Singularities of bi-Hamiltonian Systems

Lecture 2: Compatible Poisson structures and integrability. Singularities of Poisson structures

Alexey Bolsinov
Loughborough University

29 August – 2 September, 2011

Lecture 2. Main ingredients

- Singular set of a Poisson bracket
- Jordan-Kronecker decomposition theorem
- From Linear Algebra to bi-Poisson Geometry
- Compatible Poisson structures and the family $\mathcal{F}$ of commuting Casimirs
- Completeness criterion and codimension two principle
- From singularities of Poisson brackets to singularities of Lagrangian fibrations

Some basic notions and notation

Poisson manifold (M, A), Poisson structure $A = (A^{ij})$ and Poisson bracket $\{f, g\}_A = A^{ij}(x) \dfrac{\partial f}{\partial x^i} \dfrac{\partial g}{\partial x^j}$.

We set rank $A = \max_{x \in M} \operatorname{rank} A(x)$.

If rank $A <$ dim M then, as a rule, there exist Casimir functions $f \in C^\infty(M)$ such that

$$\{f, g\}_A = 0 \quad \text{for any } g \in C^\infty(M)$$

Property: f is Casimir $\Leftrightarrow$ $df(x) \in \operatorname{Ker} A(x)$ for all $x \in M$; moreover, for regular $x \in M$, the differentials $df(x)$ of (local) Casimirs generates Ker $A(x)$.

M is foliated into symplectic leaves and the Casimir functions can be characterized by the property of being constant on each symplectic leaf.

To each A we can assign its singular set

$$S_A = \{x \in M \mid \operatorname{rank} A(x) < \operatorname{rank} A\}$$

(equivalently, S_A is the union of all symplectic leaves of non-maximal dimension).

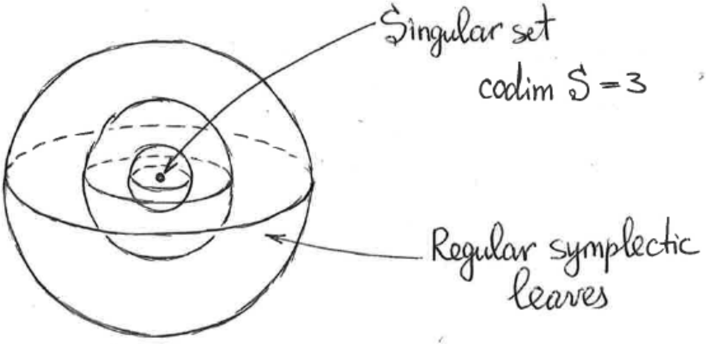
Example 1

$so(3)$–bracket: $$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

Casimir function: $F = x^2 + y^2 + z^2$

Symplectic leaves are spheres centered at the origin + one singular leaf $\{0\}$

Singular set is $S_A = \{\mathrm{rank}\, A < 2\} = \{0\}$, $\quad \mathrm{codim}\, S_A = 3$



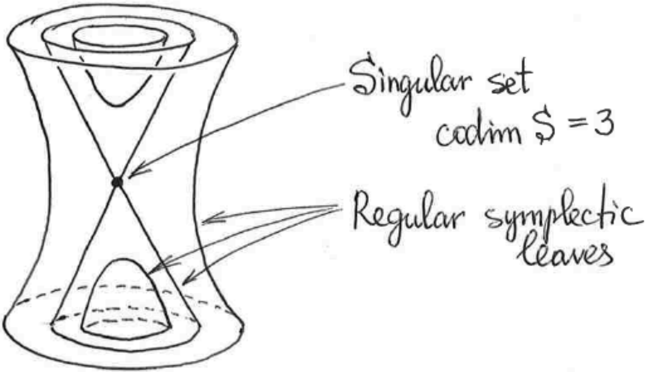
Example 2

$sl(2,\mathbb{R})$–bracket: $A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix}$

Casimir function: $F = x^2 + yz$

Symplectic leaves: hyperboloids, two halves of the cone + one singular leaf $\{0\}$

Singular set is $S_A = \{\text{rank}\, A < 2\} = \{0\}$, $\text{codim}\, S_A = 3$



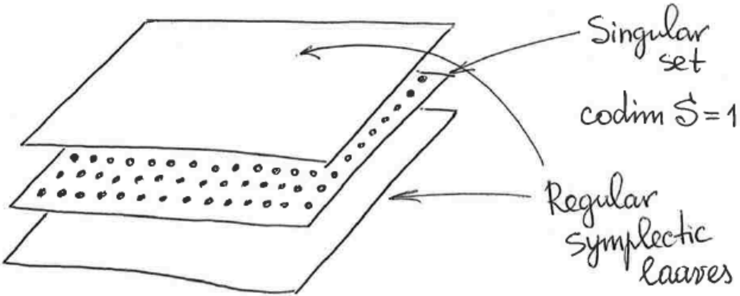
Example 3

Heisenberg–Lie bracket: $A = \begin{pmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Casimir function: $F = z$

Symplectic leaves: planes $\{z = \text{const} \neq 0\}$ $+$ points on $\{z = 0\}$

Singular set is $S_A = \{\text{rank}\, A < 2\} = \{z = 0\}$, $\quad \text{codim}\, S_A = 1$



Theorem

Let A and B be two skew-symmetric bilinear forms. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \qquad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where the pairs of the corresponding blocks A_i and B_i can be of the following three types (see next slide)

Types of blocks

	A	B
Jordan block $(\lambda \in \mathbb{R})$	$\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$
Jordan block $(\lambda = \infty)$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$
Kronecker block	$\begin{pmatrix} & \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{bmatrix} & \end{pmatrix}$	$\begin{pmatrix} & \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{bmatrix} & \end{pmatrix}$

Observation

Kronecker block case:

$$A - \lambda B = \begin{pmatrix} & \boxed{\begin{matrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{matrix}} \\ \boxed{\begin{matrix} -1 & & \\ \lambda & \ddots & \\ & \ddots & -1 \\ & & \lambda \end{matrix}} & \end{pmatrix}$$

The kernel of $A - \lambda B$ is generated by $(0, \ldots, 0, \lambda^k, \lambda^{k-1}, \ldots, \lambda, 1)$.

Proposition

Let $L = \mathrm{Span}\{\mathrm{Ker}(A - \lambda B)\}_{\lambda \in \mathbb{R}}$. Then

- L is isotropic w.r.t. every form $A - \lambda B$;
- L is maximal (!) isotropic.

More generally:

Theorem

Let $A - \lambda B$ be a pencil of skew symmetric forms,
$L = Span\{Ker(A - \lambda B)\}_{\lambda \ generic}$

- *L is isotropic w.r.t. every form $A - \lambda B$;*
- *L is maximal isotropic if and only if (the normal form of) the pencil $A - \lambda B$ has no Jordan blocks.*

Terminology: $\lambda \in \mathbb{C}$ is called generic if $\mathrm{rank}(A - \lambda B)$ is maximal in the pencil, otherwise λ is called a characteristic number of the pencil.

Relationship with Hamiltonian mechanics:

skew-symmetric form	$\longrightarrow$	Poisson structure
pencil of skew-symmetric forms	$\longrightarrow$	compatible Poisson structures
kernel of a skew-symmetric form	$\longrightarrow$	Casimir functions
maximal isotropic subspace	$\longrightarrow$	integrable system

Translation: Linear Algebra → bi-Poisson Geometry

Two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

Let M be a manifold endowed with a linear family $\mathcal{J} = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \mathcal{J}$ are degenerate so that each of them possesses non-trivial Casimir functions.

We say that $\mu \in \mathbb{R}$ is generic if rank A_μ is maximal in $\mathcal{J}$.

Proposition

Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \mathcal{J}$, then

1) the family of functions

$$\mathcal{F}_{\mathcal{J}} = \{\text{all Casimir functions of all brackets } A_\mu\}$$

consists of its first integrals;

2) these integrals commute.

Natural questions to discuss: PROPERTIES of $\mathcal{F}_{\mathcal{J}}$

- Completeness
- Set of critical points
- Equilibrium points
- Non-degeneracy conditions, types
- Codimension one singularities

Argument shift method

On the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$ there are two natural compatible Poisson brackets:

$$\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad \text{and} \quad \{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

where $a = (a_i) \in \mathfrak{g}^*$ is a fixed element.

Proposition

For each $\lambda \in \mathbb{R}$, the bracket $\{\ ,\ \}_\lambda = \{\ ,\ \} + \lambda\{\ ,\ \}_a$ is isomorphic to $\{\ ,\ \}$ (by means of translation $x \to x + \lambda a$). In particular,

- the Casimir functions of $\{\ ,\ \}_\lambda$ are of the form $f(x+\lambda a)$, where f is a coadjoint invariant of $\mathfrak{g}$;
- the singular set of $\{\ ,\ \}_\lambda$ is $\mathrm{Sing} + \lambda a$, where Sing is the set of singular coadjoint orbits of $\mathfrak{g}$;
- the Kernel of $\{\ ,\ \}_\lambda$ at the point $x \in \mathfrak{g}^*$ is the ad^*-stationary subalgebra of $x + \lambda a$, i.e., $\mathrm{ann}(x+\lambda a) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi(x+\lambda a) = 0\}$.

For this special kind of a Poisson pencil $\{\ ,\ \}_\lambda$ on $\mathfrak{g}^*$ we can construct the family of commuting functions $\mathcal{F}_a = \{f(x+\lambda a) \mid \lambda \in \mathbb{R}, f \in \mathrm{I}_{\mathrm{Ad}^*}(\mathfrak{g}^*)\}$ and ask all the above questions.

Completeness

Consider a pencil of compatible Poisson brackets $\mathcal{J} = \{A + \lambda B\}$ on M and the family of commuting Casimirs $\mathcal{F}_{\mathcal{J}}$ as above.
Question. Is $\mathcal{F}_{\mathcal{J}}$ complete, i.e., sufficient to guarantee complete integrability? How many commuting integrals do we need?

$$s = \frac{1}{2}(\dim M + \operatorname{corank} \mathcal{J})$$

Instead of computing the number of independent integrals in $\mathcal{F}_{\mathcal{J}}$ it is much better to use the following definition: $\mathcal{F}_{\mathcal{J}}$ is complete if at a generic point $x \in M$ the differentials $df(x)$, $f \in \mathcal{F}_{\mathcal{J}}$, generate a maximal isotropic subspace.

Theorem

The family $\mathcal{F}_{\mathcal{J}}$ is complete if and only if at a generic point $x \in M$ the following condition holds:

$$\operatorname{rank} A_\lambda(x) = \operatorname{rank} \mathcal{J} \quad \textit{for all } \lambda \in \overline{\mathbb{C}}.$$

Codimension two principle. Let all the brackets A_λ, $\lambda \in \overline{\mathbb{C}}$ have the same rank and $\operatorname{codim} S_\lambda \geq 2$ for almost all $\lambda \in \overline{\mathbb{C}}$. Then $\mathcal{F}_{\mathcal{J}}$ is complete.

Theorem

The family of shifts $\mathcal{F}_a$ is complete on $\mathfrak{g}^*$ iff $a \in \mathfrak{g}^*$ is regular and $\operatorname{codim} \mathrm{Sing} > 2$.

Set of critical points

Suppose that the family of commuting Casimirs $\mathcal{F}_{\mathcal{J}}$ related to a pencil $\mathcal{J} = \{A + \lambda B\}$ is complete on M. However, there are still some singular points $x \in M$ where the commuting functions from $\mathcal{F}_{\mathcal{J}}$ become dependent:

$$S_{\mathcal{J}} = \{x \in M \mid \dim D_{\mathcal{F}_{\mathcal{J}}}(x) < \frac{1}{2}(\dim M + \operatorname{corank} \mathcal{J})\}$$

where $D_{\mathcal{F}_{\mathcal{J}}}(x) \subset T_x^* M$ is the subspace spanned by the differentials of $f \in \mathcal{F}_{\mathcal{J}}$.

$S_{\mathcal{J}}$ is, by definition, the set of critical points of $\mathcal{F}_{\mathcal{J}}$ (or, equivalently the singular set of the corresponding Lagrangian fibration (see Lecture 1)).

On the other hand, for $\lambda \in \bar{\mathbb{C}}$, we can define the set of "singular points" of A_λ in M:

$$S_\lambda = \{x \in M \mid \operatorname{rank}(A_\lambda(x)) < \operatorname{rank} \mathcal{J}\}.$$

Theorem

A point x is critical for $\mathcal{F}_{\mathcal{J}}$ iff there is $\lambda \in \bar{\mathbb{C}}$ such that $x \in S_\lambda$.
In other words, the set of critical points $S_{\mathcal{J}}$ of the family $\mathcal{F}_{\mathcal{J}}$ is the union of "singular sets" S_λ over all $\lambda \in \overline{\mathbb{C}}$:

$$S_{\mathcal{J}} = \cup_{\lambda \in \overline{\mathbb{C}}} S_\lambda$$

Common equilibria

Let $\mathcal{F}$ be a family of commuting functions of a Poisson manifold M. We say that $x \in M$ is a common equilibrium point for $\mathcal{F}$ if $X_f(x) = 0$ for all $f \in \mathcal{F}$.

Theorem

$x \in M$ is a common equilibrium point for $\mathcal{F}_{\mathcal{J}}$ if and only if the kernels of all generic brackets at this point coincide: $\mathrm{Ker}\, A_\lambda(x) = \mathrm{Ker}\, A_\mu(x)$*, for all* $A_\lambda(x)$ *and* $A_\mu(x)$ *generic.*

Remark 1. These general results (completeness, set of critical points and common equilibria) are local in the sense that we always assume that the Casimir functions exist and their differentials at a generic point generate the kernel of the bracket.

Remark 2. All these results are of "zero order" in the sense that they require the forms $A(x)$ and $B(x)$ at a fixed point $x \in M$ only, but not their derivatives! Thus, so far this is Linear Algebra but not Differential Geometry.
From Differential Geometry we only need one simple thing: at a generic point $x \in M$ the differential of Casimir functions generate the kernel of the bracket. After this, everything is just a simple corollary of Jordan-Kronecker theorem.

Question to the audience: The compatibiliity condition is of the "first order". Why do we need it then?

References

- Alexey V. Bolsinov, Andrey A. Oshemkov
  "Bi-Hamiltonian structures and singularities of integrable Hamiltonian systems" Regular and Chaotic Dynamics, 14(2009), 431–454.
- Robert C. Thompson
  "Pencils of Complex and Real Symmetric and Skew Matrices"
  Linear Algebra and its Applications, 147(1991), 323–371.
- A. S. Mischenko and A. T. Fomenko
  "Euler equation on finite-dimensional Lie groups", Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 2, 396–415.
- A. V. Bolsinov
  "Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution", Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), no. 1, 68–92.
- Israel M. Gelfand and Ilya Zakharevich
  "Spectral theory for a pair of skew-symmetrical operators on S^1"
  Func. Anal. Appl. 23 (1989), no. 1, 85–93.
- Israel M. Gelfand and Ilya Zakharevich
  "Webs, Veronese curves, and bihamiltonian systems",
  J. of Func. Anal. 99 (1991), 150–178.