Singularities of bi-Hamiltonian Systems

Lecture 1: Singularities and bifurcations

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Plan

Main goal of this minicourse is to explain how a bi-Hamiltonian structure can be used for the qualitative analysis of dynamics.

- Lecture 1. Singularities and bifurcation in integrable systems: topological viewpoint
- Lecture 2. Compatible Poisson brackets from algebraic viewpoint, integrability, singularities of Poisson structures
- Lecture 3. Linearisation of Poisson pencils and general criterion of non-degeneracy
- Lecture 4. How does it work in practice? Examples and applications

- Integrable systems
- Lagrangian fibration and its singularities
- Why singularities are important?
- Non-degenerate singularities and their basic properties
- Elementary examples

Integrable systems and their singularities

Symplectic manifold (M, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$
Integrability: there exist $f_1, \ldots, f_n : M \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Singular Lagrangian fibration on M whose generic fibers are Liouville tori with quasi-periodic dynamics

Set of critical points $S = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$

SINGULARITIES ARE IMPORTANT

General problem: Describe S and its properties

Example: Neumann system on S^2

Point on the sphere S^2 in a quadratic potential
$M^4 = T^*S^2$
In sphere-conical coordinates (x_1, x_2):

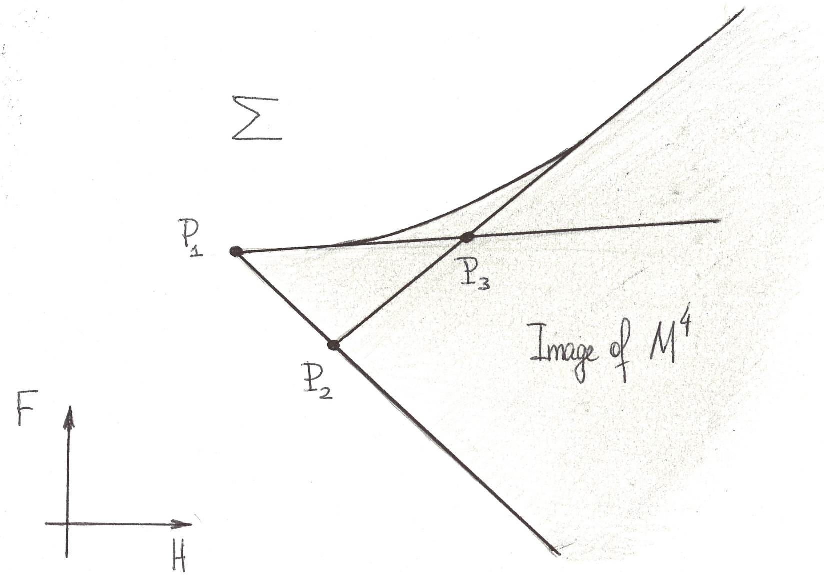
$$H = \frac{-P(x_1)p_1^2 + P(x_2)p_2^2}{x_1 - x_2} + x_1 + x_2$$

$$F = \frac{x_2 P(x_1)p_1^2 - x_1 P(x_2)p_2^2}{x_1 - x_2} - x_1 x_2$$

Momentum mapping $\Phi : T^*S^2 \to \mathbb{R}^2$
Bifurcation diagram $\Sigma \subset \mathbb{R}^2(H, F)$
Set of critical points $S = \{(x, p) \in T^*S^2 \mid \operatorname{rank} d\Phi(x, p) < 2\}$

$$(x, p) \in S \quad \text{iff} \quad \text{rank} \begin{pmatrix} \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} & \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} & \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{pmatrix} < 2 \quad \text{at } (x, p) \in T^*S^2$$

Bifurcation diagram



- Classical mechanics, topological analysis of integrable cases:
  M.P.Kharlamov, A.A.Oshemkov, M.Odin, R.Cushman
- Topological obstructions to integrability:
  V.V.Kozlov, A.T.Fomenko, I.A.Taimanov, G.Paternain, L.Butler
- Three-dimensional topology and the problem of isoenergy classification:
  A.T.Fomenko, H.Zieschang, S.V.Matveev, V.V.Sharko, A.V.Brailov
- Perturbation theory and topological invariants of dynamical systems:
  L.M.Lerman, Ya.L.Umanskii
- Theory of normal forms:
  L.Eliasson, J.Vey, H.Ito, Nguyen Tien Zung
- Symplectic manifolds with actions of Lie groups, toric manifolds:
  M.Atiyah, V.Guillemin, S.Sternberg, T.Delzant, M.Simington
- Quantization and global action-angle variables:
  J.Duistermaat, San Vu Ngoc, B.Zhilinskii

Theorem (Two degrees of freedom)

Let $\gamma(t)$ be a stable periodic solution. Then γ is singular, i.e., belongs to the singular set S (unless the system is resonant). Moreover, in the real analytic case $\gamma(t)$ is stable if and only if $\gamma(t)$ coincides with the common level of the integrals H and F:

$$\{\gamma(t), t \in \mathbb{R}\} = \{H(x) = H(x_0), F(x) = F(x_0)\}, \quad x_0 = \gamma(t_0)$$

Theorem

Let $P \in M^{2n}$ be an equilibrium point of a non-resonant integrable system. If P is stable then P is a critical point of Φ and, moreover, $\operatorname{rank} \Phi(P) = 0$, *i.e., P is a common equilibrium point for all the integrals $F_1, \ldots, F_n$.*

Strange conclusion: for stability analysis of integrable systems, we do not need to consider the Hamiltonian equations, the only important thing is the momentum mapping and its singularities (or, equivalently, the corresponding singular Lagrangian fibration).

Non-degenerate singularities

Definition

Let $x \in M^{2n}$ be a singular point of rank zero, i.e., $df_i(x) = 0$, $i = 1, \dots, n$. It is called non-degenerate, if the operators $\omega^{-1}d^2f_1, \dots, \omega^{-1}d^2f_n$ generate a Cartan subalgebra in the symplectic Lie algebra $sp(2n, \mathbb{R}) = sp(T_xM, \omega)$.

Equivalently, non-degeneracy means that the quadratic parts $d^2f_1(x), \dots, d^2f_n(x)$ are linearly independent and there is a linear combination $f = \sum a_i f_i$ such that the roots of the characteristic equation $\det(d^2f(x) - \omega) = 0$ are all distinct.

The type of a non-degenerate singular point is defined by the type of the corresponding Cartan subalgebra. Their description is given by the Williamson theorem.

Typical situations:

- elliptic type (center) $f(p,q) = p^2 + q^2$,
- hyperbolic type (saddle) $f(p,q) = p^2 - q^2$,
- focus $f_1 = p_1q_1 + p_2q_2$, $f_2 = p_1q_2 - p_2q_1$.

In general, we have combination of these three types. In other words, the type is defined by the eigenvalues of the linearised vector fields:

- a pair of imaginary numbers $i\beta$, $-i\beta$;
- a pair of real numbers α, $-\alpha$;
- four complex numbers $\alpha + i\beta$, $\alpha - i\beta$, $-\alpha + i\beta$, $-\alpha - i\beta$.

Local classification of non-degenerate singulerities

Theorem (Vey, Eliasson)

The type of a singularity is its complete topological, smooth and even symplectic invariant. In other words, two singularities having the same type (in the sense of Williamson) are locally symplectomorphic.

Topological interpretation: every non-degenerate singularity can be represented as the product of the simplest singularities, i.e., center, saddle and focus.

These statement remain true for singularities of an arbitrary rank. Moreover, it holds true in a neighborhood of a non-degenarate orbit of the Poisson action generated by the commuting integrals $F_1, \dots, F_n$ (Nguyen Tien Zung, E.Miranda).

Important property: non-degenerate singular points or rank k form an invariant symplectic submanifold of M^{2n} of dimension $2k$.

Semi-local classification: decomposition theorem

Elementary blocks

- elliptic singularity M^{ell};
- hyperbolic singularity M^{hyp};
- focus singularity M^{foc};

Singularities of direct product (type (k_1, k_2, k_3) and rank k).

$$\underbrace{M_1^{\mathrm{ell}} \times \cdots \times M_{k_1}^{\mathrm{ell}}}_{k_1} \times \underbrace{M_1^{\mathrm{hyp}} \times \cdots \times M_{k_2}^{\mathrm{hyp}}}_{k_2} \times \underbrace{M_1^{\mathrm{foc}} \times \cdots \times M_{k_3}^{\mathrm{foc}}}_{k_3} \times \underbrace{M_1^{\mathrm{reg}} \times \cdots \times M_k^{\mathrm{reg}}}_{k}$$

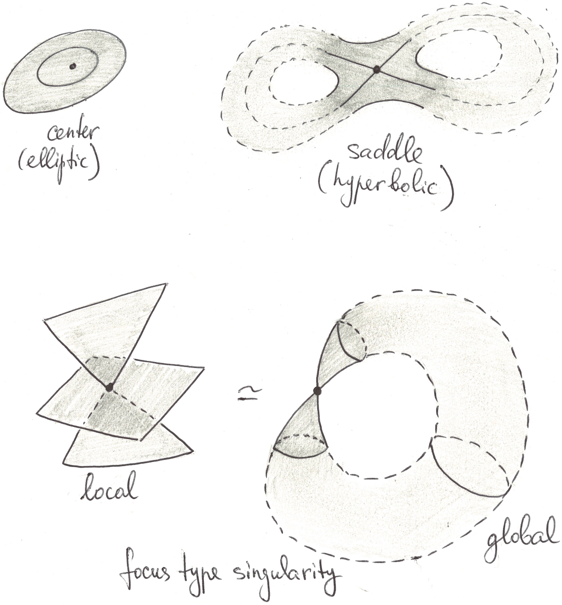
Theorem (Nguyen Tien Zung)

Every non-degenerate singularity is topologically equivalent to a singularity of almost direct product type

$$\big(\underbrace{M_1^{\mathrm{ell}} \times \cdots \times M_{k_1}^{\mathrm{ell}}}_{k_1} \times \underbrace{M_1^{\mathrm{hyp}} \times \cdots \times M_{k_2}^{\mathrm{hyp}}}_{k_2} \times \underbrace{M_1^{\mathrm{foc}} \times \cdots \times M_{k_3}^{\mathrm{foc}}}_{k_3} \times M_{\mathrm{reg}}^{2k}\big) \;/\; G,$$

where G is a finite group which acts freely, syplectically and preserves the fibration.

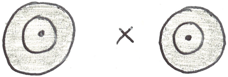
Basic singularities



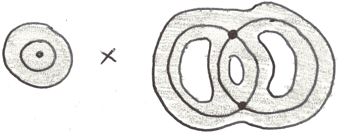
Examples

Neumann system:

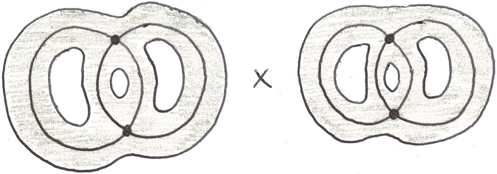
$\mathbb{P}_1$ is



$\mathbb{P}_2$ is



$\mathbb{P}_3$ is



$/\mathbb{Z}_2$

Given a Hamiltonian system $\dot{x} = X_H(x)$ with commuting integrals $\mathcal{F} = \{f_1, f_2, \dots\}$, it is natural and important to study the following

PROPERTIES of $\mathcal{F}_{\mathcal{J}}$

- Completeness
- Set of critical points
- Equilibrium points
- Non-degeneracy conditions, types
- Codimension one singularities
- Global properties

Question 1: Is there any relationship between the structure of singularities of bi-Hamiltonian systems and the propertieas of the corresponding family of compatible Poisson brackets?

Answer: Obviously YES

$$\boxed{\textit{Family of compatible Poisson brackets}}$$
$$\downarrow$$
$$\boxed{\textit{Commuting integrals}}$$
$$\downarrow$$
$$\boxed{\textit{Lagrangian fibration}}$$
$$\downarrow$$
$$\boxed{\textit{Its singularities}}$$
$$\downarrow$$
$$\boxed{\textit{Dynamical properties and stability analysis}}$$

Question 2. How close this relation is?

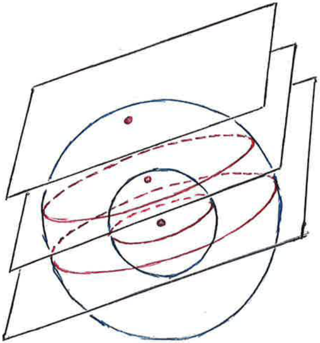
Example 1

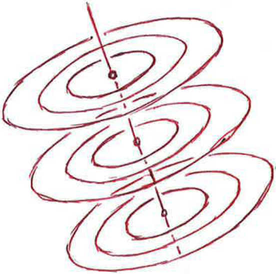
$so(3)$–bracket and constant bracket in $\mathbb{R}^3(x, y, z)$:

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + y^2 + z^2, \quad F_2 = ax + by + cz$

Lagrangian fibration:





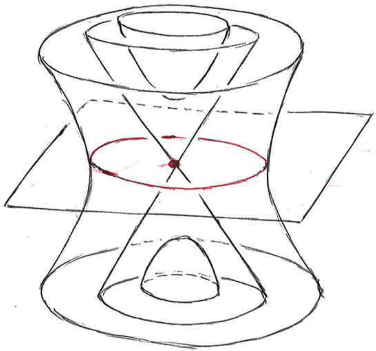
Example 2

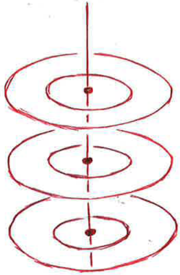
$sl(2,\mathbb{R})$–bracket and constant bracket in $\mathbb{R}^3(x,y,z)$:

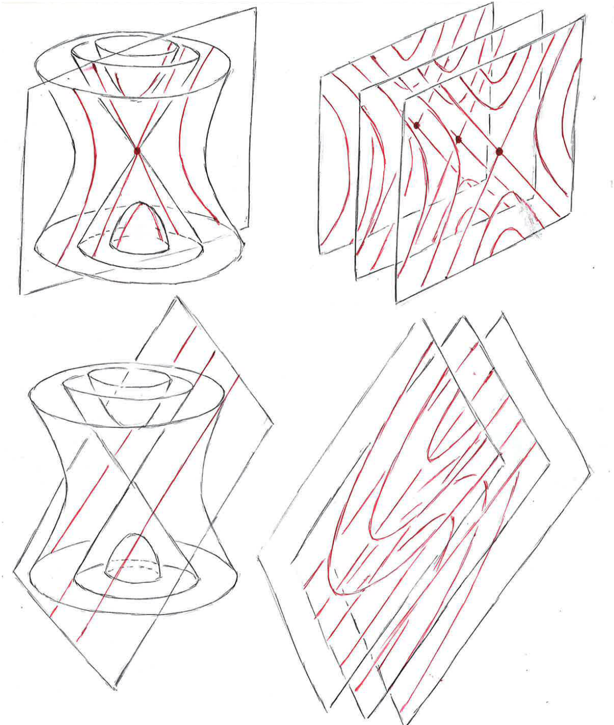
$$A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + yz,\quad F_2 = ax + by + cz$

Lagrangian fibration:







References

- Bolsinov A.V., Fomenko A.T. *Integrable Hamiltonian systems. Geometry, Topology and Classification*. CRC Press, 2004 .
- A. Bolsinov, A. Oshemkov, 'Singularities of integrable Hamiltonian systems'. In: Topological Methods in the Theory of Integrable Systems, Cambridge Scientific Publ., 2006, pp. 1-67.
  (available on my homepage `http://www-staff.lboro.ac.uk/~maab2/`)