Singularities of bi-Hamiltonian Systems
Lecture 4: How does it work?
Examples and applications

Alexey Bolsinov
Loughborough University

29 August – 2 September, 2011

Three examples

- Rubanovskii case in rigid body dynamics
- Mischenko-Fomenko systems on semisimple (compact) Lie algebras
- Euler-Manakov tops on $so(n)$

Rubanovskii case

Generalisation of the Steklov-Lyapunov case (rigid body in fluid)

Lax pair discovered by Yu. Fedorov:

$$\frac{dL(s)}{dt}L(s) = [L(s), A(s)]$$

where L and A are 3×3 skew-symmetric matrices:

$$L_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\left(\sqrt{\lambda - b_j}\,(z_\gamma + sp_\gamma) + g_\gamma/\sqrt{\lambda - b_j}\right)$$

$$A(s) = \varepsilon_{\alpha\beta\gamma}\frac{1}{s}\sqrt{(s - b_\alpha)(s - b_\beta)}(b_\gamma z_\gamma - g_\gamma)$$

Here $(z_1, z_2, z_3,\ p_1, p_2, p_3)$ are coordinates in the phase space $\mathbb{R}^6$.

The first integrals:

$$J = p_1^2 + p_2^2 + p_3^2,$$

$$F_\lambda(z, p) = \sum_{i=1}^{3}(\lambda - b_i)\left(z_i + \lambda p_i + \frac{g_i}{\lambda - b_i}\right)^2$$

Proposition

The Rubanovskii system is Hamiltonian w.r.t. the pencil generated by the following compatible Poisson brackets:

$$\Pi_0 = \begin{pmatrix} 0 & z_3 - b_3 p_3 & -z_2 + b_2 p_2 & 0 & p_3 & -p_2 \\ -z_3 + b_3 p_3 & 0 & z_1 - b_1 p_1 & -p_3 & 0 & p_1 \\ z_2 - b_2 p_2 & -z_1 + b_1 p_1 & 0 & p_2 & -p_1 & 0 \\ 0 & p_3 & -p_2 & 0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & 0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Pi_1 = \begin{pmatrix} 0 & b_3 z_3 - g_3 & -b_2 z_2 + g_2 & 0 & 0 & 0 \\ -b_3 z_3 + g_3 & 0 & b_1 z_1 - g_1 & 0 & 0 & 0 \\ b_2 z_2 - g_2 & -b_1 z_1 + g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & 0 & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & 0 & p_2 & -p_1 & 0 \end{pmatrix}$$

z, p are coordinates in the phase space $\mathbb{R}^6$, b and g are geometric parameters.

Rubanovskii case

The algebraic structure of $\Pi_1 - \lambda\Pi_0$ becomes clear if we change variables:

$$\tilde{z}_i = z_i + \lambda p_i + \frac{g_i}{\lambda - b_i}, \qquad p_i\text{'s remain the same}$$

Then:

$$\Pi_1 - \lambda\Pi_0 = \begin{pmatrix} \begin{matrix} 0 & (b_3-\lambda)\tilde{z}_3 & -(b_2-\lambda)\tilde{z}_2 \\ -(b_3-\lambda)\tilde{z}_3 & 0 & (b_1-\lambda)\tilde{z}_1 \\ (b_2-\lambda)\tilde{z}_2 & -(b_1-\lambda)\tilde{z}_1 & 0 \end{matrix} & \\ & \begin{matrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{matrix} \end{pmatrix}$$

Thus, $\Pi_1 - \lambda\Pi_0$ splits into the direct sum of two brackets, one of which is the standard $so(3)$-bracket and the other is isomorphic to either to $so(3)$, or to $sl(2)$ depending on the signs of $b_i - \lambda$, $i = 1, 2, 3$.

Question: What are the critical points for the integrals?

Answer: Those points where the rank of $\Pi_1 - \lambda\Pi_0$ drops.

Theorem (Basak, AB)

A point (z, p) is critical iff there is $\lambda \in \mathbb{C} \setminus \{b_1, b_2, b_3\}$ such that

$$z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0, \qquad i = 1, 2, 3.$$

Theorem (Basak, AB)

A point (z, p) is a common equilibrium iff $\mathrm{rank}\begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} = 1.$

Equivalently, this means that there are two different λ_1, λ_2 satisfying

$$z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0, \qquad i = 1, 2, 3.$$

How to apply the general non-degeneracy criterion?

Proposition

Consider a linear-constant Poisson pencil in $\mathbb{R}^3$ generated by

$$A = \begin{pmatrix} 0 & a_3x_3 & -a_2x_2 \\ -a_3x_3 & 0 & a_1x_1 \\ a_2x_2 & -a_1x_1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}$$

This pencil is non-degenerate iff

$$C = a_1a_2a_3 \sum_i \frac{b_i^2}{a_i} \neq 0.$$

Moreover, the singularity is elliptic if $C > 0$, and hyperbolic if $C < 0$.

Rubanovskii case

Theorem (Basak, AB)

Let γ be a critical closed trajectory passing through (z, p) with parameter λ. Then γ is non-degenerate iff

$$C = C(\lambda) = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3) \sum_{i=1}^{3} \left((\lambda - b_i) p_i - \frac{g_i}{\lambda - b_i} \right)^2 \frac{1}{\lambda - b_i} \neq 0$$

Moreover, if this expression $C > 0$ then γ is stable, and if $C < 0$ then γ is unstable.

Theorem (Basak, AB)

Let (z, p) be an equilibrium points with parameters λ_1, λ_2. Then γ is non-degenerate iff

$$C_k = C(\lambda_k) \neq 0$$

Moreover, for real λ_1, λ_2:

- *(z, p) is of ell-ell type if both $C_k > 0$;*
- *(z, p) is of hyp-hyp type if both $C_k < 0$;*
- *(z, p) is of ell-hyp type if both C_k have different signs.*

If λ_1 and λ_2 are complex conjugate then (z, p) is a focus.

Let $\mathfrak{g}$ be a finite-dimensional (real) Lie algebra and g^* its dual space endowed with the two LiePoisson bracket:

$$\{f,g\}(x) = x([df(x), dg(x)]) \quad \text{and} \quad \{f,g\}_a(x) = a([df(x), dg(x)]),$$

where $f, g : g^* \to \mathbb{R}$ are arbitrary smooth functions, $x, a \in \mathfrak{g}^*$ and a is fixed. The pencil $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ leads to the family $\mathcal{F}_a = \{f(x+\lambda a) \mid f \in I_{\mathrm{Ad}^*}(\mathfrak{g}), \lambda \in \mathbb{R}\}$ of commuting Casimirs.

Our goal: Properties of $\mathcal{F}_a$ for semisimple (and even compact) Lie algebras. In this case $g \simeq g^*$ and the family $\mathcal{F}_a$ possesses a natural basis consisting of $s = 1/2(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ homogeneous polynomials $f_1, \dots, f_s$. In other words, $\mathcal{F}_a$ is freely generated by them.

We want to study the properties of the corresponding momentum mapping $\Phi_a : \mathfrak{g} \to \mathbb{R}^s$, $\Phi(x) = (f_1(x), \dots, f_s(x))$.

MF systems are those systems on $\mathfrak{g}$ with quadratic Hamiltonians H which are bi-Hamiltonian w.r.t. this pencil or, equivalently, $H \in \mathcal{F}_a$.

Theorem (Mischenko, Fomenko)

If g is semisimple and $a \in \mathfrak{g}^*$ is regular, then the collection of commuting polynomials $\mathcal{F}_a$ is complete on $\mathfrak{g} \simeq \mathfrak{g}^*$. In other words, the basic shifts $f_1, \dots, f_s$ are functionally independent on $\mathfrak{g}$.

Proof: Codimension two principle + codim $\mathrm{Sing} = 3$ in the semisimple case

Theorem

An element $x \in g$ is a critical point of the momentum mapping Φ_a) if and only if there exists $\lambda \in \mathbb{C}$ such that $x + \lambda a$ is a singular element in $\mathfrak{g}^{\mathbb{C}}$. In other words, the set of critical points S_a of Φ_a is the (real part of the) cylinder over the set of singular elements Sing with the generating line parallel to a, that is: $S_a = (Sing + \mathbb{C} \cdot a) \cap \mathfrak{g}$.

Recall that $x \in \mathfrak{g}$ is said to be a common equilibrium point for $\mathcal{F}_a$ if for any $f \in \mathcal{F}_a$ we have $X_f(x) = [df(x), x] = 0$.

Theorem

A point $x \in \mathfrak{g}$ is a common equilibrium point for $\mathcal{F}_a$ if and only if $x \in \mathfrak{h}_a$, where $\mathfrak{h}_a$ is the Cartan subalgebra generated by $a \in \mathfrak{g}$. The number of equilibrium points on each regular orbit is the order of the Weyl group.

Let x be a common equilibrium point. We know that $x \in \mathfrak{h}_a$. Is x non-degenerate?

Theorem (Oshemkov, AB)

Let $\alpha_1, \dots, \alpha_s$ be the positive roots associated with the complexification $\mathfrak{h}_a^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, $s = 1/2(\dim g + \operatorname{ind} g)$. Consider the collection of numbers

$$\lambda_i = \frac{\alpha_i(x)}{\alpha_i(a)}$$

If all these numbers are distinct, then $x \in \mathfrak{g}$ is a non-degenerate equilibrium point. Moreover, if $\mathfrak{g}$ is compact, then x is of pure elliptic type and, therefore, is stable.

What about codimension 1 singularities?

Theorem (Oshemkov, AB)

Let $x \in \mathfrak{g}$ be a critical point of corank 1, and $\lambda \in \mathbb{R}$ the unique value of the parameter such that $x + \lambda a$ is a singular element of $\mathfrak{g}$. Assume that $x + \lambda a$ is semisimple and $\mathfrak{u}$ is the semisimple part of the centralizer of $x + \lambda a$. Consider the natural orthogonal projection $b = \mathrm{pr}_{\mathfrak{u}}\, a$ of a onto $\mathfrak{u}$. If b is semisimple and non-zero, then x is non-degenerate.
Moreover, if $(b, b) > 0$, then the singularity is hyperbolic, and if $(b, b) < 0$, then the singularity is elliptic, where $(\ ,\ )$ is the Killing form on $\mathfrak{u}$.

In particular, in the case of a compact Lie algebra g, all corank 1 singularities are non-degenerate and of elliptic type. In this case, there are no hyperbolic singularities. It follow from this that the set of regular values of Φ_a in $\mathbb{R}^s$ is connected and each non-trivial regular level $\Phi_a^{-1}(y)$, $y \in \mathbb{R}^s$, consists of one Liouville torus.

Euler-Manakov tops on $so(n)$

Euler-Manakov top: $\frac{d}{dt}X = [R(X), X]$, where $R(X)_{ij} = \frac{b_i - b_j}{a_i - a_j} X_{ij}$.
Bi-Hamiltonian structure for the E-M top:
Along with the standard commutator $[X, Y] = XY - YX$ on the space of skew-symmetric matrices, we introduce a new operation

$$[X, Y]_A = XAY - YAX$$

where A is the diagonal matrix $\mathrm{diag}(a_1, a_1, \dots, a_n)$.
Observation: E-M top is Hamiltonian w.r.t to the corresponding pencil of compatible Poisson brackets $\{\ ,\ \}_{A+\lambda E}$ on $so(n) = so(n)^*$ and, therefore, it admits a large family of commuting integrals of the form

$$\mathrm{Tr}\left(X(A+\lambda E)^{-1}\right)^k$$

which is equivalent to the standard Manakov integrals:

$$\mathcal{F}_A = \left\{\mathrm{Tr}(X + \lambda A)^k\right\}.$$

This family admits a basis that consists of exactly $s = \frac{1}{2}\left(\dim so(n) + \mathrm{ind}\, so(n)\right)$ commuting polynomials.

Euler-Manakov tops on $so(n)$

Theorem (Mischenko, Fomenko)

If the eigenvalues of A are all distinct, then the family of Manakov's integrals $\mathcal{F}_A$ is complete on $so(n)$.

Proof. Almost all brackets in the family are semisimple, so the singular set of each of them has codemention 3.

Theorem (Oshemkov, AB)

$X \in so(n)$ is a critical point of $\mathcal{F}_A$ if and only if there exists $\lambda \in \overline{\mathbb{C}}$ such that X is singular for the bracket $\{\ ,\ \}_{A+\lambda E}$. Equivalently,

$$S_A = \left(\cup_{\lambda \in \bar{\mathbb{C}}} (A+\lambda E)^{1/2} \, \mathrm{Sing} \, (A+\lambda E)^{1/2} \right) \cap so(n, \mathbb{R}).$$

where $\mathrm{Sing} \subset so(n, \mathbb{C})$ is the set of singular points.

Euler-Manakov tops on $so(n)$

Theorem (L.Féher, I.Marshall)

The set of common equilibrium points of $\mathcal{F}_A$ (with A diagonal) is the union of those Cartan subalgebras $\mathfrak{h} \subset so(n)$ which are common Cartan subalgebras for all commutators $[\ ,\ ]_{A+\lambda E}$. One of these Cartan subalgebras is standard:

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} 0 & x_{12} & & & \\ -x_{12} & 0 & & & \\ & & 0 & x_{34} & \\ & & -x_{34} & 0 & \\ & & & & \ddots \end{pmatrix}, \quad x_{i,i+1} \in \mathbb{R} \right\}.$$

All the others are obtained from $\mathfrak{h}_0$ by conjugation $\mathfrak{h}_0 \mapsto P\mathfrak{h}_0 P^{-1}$ where P is a permutation matrix.

Theorem (Oshemkov, AB)

Let X be a 2×2 block-diagonal skew-symmetric matrix (as above). For each pair $x_{i,i+1}$, $x_{j,j+1}$, consider the two roots $\lambda_{ij}, \lambda'_{ij}$ of the equation

$$\frac{x_{i,i+1}^2}{x_{j,j+1}^2} = \frac{(a_i+\lambda)(a_{i+1}+\lambda)}{(a_j+\lambda)(a_{j+1}+\lambda)}.$$

If $\lambda_{ij}, \lambda'_{ij}$ $(i \neq j,\ \ i,j = 1,3,\dots,2n-1)$ are all distinct, then X is a non-degenerate equilibrium point for $\mathcal{F}_A$.

Euler-Manakov tops on $so(n)$

For each pair of blocks $\begin{pmatrix} 0 & x_{i,i+1} \\ -x_{i,i+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, $\begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ consider the function $f(x) = \frac{(x-\lambda_1^2)(x-\lambda_2^2)}{\omega^2(\lambda_1+\lambda_2)^2}$. and let $f(\infty) = \frac{1}{\omega^2(\lambda_1+\lambda_2)^2}$.

By drawing the graphs of all these functions on the same plane $\mathbb{R}^2$, we obtain a collection of parabolas called the *parabolic diagram* $\mathcal{P}$. For simplicity we assume that n is even.

We say that this diagram is generic if any two parabolas intersect exactly at two points (including complex intersections and intersections at infinity)

Theorem (A. Izosimov)

- *The equilibrium point is non-degenerate iff the parabolic diagram $\mathcal{P}$ is generic:*
  - *each intersection point in the upper half plane corresponds to an elliptic component;*
  - *each intersection point in the lower half plane corresponds to a hyperbolic component;*
  - *each complex intersection corresponds to a focus component.*
- *If $\mathcal{P}$ is generic, all intersections are real and located in the upper half plane, then the equilibrium is stable.*
- *If there is either a complex intersection or an intersection point in the lower half plane, then the equilibrium point is unstable.*