Singularities of bi-Hamiltonian Systems
Lecture 3: Linearisation of Poisson pencils and a criterion of non-degeneracy

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Lecture 3: main ingredients

- Linearisation of a Poisson structure at a singular point
- Linearisation of a Poisson pencil
- Non-degenerate linear-constant Poisson pencils
- Classification of non-degenerate linear-constant Poisson pencils (A. Izosimov)
- General non-degeneracy criterion (A. Izosimov)

Linearisation of a Poisson structure

According to the splitting theorem (A.Weinstein), locally each Poisson structure A splits into direct product of a non-degenerate Poisson structure A_{sympl} and the transversal structure A_{transv} which vanishes at the given point:

$$A = A_{\mathrm{sympl}} \times A_{\mathrm{transv}}$$

The transversal Poisson structure A_{transv} is well defined and we can consider its linearisation just by taking the linear terms in the Taylor expansion

$$A_{\mathrm{transv}}(x) = \sum c_{ij}^{k} x_k + \dots$$

Definition

From the algebraic viewpoint, the linearisation of A at a point $x \in M$ is a Lie algebra $\mathfrak{g}_A$ defined on $\mathrm{Ker}\, A(x)$ as follows. Let $\xi, \eta \in \mathrm{Ker}\, A(x)$ and f, g be smooth functions such that $df(x) = \xi$, $dg(x) = \eta$. Then, by definition,

$$[\xi, \eta] = d\{f, g\}(x) \in \mathrm{Ker}\, A(x)$$

Remark. If $x \in M$ is a regular point, then $\mathfrak{g}_A$ is obviously trivial.

Linearisation of a Poisson pencil

$\mathcal{J} = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$. Let us take $x \in M$, fix $\lambda \in \mathbb{C}$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$ we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, the lineraisation of A_λ at the point x,
- the restriction of B onto $\operatorname{Ker} A_\lambda$.

We can think of them as two Poisson structures on $\mathfrak{g}_\lambda^*$:

- the first on is linear, i.e., the standard Lie-Poisson structure related to $\mathfrak{g}_\lambda$,
- the second one is constant $B|_{\mathfrak{g}_\lambda}$.

Proposition

These two Poisson structures are compatible, i.e. generate, a Poisson pencil $\Pi = \Pi(\lambda, x)$.

Definition

This Poisson pencil Π is called the λ-linearisation of the pencil $\mathcal{J}$ at $x \in M$.

How to find the λ-linearisation in practice?

For simplicity $\lambda = 0$.
Choose a coordinate system $x_1, \dots, x_k, x_{k+1}, \dots x_n$ such that

$$A(x) = \begin{pmatrix} A_1(x) & A_2(x) \\ -A_2^\top(x) & A_3(x) \end{pmatrix}, \qquad B(x) = \begin{pmatrix} B_1(x) & B_2(x) \\ -B_2^\top(x) & B_3(x) \end{pmatrix}$$

where $A_1(0) = 0$, $A_2(0) = 0$ and $A_3(0)$ is non-degenerate (in other words, the first coordinates $x_1, \dots, x_k$ "generates" the kernel of A at $x = 0$).

Then the linear terms of $A_1(x)$ do not depend on $x_{k+1}, \dots x_n$ and form a linear Poisson bracket. The constant bracket is simply $B_1(0)$.

Thus, the linearisation of this pencil at $x = 0$ is defined by the linear part of A_1 and the constant part of B_1.

In practice (see next lecture), such a coordinate system often can be found explicitly.

Example: Argument shift method

Consider the standard "shift argument" pencil $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ corresponding to a Lie algebra $\mathfrak{g}$ and $a \in \mathfrak{g}^*$.

Let $\operatorname{codim} S \geq 2$ so that the argument shift method gives a complete family $\mathcal{F}_a$ of commuting polynomials.

Let $x \in \mathfrak{g}^*$ be a singular point for $\mathcal{F}_a$. This means that $x + \lambda a$ is a singular element of $\mathfrak{g}^*$.

Question. What is the λ-linearization $\Pi = \Pi(\lambda, x)$ of this pencil at this point?

Answer is very natural:
$\mathfrak{g}_\lambda$ is the ad*-stationary subalgebra of $x + \lambda a \in \mathfrak{g}^*$:

$$\mathfrak{g}_\lambda = \operatorname{ann}(x + \lambda a) = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi(x + \lambda a) = 0\}$$

and the constant bracket on $\mathfrak{g}^*_\lambda$ is $\{\ ,\ \}_{\operatorname{pr}(a)}$ where $\operatorname{pr}(a)$ is the natural projection of a from $\mathfrak{g}^*$ to $\operatorname{ann}(x + \lambda a)^*$ induced by the inclusion $\operatorname{ann}(x + \lambda a) \subset \mathfrak{g}$.

Linear-constant pencils and generalised "shift of argument"

Consider two compatible Poisson brackets on a vector space V:

linear A + constant B.

What are "compatibility conditions" for this kind of brackets?

Standard situation is "shift of argument" construction:

The brackets $\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$, $\{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ are compatible for each $a = (a_i) \in V$.

Situation can be different:

For $\{f,g\}_A(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ there may exist constant compatible brackets

$$\{f,g\}_B(x) = \sum B_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

which are not of the above type. The compatibility condition can be written as

$$B([\xi,\eta],\zeta) + B([\eta,\zeta],\xi) + B([\zeta,\xi],\eta) = 0.$$

This identity has a natural cohomological interpretation.

Remark 1. If the corresponding Lie algebra is semisimple, then the constant bracket must have the above form $\{\ ,\ \}_a$ for some $a \in V$.

Remark 2. Ker B is a subalgebra of $\mathfrak{g}$.

Non-degenerate linear-constant pencils

Consider two compatible Poisson brackets on a vector space V:

linear A + constant B

and the corresponding linear-Poisson pencil $\Pi = \{A + \lambda B\}$.

For this pencil $\Pi = \{A + \lambda B\}$ we can construct the family of commuting Casimirs $\mathcal{F}_\Pi$ and ask the question about the structures of singular points. We will say that Π is complete, if $\mathcal{F}_\Pi$ is complete.

It is easy to see that $0 \in V$ is a singular point of $\mathcal{F}_\Pi$ and, moreover, it is a common equilibrium.

Definition

We say that a complete linear-constant pencil $\Pi = \{A + \lambda B\}$ is *non-degenerate*, if $0 \in V$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$.

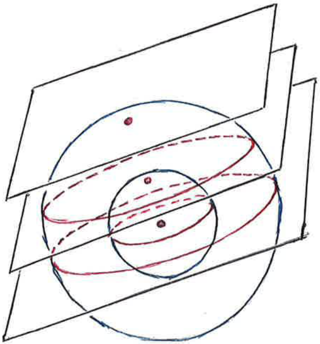
Examples: semisimple case $so(3)$

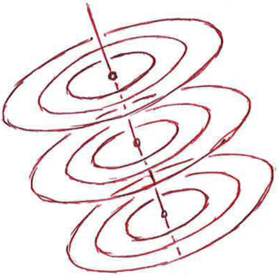
Example

If $A \simeq so(3)$ and B is arbitrary, then $\Pi = \{A + \lambda B\}$ is non-degenerate.

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + y^2 + z^2, \quad F_2 = ax + by + cz$





Examples: semisimple case $sl(2,\mathbb{R})$

Example

$sl(2,\mathbb{R})$–bracket A and constant bracket B defined by an element $\xi \in sl(2,\mathbb{R}) \simeq sl(2,\mathbb{R})^*$:

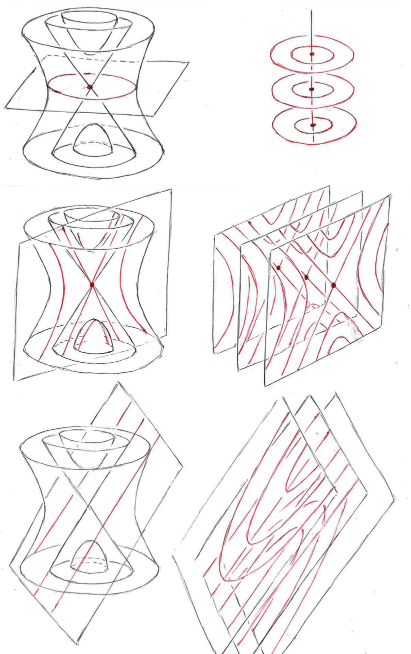
$$A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $\quad F_1 = x^2 + yz, \quad F_2 = ax + by + cz$

Is this pencil non-degenerate?

The answer depends on ξ: see next slide

Examples: semisimple case $sl(2, \mathbb{R})$



Examples: semisimple case $sl(2, \mathbb{R})$

Question.
Why are there 3 different cases? How to distinguish them?

Answer.
There are non-trivial elements $\xi \in sl(2, \mathbb{R})$ of three types:

- elliptic (eigenvalues are pure imaginary $i\lambda, -i\lambda$) ;
- hyperbolic (eigenvalues are real $\lambda, -\lambda$);
- nilpotent (both eigenvalues are zero).

We can distinguish them by using the Killing form:

- elliptic: $(\xi, \xi) < 0$;
- hyperbolic: $(\xi, \xi) > 0$;
- nilpotent: $(\xi, \xi) = 0$.

Equivalently, one may use the sign of $\operatorname{Tr} \xi^2 = -2 \det \xi$ in the standard 2×2 representation.

Conclusion.
Non-degeneracy $\Leftrightarrow$ ξ is semisimple $\Leftrightarrow$ $\operatorname{Ker} B_\xi$ is a Cartan subalgebra

Examples: non-semisimple case

Example

Consider the Lie algebra $e(2) = so(2) +_\phi \mathbb{R}^2 = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{pmatrix} \right\}$

The corresponding Lie-Poisson bracket: $A = \begin{pmatrix} 0 & x & -y \\ -x & y & 0 \\ y & 0 & 0 \end{pmatrix}$

where x, y, z are dual coordinates to α, β, θ.

Constant bracket: $B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$

Linear-constant pencil $\Pi = \{A + \lambda B\}$

Casimir functions on $e(2)^*$: $F_A = x^2 + y^2, \quad F_B = ax + by + cz.$

These functions give a non-degenerate singularity iff $z \neq 0$.

The algebraic reformulation:

Non-degeneracy $\Leftrightarrow$ Ker B is a Cartan subalgebra of $e(2)$.

Problem.
Describe all "good" Lie algebras $\mathfrak{g}$ (equivalently, Lie-Poisson brackets A) which may produce non-degenerate linear-constant pencils and then for these Lie algebras find necessary and sufficient condition for a constant bracket B to give indeed a non-degenerate pencil $\Pi = \Pi(\mathfrak{g}, B) = \{A + \lambda B\}$.

Such Lie algebras are called non-degenerate too.

Theorem (A. Izosimov)

A linear-constant pencil $\Pi = \Pi(g, B)$ is non-degenerate (in the complex case) if and only if the Lie algebra $\mathfrak{g}$ associated with the linear bracket A is isomorphic to

$$\left(\bigoplus so(3,\mathbb{C})\right) \oplus \left(\left(\bigoplus \mathfrak{D}\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{C}\right)$$

where $\mathfrak{D}$ is the diamond Lie algebra, $\mathfrak{h}_0$ is a commutative ideal which belongs to the center of $(\bigoplus \mathfrak{D})$, and $\operatorname{Ker} B$ is a Cartan subalgebra of $\mathfrak{g}$.

What is the diamond Lie algebra $\mathfrak{D}$?

$\mathfrak{D}$ is a four dimensional Lie algebra generated by x, y, z, u with the following relations

$$[z, x] = y, \quad [z, y] = -x \quad \text{and} \quad [x, y] = u, \quad [u, \mathfrak{D}] = 0. \tag{1}$$

In other words, $\mathfrak{D}$ (as a complex Lie algebra) is the non-trivial central extension of $e(2, \mathbb{C})$.
Matrix representation:

$$\alpha x + \beta y + \theta z + \gamma u \quad \mapsto \quad \begin{pmatrix} 0 & \alpha & \beta & 2\gamma \\ 0 & 0 & -\theta & \beta \\ 0 & \theta & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Casimir functions: $\qquad F_1 = x^2 + y^2 + 2zu, \quad F_2 = u.$

The complex diamond Lie algebra $\mathfrak{D}$ has 2 different real forms

- $\mathfrak{g}_{ell}$ defined by (1) and
- $\mathfrak{g}_{hyp}$ defined by $[z, x] = x$, $[z, y] = -y$, and $[x, y] = u$.

Theorem (A. Izosimov)

A real Lie algebra $\mathfrak{g}$ is non-degenerate iff

$$\mathfrak{g} \simeq \left(\bigoplus so(3,\mathbb{R})\right) \oplus \left(\bigoplus sl(2,\mathbb{R})\right) \oplus \left(\bigoplus so(3,\mathbb{C})\right) \oplus \left(\left(\left(\bigoplus \mathfrak{g}_{ell}\right) \oplus \left(\bigoplus \mathfrak{g}_{hyper}\right) \oplus \left(\bigoplus \mathfrak{g}_{foc}\right)\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{R}\right)$$

where

- $\mathfrak{g}_{ell}$ *and* $\mathfrak{g}_{hyp}$ *are the non-trivial central extensions of* $e(2)$ *and* $e(1,1)$ *(equivalently, they are real forms of* $\mathfrak{D}$*),*
- $\mathfrak{g}_{foc} = \mathfrak{D}$ *treated as real Lie algebra,*
- $\mathfrak{h}_0$ *is a commutative ideal which belongs to the center.*

A linear-constant pencil $\Pi(\mathfrak{g}, B)$ *is non-degenerate if* $\mathfrak{g}$ *is non-degenerate and* Ker B *is a Cartan subalgebra of* $\mathfrak{g}$*.*
The type of the singularity is naturally defined by the "number" of elliptic, hyperbolic and focus components in the above decomposition.

Now let $J = \{A + \lambda B\}$ be an arbitrary pencil of compatible Poisson brackets. We consider the commutative family of functions $\mathcal{F}_J$ and a singular point $x \in S_J$.

This means, that at this point there are non-trivial characteristic numbers λ_i for the pencil $J(x) = \{A(x) + \lambda B(x)\}$.

For each of them we can consider the λ_i-linearisation.

Is x non-degenerate?

Theorem (A. Izosimov)

Let $J = \{A + \lambda B\}$ be a pencil of compatible Poisson brackets, $\mathcal{F}_J$ be the corresponding family of commuting Casimirs and $x \in M$ singular point for $\mathcal{F}_J$. This point is non-degenerate if and only if for every characteristic number λ_i,

1. *the λ_i-linearisation of J at x is non-degenerate;*
2. *the pencil $J(x) = \{A + \lambda B\}$ is diagonalisable (i.e. all the Jordan blocks are 2×2).*