let $M = \mathbb{R}^4, H = (1/2)(p_1^2 + p_2^2 + q_1^2 + q_2^2), \omega = dp \wedge dq$. The hamiltonian vector field is $\eta(H) = -p_1\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial q_2} + q_1\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial p_2}$. Obviously $\eta(H)f = 0$ for $f := q_1q_2 + p_1p_2$ so this is a Liouville–Arnold integrable system.

# Lecture III

*References:* [Mag78, GZ89, Bol91]

**A Poisson pencil on $M$:** Let a pair $(\eta_1, \eta_2)$ of linearly independent bivectors on a manifold $M$ be given. Assume $\eta^t := t_1\eta_1 + t_2\eta_2$ is a Poisson structure for any $t = (t_1, t_2) \in \mathbb{R}^2$. We say that the Poisson structures $\eta_1, \eta_2$ are *compatible* (or form a *bihamiltonian structure* or a *Poisson pair*) and that the whole family $\Theta := \{\eta^t\}_{t\in\mathbb{R}^2}$ is a *Poisson pencil.*

*Exercise:* Show that the following conditions are equivalent:

1. $\eta^t$ is Poisson, i.e. $[\eta^t, \eta^t]_S = 0$, for any $t \in \mathbb{R}^2$ (here $[,]_S$ is the Schouten bracket);

2. $[\eta^t, \eta^t]_S = 0$ for any three pairwise nonproportional values of $t \in \mathbb{R}^2$;

3. $[\eta_1, \eta_1]_S = 0, [\eta_1, \eta_2]_S = 0, [\eta_2, \eta_2]_S = 0$.

**Example 1:** Let $\eta_1, \eta_2$ be bivectors on $\mathbb{R}^n$ with constant coefficients. Then they form a Poisson pair (recall that, given a bivector $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have $[\eta,\eta]_S^{ijk} := \sum_{c.p.\,i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$).

**Example 2:** Let $\mathfrak{g}$ be a Lie algebra and $\eta_{\mathfrak{g}}$ the Lie–Poisson structure on $\mathfrak{g}^*$. Let $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a 2-cocycle on $\mathfrak{g}$, i.e. $c$ is skew-symmetric and $\sum_{c.p.\ v,w,u} c([v,w],u) = 0$ for any $v, w, u \in \mathfrak{g}$. Then $c \in (\mathfrak{g} \wedge \mathfrak{g})^* \cong \mathfrak{g}^* \wedge \mathfrak{g}^*$ can be regarded as a bivector on $\mathfrak{g}^*$ with constant coefficients. It turns out that $(\eta_1, \eta_2)$, where $\eta_1 := \eta_{\mathfrak{g}}, \eta_2 := c$, is a Poisson pair.

Indeed, it is easy to see that the bracket $[(v, \alpha), (w, \beta)]' := ([v, w], c(v, w))$ defines a Lie algebra structure on $\mathfrak{g}' := \mathfrak{g} \times \mathbb{R}$ (*Exercise:* check this). The $\mathbb{R}$-component lies in the centre of $\mathfrak{g}'$, we say that $\mathfrak{g}'$ is a *central extension* of $\mathfrak{g}$. The affine subspaces $\mathfrak{g}^*_{x_0} := \mathfrak{g}^* \times x_0 \subset (\mathfrak{g}')^* = \mathfrak{g}^* \times \mathbb{R}$ are Poisson submanifolds of the Poisson manifold $((\mathfrak{g}')^*, \eta_{\mathfrak{g}'})$. The restriction $\eta_{\mathfrak{g}'}|_{\mathfrak{g}^*_{x_0}}$ coincides with $\eta_1 + x_0\eta_2$, i.e. the last bivector is Poisson at least for three different values of $x_0$. We conclude that $(\eta_1, \eta_2)$ is a Poisson pair.

In coordinates this looks as follows. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ and $[e_i, e_j] = c^k_{ij} e_k, c(e_i, e_j) = c_{ij}, i, j, k = 1, \ldots, n$, for some constants $c^k_{ij}, c_{ij} \in \mathbb{R}$. Put $\eta'_0 := (0, 1), \eta'_i := (\eta_i, 0) \in \mathfrak{g}', i = 1, \ldots, n$, and let $x'_0, \ldots, x'_n$ denote the same elements regarded as coordinates on $(\mathfrak{g}')^*$. Then $\eta_{\mathfrak{g}'} = (c^k_{ij} x'_k + x'_0 c_{ij}) \frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j}$ and $\eta^t = (t_1 c^k_{ij} x_k + t_2 c_{ij}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Here $x_1, \ldots, x_n$ are coordinates on $\mathfrak{g}^*$ corresponding to $e_1, \ldots, e_n$.

**Example 3:** In a particular case when the cocycle $c$ is trivial, i.e. $c(v, w) = a([v, w])$ for some $a \in \mathfrak{g}^*$ we get a Poisson pencil $\{\eta^t\}, \eta^t := (t_1 c^k_{ij} x_k + t_2 c^k_{ij} a_k) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, here $a_1, \ldots, a_n$ are coordinates of $a$ in the dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. In the corresponding Poisson pair $(\eta_1, \eta_2)$ the first bivector is the Lie-Poisson one, $\eta_{\mathfrak{g}}$, and the second one is $\eta_{\mathfrak{g}}(a)$, the Lie-Poisson bivector "frozen" at $a$.

**Example 4:** Let $\mathfrak{g} : \mathfrak{gl}(n, \mathbb{R})$ and $A \in \mathfrak{g}$. Put $[x, y]_A := xAy - yAx$. It is easy to see that $[,]_A$ is a Lie bracket on $\mathfrak{g}$ for any $A$ (*Exercise*: check this). In particular, for a fixed $A \in \mathfrak{g}$ the bracket $[,]^t := t_1[,] + t_2[,]_A = [,]_{t_1 I + t_2 A}$ is a Lie bracket for any $t \in \mathbb{R}^2$ (any family of Lie brackets linearly spanned by two fixed brackets will be called a *Lie pencil*). Denote $\mathfrak{g}^t := (\mathfrak{g}, [,]^t)$. The Lie–Poisson structures $\eta_{\mathfrak{g}^t}$ form a Poisson pencil on $\mathfrak{g}^*$.

We get a generalization of this example taking $\mathfrak{g} := \mathfrak{so}(n, \mathbb{R})$ and $A$ a symmetric $n \times n$-matrix.

**I mechanism of constructing functions in involution (the Magri–Lenard scheme):** Let $(\eta_1, \eta_2)$ be a pair of Poisson structures (not necessarily compatible). Assume we can found a sequence of functions $H_0, H_1, \ldots \in \mathcal{E}(M)$ satisfying

$$\begin{array}{rcl} \eta_1(H_0) & = & \eta_2(H_1) \\ \eta_1(H_1) & = & \eta_2(H_2) \\ & \vdots & . \end{array} \tag{1}$$

PROPOSITION. *For any indices $i, j$ the following equality holds:*

$$\{H_i, H_j\}_{\eta_1} = \{H_{i+1}, H_{j-1}\}_{\eta_1}.$$

*Proof* $\eta_1(H_i)H_j = \eta_2(H_{i+1})H_j = -\eta_2(H_j)H_{i+1} = -\eta_1(H_{j-1})H_{i+1} = \eta_1(H_{i+1})H_{j-1}$ □

Now assume $i < j$. If $j - i = 2k$, we can apply the proposition $k$ times and get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k}\}_{\eta_1} = 0$. If $j - i = 2k + 1$, we get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k+1}\}_{\eta_1} = \eta_1(H_{i+k})H_{i+k+1} = \eta_2(H_{i+k+1})H_{i+k+1} = 0$. Hence the sequence $H_0, H_1, \ldots$ is a family of first integrals in involution for any of vector fields $v_i := \eta_1(H_i), i = 0, 1, \ldots$ Note that all these vector fields are "bihamiltonian", i.e. hamiltonian with respect to both the Poisson structures $\eta_1, \eta_2$.

In general it is hard to find the sequences of functions $H_0, H_1, \ldots$ with the required properties. However, if we assume additionally that $(\eta_1, \eta_2)$ is a Poisson pair, there are some cases, when such sequences naturally appear. For instance, assume that all the bivectors $\eta^t := t_1\eta_1 + t_2\eta_2$ of the corresponding Poisson pencil are degenerate. Let $\eta^\lambda := \lambda\eta_1 + \eta_2, \lambda := t_1/t_2$, and let $f^\lambda$ be a Casimir function of $\eta^\lambda$. It turns out that $f^\lambda$ depends smoothly, let $f^\lambda = f_0 + \lambda f_1 + \lambda^2 f_2 + \cdots$ be the corresponding Tailor expansion. Then we deduce from the equality $\eta^\lambda(f^\lambda) = 0$ that $0 = \eta_2(f_0), \eta_1(f_0) + \eta_2(f_1), \eta_1(f_1) + \eta_2(f_2), \ldots$ (coefficients of different powers of $\lambda$). Thus we can put $H_0 := f_0, H_1 := -f_1, H_2 := f_2, \ldots$ Note that such a Magri–Lenard chain starts from a Casimir function of $\eta_2$. If $g^\lambda = g_0 + \lambda g_1 + \cdots$ is another Casimir function of $\eta^\lambda$, we get another sequence of functions in involution. A question arises, is it true that $\{f_i, g_j\}_{\eta_k} = 0$? Another important question concerns the *completeness* of the obtained family of functions.

**II mechanism of constructing functions in involution (based on the Casimir functions of a Poisson pencil):** Let $\{\eta^t\}_{t\in\mathbb{R}^2}$ be a Poisson pencil on $M$. Denote by $\mathcal{C}^t(M)$ the space of Casimir functions of $\eta^t$.

PROPOSITION. *Let $t', t'' \in \mathbb{R}^2$ be linearly independent and let $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M)$. Then*

$$\{f, g\}_{\eta^t} = 0$$

*for any $t \in \mathbb{R}^2$.*

*Proof* Indeed for any $t \in \mathbb{R}^2$ there exist $c', c'' \in \mathbb{R}$ such that $t = c't' + c''t''$. Then $\{f, g\}_{\eta^t} = \eta^t(f)g = (c'\eta^{t'} + c''\eta^{t''})(f)g = c''\eta^{t''}(f)g = -c''\eta^{t''}(g)f = 0$. □

It is not clear from this fact whether $\{f, g\}_{\eta^t} = 0$ if $f, g$ are Casimir functions of the *same* bivector $\eta^{t'}$. We will discuss this question in the next lecture.

**The Jordan–Kronecker decomposition of a pair of bivectors:** A bivector $b$ on a vector space $V$ is an element of $\bigwedge^2 V$. We will view a bivector $b$ sometimes as a skew-symmetric map $V^* \to V$ (then its value at $x \in V^*$ will be denoted by $b(x)$) and sometimes as a skew-symmetric bilinear form on $V^*$ (then its value at $x, y \in V^*$ will be denoted by $b(x, y)$). In particular, $b(x, y) = \langle b(x), y\rangle$.

THEOREM. *(Gelfand–Zakharevich, 1989) Given a finite-dimensional vector space $V$ over $\mathbb{C}$ and a pair of bivectors $(b^{(1)}, b^{(2)}), b^{(i)} : \bigwedge^2 V^* \to \mathbb{C}$, there exists a direct decomposition $V^* = \oplus_{m=1}^k V_m^*$ such that $b^{(i)}(V_l^*, V_m^*) = 0$ for $i = 1, 2, l \neq m$, and the triples $(V_m^*, b_m^{(1)}, b_m^{(2)})$, where $b_m^{(i)} := b^{(i)}|_{V_m^*}$, are from the following list:*

1. [the Jordan block $\mathbf{j}_{2j_m}(\lambda)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of $V_m^*$ the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*

$$\begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & J_{j_m}(\lambda) \\ -J_{j_m}(\lambda)^T & \mathbf{0} \end{bmatrix}$$

*where $I_{j_m}$ is the unity $j_m \times j_m$-matrix and*

$$J_{j_m}(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

*is the Jordan $j_m \times j_m$-block with the eigenvalue $\lambda$;*

2. [the Jordan block $\mathbf{j}_{2j_m}(\infty)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of $V_m^*$ the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*

$$\begin{bmatrix} \mathbf{0} & J_{j_m}(0) \\ -J_{j_m}(0)^T & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m}^T & \mathbf{0} \end{bmatrix};$$

3. [the Kronecker block $\mathbf{k}_{2k_m+1}$]: $\dim V_m^* = 2k_m + 1$ *and in an appropriate basis of $V_m^*$ the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*

$$K_{1,k_m} := \begin{bmatrix} \mathbf{0} & B_{1,k_m} \\ -B_{1,k_m}^T & \mathbf{0} \end{bmatrix}, K_{2,k_m} := \begin{bmatrix} \mathbf{0} & B_{2,k_m} \\ -B_{2,k_m}^T & \mathbf{0} \end{bmatrix},$$

*where*

$$B_{1,k_m} := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, B_{2,k_m} := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

*($k_m \times (k_m + 1)$-matrices).*

**Kronecker Poisson pencils:** Let $\{\eta^t\}_{t\in\mathbb{R}^2}, \eta^t := t_1\eta_1 + t_2\eta_2$, be a Poisson pencil on $M$. We say that it is *Kronecker at a point* $x \in M$, if the Jordan–Kronecker decomposition of the pair of bivectors $\eta_1|_x, \eta_2|_x$ (regarded as elements of $\bigwedge^2 T_x^{\mathbb{C}} M$, here $T_x^{\mathbb{C}} M$ is the complexified tangent space) does not contain Jordan blocks.

PROPOSITION. $\{\eta^t\}_{t\in\mathbb{R}^2}$ *is Kronecker at $x$ if and only if*

$$\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x) = const, (t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}.$$

*Proof* It is easy to see that any nontrivial linear combination of matrices $K_{1,k_m}, K_{2,k_m}$ has constant rank equal to $2k_m$. So the rank can "jump" at some $t \neq 0$ if and only if there are Jordan blocks in the decomposition. □

We say that a Poisson pencil $\Theta$ on $M$ is *Kronecker* if there exists an open dense set $U \subset M$ such that $\Theta$ is Kronecker at any $x \in U$.

**Involutivity of Casimir functions for Kronecker Poisson pencils:** We have already proven that, if $t', t'' \in \mathbb{R}^2$ are linearly independent, then $\{f, g\}_{\eta^t} = 0$ for any $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M), t \in \mathbb{R}^2$. In the same way one can prove that $\eta^t|_x(\alpha, \beta) = 0$ for any $\alpha \in \ker \eta^{t'}|_x, \beta \in \ker \eta^{t''}|_x, t \in \mathbb{R}^2$.

PROPOSITION. *Let $\{\eta^t\}_{t\in\mathbb{R}^2}$ be Kronecker and let $t' \in \mathbb{R}^2, t' \neq 0$. Then $\{f,g\}_{\eta^t} = 0$ for any $f, g \in \mathcal{C}^{t'}(M), t \in \mathbb{R}^2$.*

*Proof* Fix $x \in U$. Let $t_{(n)} \in \mathbb{R}^2$ be such that $t_{(n)}$ is linearly independent with $t'$ and $t_{(n)} \overset{n\to\infty}{\longrightarrow} t'$. The kernel of the map $\eta^t|_x : T_x^*M \to T_xM$ continuously depend on $t \in \mathbb{R}^2 \setminus \{0\}$ and is of constant dimension. Consequently we can find a sequence of covectors $\alpha_n \in \ker \eta^{t_{(n)}}|_x$ such that $\alpha_n \overset{n\to\infty}{\longrightarrow} d_xg$. We get $\eta^t|_x(d_xf, \alpha_n) = 0$ and by continuity we conclude that $\eta^t|_x(d_xf, d_xg) = 0$. In other words, $\{f,g\}_{\eta^t}(x) = 0$ for any $x \in U$. Since $U$ is dense, using again the continuity argument we get the proof. □

Summarizing, we get the following result.

PROPOSITION. *Let $\Theta = \{\eta^t\}_{t\in\mathbb{R}^2}$ be a Kronecker Poisson pencil and let*

$$\mathcal{C}^\Theta(M) := \operatorname{Span}\{\bigcup_{t\in\mathbb{R}^2\setminus\{0\}} \mathcal{C}^t(M)\}.$$

*Then $\mathcal{C}^\Theta(M)$ is a family of functions in involution with respect to any Poisson bivector $\eta^t$.*

*Remark:* It can be shown that in the Kronecker case the family of functions in involution obtained by the Magri-Lenard scheme starting from Casimir functions coincide with the family $\mathcal{C}^\Theta(M)$.

**Completeness of Casimir functions for Kronecker Poisson pencils:** Let $(M, \eta)$ be a Poisson structure. We say that an open set $W \subset M$ is *correct* for $\eta$ if the set $W' := W \setminus (W \cap \operatorname{Sing}\eta)$ is nonempty and the common level sets of the functions from $\mathcal{C}^\eta(W')$ coincide with the symplectic foliation of $\eta$ on the set $W'$. In other words, the set $W$ is correct if the Poisson structure does not have regular symplectic leaves dense in $W$. Equivalent definition: $W$ is correct if $\{d_xf \mid f \in \mathcal{C}^\eta(W)\} = \ker \eta_x$ for any $x \in W'$. Note that in analytic category any sufficiently small open set is correct.

PROPOSITION. *Let $\Theta = \{\eta^t\}_{t\in\mathbb{R}^2}$ be a Kronecker Poisson pencil. Assume $W \subset M$ is an open set that is correct for $\eta^t$ for a countable set $\{t_{(1)}, t_{(2)}, \ldots\}$ of pairwise linearly independent values of the parameter $t$ and the set $W' := W \setminus \bigcup_{i=1}^\infty \operatorname{Sing}\eta^{t_{(i)}}$ is nonempty. Then the set of functions in involution $\mathcal{C}^\Theta(W')$ is* complete *with respect to any $\eta^t, t \neq 0$.*

*Proof* Fix $x \in U \cap W'$. Let us first prove that the set $C_x := \{d_xf \mid f \in \mathcal{C}^\Theta(W')\} \subset T_x^*M$ coincides with the set $L_x := \operatorname{Span}\{\bigcup_{t\in\mathbb{R}^2\setminus\{0\}} \ker \eta_x^t\}$. Indeed, the vector space $L_x$ is finite-dimensional, hence is generated by a finite number of kernels $\ker \eta_x^t = \{d_xf \mid f \in \mathcal{C}^t(W)\}$. Hence $L_x \subset C_x$. The same considerations show that $C_x \subset L_x$.

It is easy to see that the set $L_x$ is of dimension $(1/2)\operatorname{rank}\eta_x^t + \dim M - \operatorname{rank}\eta_x^t$. Assume for a moment that the Jordan–Kronecker decomposition of the pair $\eta_1|_x, \eta_2|_x$ consists of one Kronecker block $\mathbf{k}_{2k_m+1}$. The kernel of the matrix $\lambda K_{1,k_m} + K_{2,k_m}$ is 1-dimensional and is spanned by the vector $[0, \ldots, 0, 1, -\lambda, \ldots, (-\lambda)^{k_m}]$. Taking $k_m + 1$ different values of $\lambda$ we get $k_m + 1 = (1/2)\operatorname{rank}\eta_x^t + \dim M - \operatorname{rank}\eta_x^t$ linearly independent vectors (recall the Vandermonde determinant) spanning the set $L_x$. In the case of several Kronecker blocks you repeat these considerations for each block. □

*Remark:* In fact it is sufficient to require that $W$ is correct for a finite number of $\eta^t$. However, this number depends on the number and dimension of the Kronecker blocks, so we make a bit stronger assumption (which in practice is always satisfied).

**Example (method of the argument translation):** Let $M := \mathfrak{g}^*, \eta_1 := \eta_{\mathfrak{g}}, \eta_2 := \eta_{\mathfrak{g}}(a), S := \operatorname{Sing} \eta_{\mathfrak{g}}$, where $a \in \mathfrak{g}^* \setminus S$. Assume that $\operatorname{codim} S \geqslant 2$ (if $\mathfrak{g}$ is semisimple it is known that $\operatorname{codim} S \geqslant 3$). Note that $S$ is an algebraic set, i.e. it is defined by a finite number of algebraic equations $f_1(x) = 0, \ldots, f_m(x) = 0$ on $\mathfrak{g}^*$. Any algebraic set in a neighbourhood of its generic point is diffeomorphic to a manifold, hence its dimension is correctly defined.

If $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ and the corresponding structure constants are defined by $[e_i, e_j] = c_{ij}^k e_k$, the polynomials $f_1, \ldots, f_m$ are the $r \times r$-minors of the matrix $c_{ij}(x) = c_{ij}^k x_k$, where $r = \max_x \operatorname{rank} [c_{ij}(x)]$. Here $x_1 = e_1, \ldots, x_n = e_n$ are the corresponding coordinates on $\mathfrak{g}^*$.

In order to check the condition of Kroneckerity we need to consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of the initial Lie algebra. It can be regarded as a vector space $\operatorname{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\} \cong \mathbb{C}^n$ with the Lie bracket defined by the same structure constants. The set $S_{\mathbb{C}} := \{(z_1, \ldots, z_n) \in \mathfrak{g}_{\mathbb{C}}^* \cong \mathbb{C}^n \mid \operatorname{rank} c_{ij}^k z_k < \max_{z \in \mathbb{C}^n} \operatorname{rank} c_{ij}^k z_k\}$ is a complex algebraic set defined by the equations $f_1(z) = 0, \ldots, f_m(z) = 0$, where $f_1, \ldots, f_m$ are the same polynomials as above. In particular, the set $S_{\mathbb{C}}$ is of complex codimension at least 2.

We know that $t_1\eta_1|_x + t_2\eta_2|_x = c_{ij}^k(t_1 x_k + t_2 a_k), t_1, t_2 \in \mathbb{C}$. Thus $\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x)$ is maximal (over $t$) and independent of $t \in \mathbb{C}^2 \setminus \{0\}$ if and only if $t_1 x + t_2 a \in \mathfrak{g}_{\mathbb{C}}^* \setminus S$ if and only if $x \notin \overline{a, S_{\mathbb{C}}}$, where $\overline{a, S_{\mathbb{C}}} := \{z \in \mathfrak{g}_{\mathbb{C}}^* \mid \exists (t_1, t_2) \in \mathbb{C}^2 \setminus \{0\} \colon t_1 z + t_2 a \in S_{\mathbb{C}}\}$.

Note that the set $S_{\mathbb{C}}$ is homogeneous (stable under rescaling). Passing to the projectivization the set $\overline{a, S_{\mathbb{C}}}$ becomes a cone in $\mathbb{CP}^{n-1}$ over the projectivization of $S$. This shows that the set $\overline{a, S_{\mathbb{C}}}$ is also algebraic (by the standard arguments from algebraic geometry) and, moreover, $\dim_{\mathbb{C}} \overline{a, S_{\mathbb{C}}} = \dim_{\mathbb{C}} S_{\mathbb{C}} + 1$. In particular $\operatorname{codim}_{\mathbb{C}} \overline{a, S_{\mathbb{C}}} \geqslant 1$ and we can put $U := \mathfrak{g}^* \setminus (\mathfrak{g}^* \cap \overline{a, S_{\mathbb{C}}}) = \mathfrak{g}^* \setminus (\overline{a, S})$. Here $\overline{a, S} := \{x \in \mathfrak{g}^* \mid \exists (t_1, t_2) \in \mathbb{R}^2 \setminus \{0\} \colon t_1 x + t_2 a \in S\}$ and $\operatorname{codim}_{\mathbb{R}} \overline{a, S} \geqslant 1$. The set $U$ is an open dense set in $\mathfrak{g}^*$ such that $\{\eta^t\}$ is Kronecker at any $x \in U$.

Finally assume that $\mathfrak{g}$ is semisimple. Then $\eta_{\mathfrak{g}}$ has enough global Casimir functions and the whole space $\mathfrak{g}^*$ is a correct set for $\eta_{\mathfrak{g}}$. In particular, the assumptions of the proposition above are satisfied and we get a complete set $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ of functions in involution (with respect to any $\eta^t$). This set is generated by the "translations" $f(x + \lambda a), \lambda \in \mathbb{R}$, of the Casimir functions $f$ of $\eta_{\mathfrak{g}}$.

**III mechanism of constructing functions in involution (based on eigenvalue functions of a Poisson pencil):**

THEOREM. *Let $\{\eta^t\}$ be a Poisson pencil on $M$, $w_1(x), w_2(x)$ two eigenvalues of Jordan blocks. Then*

$$\{w_1, w_2\}_{\eta^t} = 0 \quad \forall\, t \in \mathbb{R}^2.$$

LEMMA. 1 *If $w(x)$ is an egenvalue of a Jordan block (in other words, $\operatorname{rank}(\eta_1(x) - w(x)\eta_2(x)) < \max_{v \neq w(x)} \operatorname{rank}(\eta_1(x) - v\eta_2(x))$), then $d_x w \in \ker(\eta_1 - v_0\eta_2)(x)$ for any $x \in M_{v_0} := \{x \mid w(x) = v_0\}$.*

*In particular, $\{w, f\}^{(1, -v_0)}|_{M_{v_0}} = 0$ for any function $f$.*

*Proof* Consider a Poisson structure $\eta_1 - v_0\eta_2$ and a point $x \in M_{v_0}$. A symplectic leaf $S, \dim S < \dim M$ passes through $x$. The function $w$ is constant on $S$. Indeed, if $y \in S$ is close to $x$, then rank $(\eta_1(y) - v_0\eta_2(y)) < \max_{v\neq v_0} \operatorname{rank}(\eta_1(y) - v\eta_2(y))$, i.e. $v_0$ must be an eigenvalue of "the same" jordan block at a point $y$, hence $w(y) = v_0$.

*Proof of the theorem* Let $w_1(x), w_2(x)$ be functionally independent. Then there exists a local coordinate system on $M$ of the form $w_1, w_2, x_3, \dots, x_m$.

let $v_1 \neq v_2$. Then there exist $\alpha(v_1, v_2), \beta(v_1, v_2) \in \mathbb{R}$ such that $\eta^\lambda := \lambda_1\eta_1 + \lambda_2\eta_2 = \alpha(v_1, v_2)(\eta_1 - v_1\eta_2) + \beta(v_1, v_2)(\eta_1 - v_2\eta_2)$. Thus

$$\{w_1, w_2\}^\lambda|_{(v_1,v_2,x_3,\dots,x_m)} = ((\lambda_1\eta_1 + \lambda_2\eta_2)(dw_1)w_2)|_{(v_1,v_2,x_3,\dots,x_m)} =$$

$$\alpha(v_1, v_2)\{w_1, w_2\}^{(1,-v_1)}|_{(v_1,v_2,x_3,\dots,x_m)} + \beta(v_1, v_2)\{w_1, w_2\}^{(1,-v_2)}|_{(v_1,v_2,x_3,\dots,x_m)}$$
$$= 0 - \beta(v_1, v_2)\{w_2, w_1\}^{(1,-v_2)}|_{(v_1,v_2,x_3,\dots,x_m)} = 0.$$

By continuity we also have 0 for $v_1 = v_2$.

# References


[Arn73] V. I. Arnold, *Ordinary differential equations*, The MIT, 1973.

[Arn89] ______, *Mathematical methos of classical mechanics*, Springer, 1989.

[Bol91] Alexey Bolsinov, *Compatible Poisson brackets on Lie algebras and completeness of families of functions in involution*, Izv.Akad. Nauk SSSR, Ser. Mat. **55** (1991), In Russian.

[dSW99] Ana Canas da Silva and Alan Weinstein, *Geometric models for noncommutative algebras*, Providence, 1999, available on http://math.berkeley.edu/ alanw/Models.pdf.

[GZ89] Israel Gelfand and Ilya Zakharevich, *Spectral theory for a pair of skew-symmetrical operators on $S^1$*, Funct. Anal. Appl. **23** (1989), no. 2, 85–93, Translation from the Russian.

[Mag78] Franco Magri, *A simple model of the integrable hamiltonian equation*, J. Math. Phys. **19** (1978), 1156–1162.

[Nag66] Tadashi Nagano, *Linear differential systems with singularities and an application to transitive lie algebras*, J. Math. Soc. Japan **18** (1966), 398–404.