Symplectic realizations of bihamiltonian structures

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To my wife Larysa

0 Introduction

A smooth manifold $M$ is endowed by a Poisson pair if two linearly independent bivectors $c_1, c_2$ are defined on $M$ and moreover $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. A bihamiltonian structure $J = \{c_\lambda\}$ is the whole 2-dimensional family of bivectors. The structure $J$ (the pair $(c_1, c_2)$) is degenerate if $\text{rank } c_\lambda < \dim M, \lambda \in \mathbb{R}^2$.

The degenerate bihamiltonian structures play important role in the theory of completely integrable systems due to the following fact. Let $Z_{c_\lambda}$ denote the set of Casimir functions for $c_\lambda$ and let $J_0 \subset J$ be the subfamily formed by the Poisson bivectors of maximal rank. It is easy to see that the functions from $\mathcal{F}_0 = \text{Span}_\mathbb{R}(\bigcup_{c_\lambda \in J_0} Z_{c_\lambda})$ are in involution with respect to any $c_\lambda$. If $\mathcal{F}_0$ contains enough independent functions there exists a chance of getting the completely integrable system ([2]) on symplectic leaves of $c_\lambda$. This is possible, for example, in the following situation. Consider a fixed Poisson bivector $c_\lambda \in J_0$ and a point $x$ on its symplectic leaf $S(x)$ of maximal dimension. Assume that for any $c_\lambda$ differentials of functions from $Z_{c_\lambda}$ generate the kernel of $c_\lambda(x)$. The family $\mathcal{F}_0$ is complete at $x$ with respect to $c_\lambda$, i.e. defines a lagrangian foliation in some neighbourhood of $x$ in $S(x)$, if and only if

∗Partially supported by the Polish grant KBN 2 PO3A 135 16.
rank\(_C(\lambda_1 c_1 + \lambda_2 c_2)(x)\) is maximal for any \(\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{0\}\).

This criterion of A. Brailov and its generalizations can be found in [4].

Degenerate bihamiltonian structures \(J = \{c_\lambda\}\) (Poisson pairs \((c_1, c_2)\)) satisfying (*) on an open dense subset of \(M\) will be called complete.

An intensive study of such objects was done by I.M. Gelfand and I.S. Zakharevich ([10], [11], [12]) in a particular case of bihamiltonian structures in general position on an odd-dimensional \(M\) (the corresponding Poisson pairs are necessarily degenerate: \(\text{rank } c_\lambda = 2n, \lambda \in \mathbb{R}^2 \setminus \{0\}\), if \(\dim M = 2n + 1\); they are also complete). In [11] there was introduced a notion of a Veronese web, i.e., a 1-parameter family of 1-codimensional foliations such that the corresponding family of annihilators is represented by the Veronese curve in the cotangent space at each point. It turns out that Veronese webs form a complete system of local invariants for bihamiltonian structures of general position. More precisely, it was proved in [11] that any such structure \(J = \{c_\lambda\}\) in \(\mathbb{R}^{2n+1}\) admits a local reduction to a Veronese web \(\mathcal{V}_J\) and that for any Veronese web \(\mathcal{V}\) one can locally construct a bihamiltonian structure \(J(\mathcal{V})\) of general position in \(\mathbb{R}^{2n+1}\) with the reduction equal to \(\mathcal{V}\). In the real analytic case \(J\) and \(J(\mathcal{V}_J)\) are isomorphic.

Complete Poisson pairs of higher corank are also of great interest. For example, the argument translation method of A.S. Mishchenko and A.T. Fomenko ([9]) uses the standard linear Poisson bivector \(c\) on a dual space \(g^*\) to a semisimple Lie algebra \(g\) and the constant Poisson bivector \(c(a)\), where \(a \in g^*\) is a fixed point on some symplectic leaf of maximal dimension for \(c\). The pair \((c, c(a))\) is a complete Poisson pair with corank equal to \(\text{rank } g\).

In this paper a method of constructing the complete bihamiltonian structures is presented. Essentially, it is the reduction of complex symplectic manifolds \((M, \omega)\) by a symplectic action of a real Lie group. The bivectors \(c'_1, c'_2\) generating the so built bihamiltonian structure \(J'\) are obtained as the reductions of the real and imaginary parts of the corresponding holomorphic Poisson bivector \(c = \omega^{-1}\) (it is easy to see that such \(c'_1, c'_2\) form a Poisson pair, provided that they are linearly independent). The necessary and sufficient conditions for the completeness of \(J'\) are obtained. The method is illustrated by a series of examples: it is proved that, given a complex semisimple Lie group \(G\) and its generic coadjoint orbit \((M, \omega)\) with the standard symplectic form \(\omega\), the reduction of \(\omega\) with respect to a compact real form \(G_0 \subset G\) produces a complete bihamiltonian structure \((M/G_0\) necessarily has singularities;
the formulated result concerns only the smooth part).

The paper is organized as follows. In Section 1 we recall some definitions and facts from the theory of Poisson manifolds and introduce the class of complex Poisson structures, which are generalizations of the standard ones to the case of the complexified tangent bundle. Holomorphic Poisson structures are strictly contained in this class.

Section 2 is devoted to bihamiltonian structures and their relations with the completely integrable systems. We define complete bihamiltonian structures and show that they include the mentioned case of general position.

In Subsections 3.1-3.9 we prove that the Poisson reduction \((c'_1, c'_2)\) of a Poisson pair \((c_1, c_2)\) is again a Poisson pair under the requirement of the linear independence for \(c'_1, c'_2\). This result follows from the natural behavior of the Schouten bracket with respect to the reduction and is preparational. The main theorem of Section 3 (see 4.4) deals with the reductions \((M, \omega) \longrightarrow (M', (c'_1, c'_2))\), where \((M, \omega)\) are complex symplectic manifolds and \(c'_1, c'_2\) are the push-forwards of \(c_1 = \text{Re}(\omega)^{-1}, c_2 = \text{Im}(\omega)^{-1}\) (such \((M, \omega)\) are called realizations for \((M', (c'_1, c'_2))\)). It presents the necessary and sufficient conditions for the completeness of the Poisson pair \((c'_1, c'_2)\) under the assumption that the leaves of the submersion \(M \longrightarrow M'\) are the generic \(CR\)-submanifolds in \(M\). Considering of nongeneric case is also meaningful but will not be touched in this paper. In the end of this section a notion of minimal realization is discussed.

The goal of Section 4 is to build examples of reductions \((M, \omega) \longrightarrow (M', (c'_1, c'_2))\) such that \((c'_1, c'_2)\) are the complete Poisson pairs. The corresponding complex symplectic manifolds \((M, \omega)\) will be coadjoint orbits of a complex semisimple Lie group \(G\) with the standard symplectic structure. The central result of this section (Theorem 6.7) establishes the completeness of \((c'_1, c'_2)\) on the smooth part of the reduction \(M/G_0\), where \(M\) is a coadjoint orbit of general position and \(G_0\) is a compact real form of \(G\). The proof of this result (Subsection 6.8) requires some preliminar work that is done in 6.2-6.5. Some of presented there results are devoted to the \(CR\)-geometry of the coadjoint \(G_0\)-orbits and seem to be of independent interest. Subsections ??-?? are intended to explain the proof as a generalization of the above mentioned method of argument translation. Moreover the last theorem of Section 4 shows that this proof works at least locally for arbitrary finite-dimensional Lie algebras with \(\text{codim}\, \text{Sing} \mathfrak{g}^* \geq 2\), where \(\text{Sing} \mathfrak{g}^*\) denotes the sum of the coadjoint orbits of nonmaximal dimension.
Off course, the inspiration for this paper is the theory of symplectic realizations for Poisson structures ([19]). The last section contains a discussion of open questions mainly motivated by references [11] and [19].

that there are no (anti)holomorphic functions constant along \( \mathcal{K} \), i.e. that among constants on \( \mathcal{K} \) there are no Casimir functions of \( c \) and \( \tilde{c} \) (the only, up to rescaling, degenerate bivectors in the family \( J \)). On the infinitesimal level this

1 Complex Poisson structures and other preliminaries

1.1. Let \( M \) be a \( C^\infty \)-manifold; denote by \( \mathcal{E}(M) \) (\( \mathcal{E}^\mathbb{C}(M) \)) a space of \( C^\infty \)-smooth real (complex) valued functions on \( M \). For a \( C^\infty \) vector bundle \( \pi : N \to M \), let \( \Gamma(N) \) denote the space of \( C^\infty \)-smooth sections of \( \pi \). Elements of \( \Gamma(\wedge^2 TM) \) (\( \Gamma(\wedge^2 T^\mathbb{C}M) \)) will be called (complex) bivectors for short.

1.2. Definition A (complex) bivector \( c \in \Gamma(\wedge^2 TM) \) (\( \Gamma(\wedge^2 T^\mathbb{C}M) \)) is called Poisson if

\[
[\ c, c \ ] = 0.
\]

Here \([\ , \ ]\) denotes the complex extension of the Schouten bracket which associates a trivector field \( [c_1, c_2] \in \Gamma(\wedge^3 T^\mathbb{C}M) \) to two bivectors \( c_1, c_2 \in \Gamma(\wedge^2 T^\mathbb{C}M) \). The corresponding local coordinate formula looks as follows:

\[
[c_1, c_2]_{ijk}^\alpha(x) = \frac{1}{2} \sum_{\scriptscriptstyle \mathrm{c.p.}ijk} (c_1^{ir}(x) \frac{\partial}{\partial x^r} c_2^{jk}(x) + c_2^{ir}(x) \frac{\partial}{\partial x^r} c_1^{jk} (x)), \quad (1.2.1)
\]

where \( c_\alpha = c_\alpha^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \alpha = 1, 2 \). \( \sum_{\scriptscriptstyle \mathrm{c.p.}ijk} \) denotes the sum over the cyclic permutations of \( i, j, k \) and the summation convention over repeated indices is used (the latter will be used systematically in this paper).

1.3. Definition Let \( M \) be a complex manifold. A holomorphic section of the bundle \( \wedge^2 T^{1,0} M \subset T^\mathbb{C}M \) will be called holomorphic bivector. In particular, holomorphic bivectors can be considered as complex ones and they will be called holomorphic Poisson if, in addition, they are Poisson in the sense of previous definition.
1.4. Definition A hamiltonian vector field \( c(f) \) corresponding to a function \( f \in \mathcal{E}(M) (\mathcal{E}^C(M)) \) is obtained by the contraction of the differential \( df \) and the Poisson bivector \( c \) with respect to the first index.

1.5. Proposition A (complex) bivector \( c \) is Poisson if and only if an operation \( \{ , \}_c : \mathcal{E}(M) \times \mathcal{E}(M) \rightarrow \mathcal{E}(M) (\mathcal{E}^C(M) \times \mathcal{E}^C(M) \rightarrow \mathcal{E}^C(M)) \) given by

\[
\{ f, g \}_c = c(f)g, \quad f, g \in \mathcal{E}(M) (\mathcal{E}^C(M))
\]

is a Lie algebra bracket over \( \mathbb{R} (\mathbb{C}) \).

If \( c \) is Poisson, then the map \( f \mapsto c(f) : \mathcal{E}(M) \rightarrow \text{Vect} \), where \( \text{Vect}(M) \) is a Lie algebra of smooth vector fields on \( M \) with the commutator bracket, is a Lie algebra homomorphism.

1.6. Definition \( \{ , \}_c \) is called a Poisson bracket corresponding to the (complex) Poisson bivector \( c \). A family of functions \( F \subset \mathcal{E}(M) (\mathcal{E}^C(M)) \) is involutive with respect to \( c \) if \( \{ f, g \}_c = 0 \) for each two functions \( f, g \in F \).

1.7. Definition Let \( c \) be a (complex) bivector; set \( P_{c,x} = \{ c(f)(x); f \in \mathcal{E}(M) \} \subset T_xM \) \( (P_{c,x} = \{ c(f)(x); f \in \mathcal{E}^C(M) \} \subset T_x^C M) \). A generalized distribution of subspaces \( P_c \subset TM (T^C M) \) is said to be a characteristic distribution for the bivector \( c \).

1.8. Definition Consider a (complex) bivector \( c \) at \( x \in M \) as a map \( c^x_\sharp : T^*_x M \rightarrow T_xM \left( (T^C_x M)^* \rightarrow T_x^C M \right) \) evaluating the first argument of the bivector on a 1-form. Kernel \( \ker c(x) \) and rank \( \text{rank } c(x) \) of \( c \) at \( x \) are defined as that of \( c^x_\sharp \). We say that \( c \) is nondegenerate if \( c^x_\sharp \) is an isomorphism. A complex bivector is called nondegenerate in the holomorphic sense if the restricted sharp map \( c^x_\sharp : (T^{1,0}M)^* \rightarrow T^{1,0}M \) is an isomorphism.

Obviously, \( P_{c,x} = c^x_\sharp (T^*_x M) \) \( (P_{c,x} = c^x_\sharp (T^C_x M)^*) \). Note that a complex bivector nondegenerate in holomorphic sense is not nondegenerate since \( P_{c,x} \subset T^{1,0}M \neq T^C M \). We shall usually understand the nondegeneracy of holomorphic bivectors in the holomorphic sense.
1.9. Theorem ([14]) Let $c$ be a real Poisson bivector. The generalized distribution $P_c$ is completely integrable, i.e. there exists a tangent to $P_c$ generalized foliation $\{S_\alpha\}_{\alpha \in I}$ on $M$: $T_x S_\alpha = P_{c,x}$ for any $\alpha \in I$ and for any $x \in S_\alpha$. The restriction of $c$ to each $S_\alpha$ is a nondegenerate Poisson bivector; thus $S_\alpha$ are symplectic manifolds with symplectic forms $\omega_\alpha = (c|_{S_\alpha})^{-1}$.

Here and subsequently the 2-form $\omega$ inverse to a nondegenerate bivector $c$ is defined as follows. If $\wedge^2 c^\sharp$ is the extension of the above defined sharp map to the second exterior power of $T^*M$, then $\omega = (\wedge^2 c^\sharp)^{-1}(c)$. The inverse to a nondegenerate 2-form bivector is defined similarly.

The above Theorem is also true in the complex analytic category if we understand $P_c$ as a holomorphic subbundle in $T^{1,0}M$ and the non-degeneracy in the holomorphic sense. The definition of inverse objects in this case is analogous to real one.

1.10. Definition The submanifolds $S_\alpha$ are called symplectic leaves of a Poisson bivector $c$.

1.11. Proposition Given a complex Poisson bivector $c \in \Gamma(\wedge^2 T^C M)$, its characteristic distribution $P_c \subset T^C M$ is involutive, i.e.

$$[v, w](x) \in P_{c,x}$$

for any complex valued vector fields $v, w$ such that $v(x), w(x) \in P_{c,x}, x \in M$.

In general, one can say nothing about the complete integrability of $P_c$ even if one understands this in spirit of the Newlander-Nierenberg theorem. A nonconstant rank of the subspaces $P_{c,x}$ or $P_{c,x} \cap \overline{P_{c,x}}$ (the overline means the complex conjugation) may be the obstruction here as well as some other reasons (see [18]).

1.12. Convention In the sequel, all Poisson bivectors will be assumed to have maximal rank on an open dense subset in $M$. For real Poisson bivectors this is equivalent to the following: the union of symplectic leaves of maximal dimension is dense in $M$.

1.13. Definition Let $c$ be a (complex) Poisson bivector. Define $\text{rank } c$ as $\max_{x \in M} \text{rank } c(x)$. 
1.14. Definition A Casimir function \( f \in \mathcal{E}(U) (\mathcal{E}^c(U)) \) over an open set \( U \subset M \) for a (complex) Poisson bivector \( c \) is defined by the condition \( c(f) = 0 \). A space of all Casimir functions over \( U \) for \( c \) is denoted by \( Z_c(U) \) and \( Z_{c,x} \) stands for the space of germs of Casimir functions at \( x, x \in M \).

Note that if \( c \) is real and \( \text{rank } c < \dim M \) there exist local nontrivial Casimir functions and their differentials at \( x \) span \( \ker c(x) \), provided that \( x \) is taken from a symplectic leaf of maximal dimension. This is not true concerning the global Casimir functions: it is easy to construct a Poisson bivector \( c \) with \( \text{rank } c < \dim M \) possessng only trivial ones.

1.15. Definition Let \((M, \omega), \dim M = 2n, \) be a symplectic manifold. A submanifold \( L \subset M \) is called

- coisotropic if \( (T_xL) \perp \omega(x) \subset T_xL \) for any \( x \in L \);
- isotropic if \( (T_xL) \perp \omega(x) \supset T_xL \) for any \( x \in L \);
- lagrangian if \( (T_xL) \perp \omega(x) = T_xL \) for any \( x \in L \).

A foliation \( L \) on \( M \) is coisotropic (isotropic, lagrangian) if so is its every leaf.

Here \( \perp \omega(x) \) stands for a skew-orthogonal complement in \( T_xM \) with respect to \( \omega(x) \). For the third case the following definition is equivalent: \( \dim L = n \) and \( \omega|_{TL} \equiv 0. \)

1.16. We shall need a specific generalization of this definition in the complex case. Let \( M \) be a complex manifold with the complex structure \( J : TM \to TM \). Consider a \( C^\infty \)-smooth submanifold \( L \subset M \). Write \( T_x^{CR}L \) for \( T_xL \cap JT_xL \) and \( T_x^{1,0}L \) for \( T_x^cL \cap T_x^{1,0}M, x \in L. \) Another definition for \( T_x^{1,0}L \) is the following: \( T_x^{1,0}L = \{ v - iJv; v \in T_x^{CR}L \}. \)

1.17. Definition ([18],[3]) \( L \) is called a CR-submanifold in \( M \) if \( \dim T_x^{1,0}L \) is constant along \( L \); we say that this number is \( CR \)-dimension of \( L \). \( L \) is generic (completely real) if \( T_xL + JT_xL = T_xM (T_xL \oplus JT_xL = T_xM) \) for each \( x \in L. \)
If a generic $CR$-submanifold $L$ is given by the equations \{\(f_1 = \alpha_1, \ldots, f_k = \alpha_k\), \(f_i \in \mathcal{E}(M)\), such that \(df_1 \wedge \cdots \wedge df_k \neq 0\) along $L$, then $\partial f_1 \wedge \cdots \wedge \partial f_k \neq 0$ along $L$ and $T^{1,0}_x L = \{\partial f_1(x), \ldots, \partial f_k(x)\}^{\perp,1,0}$, where $\partial$ is the (1,0)-differential on $M$, $\perp 1, 0$ denotes the annihilator in $T^{1,0}_x M$.

### 1.18. Definition
A foliation $\mathcal{L}$ on $M$ is a generic (completely real) $CR$-foliation if its each leaf is a generic (completely real) $CR$-submanifold.

### 1.19. Definition
Let $(M, \omega)$ be a holomorphic symplectic manifold. A $CR$-submanifold $L \subset M$ is
- $CR$-coisotropic if $(T^{1,0}_x L)^{\perp, \omega(x)} \subset T^{1,0}_x L$ for any $x \in L$;
- $CR$-isotropic if $(T^{1,0}_x L)^{\perp, \omega(x)} \supset T^{1,0}_x L$ for any $x \in L$;
- $CR$-lagrangian if $(T^{1,0}_x L)^{\perp, \omega(x)} = T^{1,0}_x L$ for any $x \in L$.

A $CR$-foliation $\mathcal{L}$ on $M$ is said to be $CR$-coisotropic ($CR$-isotropic, $CR$-lagrangian) if so is its every leaf.

Here $\perp \omega(x)$ denotes a skew-orthogonal complement in $T^{1,0}_x M$ with respect to the $(2,0)$-form $\omega(x)$.

Suppose $\mathcal{L}$ is generic and consits of the common level sets of the functions $f_1, \ldots, f_k \in \mathcal{E}(M), k \leq n = (1/2) \dim \mathbb{C} M$. Then $\mathcal{L}$ is $CR$-coisotropic if and only if the family $\{f_1, \ldots, f_k\}$ is involutive with respect to the holomorphic Poisson bivector $c = (\omega)^{-1}$. In particular, if $k = n$ one gets $CR$-lagrangian foliation.

### 1.20. Definition
Let $(M, \omega), \dim M = 2n$, be a symplectic manifold. A completely integrable system on $M$ is defined as a family of functions $\mathcal{F} \subset \mathcal{E}(M)$ involutive with respect to $c = (\omega)^{-1}$ and containing a subfamily of $n$ functions that are functionally independent almost everywhere on $M$. In other words, a completely integrable system on $M$ is a lagrangian foliation $\mathcal{L}$ on an open dense subset in $M$.

We conclude this section by recalling main definitions concerning hamiltonian actions of Lie groups (see [8] for details).

### 1.21. Definition
Let $G$ be a connected Lie group with the Lie algebra $\mathfrak{g}$. Assume it is acting on a Poisson manifold $M$ with the Poisson bivector $c$, i.e. a Lie algebra homomorphism $\rho : \mathfrak{g} \to \text{Vect}(M)$ is given. The
action is called Hamiltonian if there exists a Lie algebra homomorphism
\( \psi : \mathfrak{g} \to \mathcal{E}(M) \) such that the following diagram is commutative
\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\psi} & \mathcal{E}(M) \\
\| & & \downarrow c(\cdot) \\
\mathfrak{g} & \xrightarrow{\rho} & \text{Vect}(M),
\end{array}
\]
where \( c(\cdot) \) is a Lie algebra homomorphism of taking the hamiltonian
vector field (see Proposition 1.5).

The map \( x \mapsto \varphi_x : M \to \mathfrak{g}^* \) defined by \( \varphi_x(v) = \psi(v)(x), v \in \mathfrak{g} \) is
called the moment map.

2 Bihamiltonian structures and completeness

Let \( M \) be a \( C^\infty \)-manifold.

2.1. Definition Two linearly independent (complex) Poisson bivectors \( c_1, c_2 \) on \( M \) form a Poisson pair if \( c_\lambda = \lambda_1 c_1 + \lambda_2 c_2 \) is a (complex) Poisson bivector for any \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 (\mathbb{C}^2) \).

2.2. Proposition A pair of linearly independent (complex) Poisson bivectors \( (c_1, c_2) \) is Poisson if and only if \( [c_1, c_2] = 0 \).

2.3. Definition Let \( M \) be a \( C^\infty \)-manifold. A (complex) bihamiltonian structure on \( M \) is defined as a two-dimensional linear subspace \( J = \{ c_\lambda \}_{\lambda \in \mathcal{S}} \) of (complex) Poisson bivectors on \( M \) parametrized by a two-dimensional vector space \( \mathcal{S} \) over \( \mathbb{R} (\mathbb{C}) \).

It is clear that every Poisson pair generates a bihamiltonian structure
and the transition from the latter one to a Poisson pair corresponds to
a choice of basis in \( \mathcal{S} \). We shall write \( (J, c_1, c_2) \) for a bihamiltonian structure \( J \) with a chosen Poisson pair \( (c_1, c_2) \) generating \( J \).

2.4. Definition Let \( (J, c_1, c_2) \) be a bihamiltonian structure. A complex bihamiltonian structure
\[
J^\mathbb{C} = \{ \lambda_1 c_1 + \lambda_2 c_2; (\lambda_1, \lambda_2) \in \mathbb{C}^2 \}
\]
is called the complexification of \( J \).
2.5. Proposition A complex bihamiltonian structure $J$ is the complexification of some real one if and only if one can choose a generating $J$ Poisson pair $(c, \bar{c})$, where $c \in \wedge^2(T^\mathbb{C}M)$, the bar stands for the complex conjugation.

**Proof.** Generate $J'$ be $\text{Re} \, c, \text{Im} \, c$. Then $J = (J')^\mathbb{C}$. Conversely, one checks that for $(J', c_1, c_2)$ the complexification $(J')^\mathbb{C}$ is generated by $c_1 \pm ic_2$. q.e.d.

2.6. Definition Let $J$ be a (complex) bihamiltonian structure and let $J_0 \subset J$ be a subfamily of (complex) Poisson bivectors of maximal rank $R_0$ (the set $J \setminus J_0$ is at most a finite sum of 1-dimensional subspaces). We say that $J$ is symplectic if $\text{rank} \, c_\lambda = \dim M$ for any $c_\lambda \in J_0$ and that $J$ is degenerate otherwise.

2.7. Example Consider a family $J^\mathbb{C}$ generated by a pair $(c, \bar{c})$, where $c = (\omega)^{-1}$ is a complex Poisson bivector inverse to a holomorphic symplectic form $\omega$ on a complex symplectic manifold $M$. Since $c$ is holomorphic and $\bar{c}$ is antiholomorphic, we have $[c, \bar{c}] = 0$. Thus $J^\mathbb{C}$ is a bihamiltonian structure. By Proposition 2.5 it is the complexification of the real bihamiltonian structure $(J, \text{Re} \, c, \text{Im} \, c)$. This example is fundamental for the paper and we shall need the following fact.

2.8. Theorem Let $M, \omega, c$ and $J^\mathbb{C}$ be as in Example 2.7. Then $J^\mathbb{C}$ is symplectic and the only degenerate bivectors in $J^\mathbb{C}$ are those proportional to $c$ and $\bar{c}$. Moreover, $\text{rank}_\mathbb{C} c = \text{rank}_\mathbb{C} \bar{c} = \frac{1}{2} \dim_\mathbb{C} M$, $P_c = T^{1,0}M$, $P_{\bar{c}} = T^{0,1}M$.

**Proof.** The last assertion is obvious as well as the following equality

$$c_\downarrow \omega = 0,$$

where $\downarrow$ stands for the contraction with respect to the first index. For $\omega_1 = \text{Re} \, \omega, \omega_2 = \text{Im} \, \omega, c_1 = \text{Re} \, c, c_2 = \text{Im} \, c$ this implies

$$c_1 \downarrow \omega_1 + c_2 \downarrow \omega_2 = 0, c_2 \downarrow \omega_1 - c_1 \downarrow \omega_2 = 0.$$

We have for $\lambda \in \mathbb{C}$

$$(c_1 + \lambda c_2) \downarrow (\omega_1 - \lambda \omega_2) = c_1 \downarrow \omega_1 - \lambda^2 c_2 \downarrow \omega_2 - \lambda(c_1 \downarrow \omega_2 - c_2 \downarrow \omega_1) = (1 + \lambda^2)c_1 \downarrow \omega_1 = (1 + \lambda^2)\frac{1}{4} \text{id}_TM.$$
The last equality is verified directly in the Darboux local coordinates. Thus \((c_1 + \lambda c_2)^{-1} = \frac{4}{1 + \pi^2}(\omega_1 - \lambda \omega_2).\) Since \(c_1 \omega_1 = -c_2 \omega_2,\) \(c_2\) is also nondegenerate. q.e.d.

2.9. Definition Let \((M, \omega)\) be a complex symplectic manifold. The bihamiltonian structure \(J\) and its complexification \(J^c\) from Example 2.7 are called holomorphic symplectic.

2.10. Given a (complex) bihamiltonian structure \(J,\) let \(F_0\) denote the space \(\text{Span}_R(\bigcup_{c \in J_0} Z_c(M)).\) We take \(\text{Span}\) in order to obtain a vector space: a sum of two Casimir functions for different \(c_1, c_2 \in J_0\) need not be a Casimir function. However, \(\text{Span}_R\) is enough for both the real and complex cases.

The following theorem shows how the degenerate bihamiltonian structures can be applied for constructing the completely integrable systems.

2.11. Theorem Let \(J\) be a degenerate (complex) bihamiltonian structure on \(M.\) A family \(F_0\) is involutive with respect to any \(c_\lambda \in J.\)

Proof. Let \(c_1, c_2 \in J_0\) be linearly independent, \(f_i \in Z_{c_i}, i = 1, 2.\) Then

\[
\{f_1, f_2\}_{c_\lambda} = (\lambda_1 c_1(f_1) + \lambda_2 c_2(f_1)) f_2 = -\lambda_2 c_2(f_2) f_1 = 0. \tag{2.11.1}
\]

Now it remains to prove that for any \(c \in J_0, f_i \in Z_{c_i}, i = 1, 2,\) one has \(\{f_1, f_2\}_{c_\lambda} = 0.\) For that purpose we first rewrite (2.11.1) as

\[
c_{\lambda}(x)(\phi_1, \phi_2) = 0, \tag{2.11.2}
\]

where \(\phi_i \in \ker c_i(x), i = 1, 2, x \in M,\) and the lefthand side denotes a contraction of the bivector with two covectors. Second, we fix \(x\) such that \(\text{rank} c(x) = R_0\) and approximate \(df_2|_x\) by a sequence of elements \(\{\phi_i^t\}_{t=1}^{\infty},\phi_i^t \in \ker c_i(x),\) where \(c_i \in J_0, i = 1, 2, \ldots,\) is linearly independent with \(c.\) Finally, by (2.11.2) we get \(c_{\lambda}(x)(df_1|_x, \phi_i^t) = 0\) and by the continuity \(\{f_1, f_2\}_{c_\lambda}(x) = 0.\) Since the set of such points \(x\) is dense in \(M,\) the proof is finished. q.e.d.

In fact this theorem is true for the local Casimir functions (for the germs of Casimir functions).

2.12. Definition The functions from the family \(F_0\) (see 2.10) are called (global) first integrals of the bihamiltonian structure \(J.\) The family of functions \(\text{Span}_R(\bigcup_{c \in J_0} Z_c(U))\) \((\text{Span}_R(\bigcup_{c \in J_0} Z_{c,x}))\) is denoted by \(F_0(U)\) \((F_{0,x})\) and its elements are called local first integrals over an open \(U \subset M\) (germs of first integrals at \(x \in M\)).
In order to obtain a completely integrable system from Casimir functions one should require additional assumptions on the bihamiltonian structure $J$. Of course, the condition of completeness given below concerns the local Casimir functions (in fact their germs) and may be insufficient for obtaining the completely integrable system. However, it is of use if the local Casimir functions are restrictions of the global ones (see Example 2.18, below).

Given a characteristic distribution $P_c \subset TM$ ($T^C M$) of some (complex) Poisson bivector and a point $x \in M$, let $P_{c,x}$ denote a dual space to $P_{c,x}$. Any functional $\phi \in T^*_x M$ ($(T^C_x M)^*$) can be regarded as an element of $P_{c,x}^*$ called the restriction of $\phi$ to $P_{c,x}$.

2.13. Definition ([4]) Let $J$ be a (complex) bihamiltonian structure; fix some $c_\lambda \in J$.

$J$ is called complete at a point $x \in M$ with respect to $c_\lambda$ if a linear subspace of $P_{c_\lambda,x}^*$ generated over $\mathbb{R}$ ($\mathbb{C}$) by the differentials of the germs $f \in \mathcal{F}_{0,x}$ restricted to $P_{c_\lambda,x}$ has dimension $\frac{1}{2} \dim \mathbb{C} P_{c_\lambda,x} (\frac{1}{2} \dim \mathbb{R} P_{c_\lambda,x})$.

2.14. Proposition A (complex) bihamiltonian structure $J$ is complete with respect to $c_\lambda \in J$ at a point $x \in M$ if and only if $\dim(\bigcap_{c \in J_0} P_{c,x}) = \frac{1}{2} \dim \mathbb{R} P_{c_\lambda,x}$.

Proof is obvious.

The following theorem is due to A.Brailov (see [4], Theorem 1.1 and Remark after it).

2.15. Theorem A (complex) bihamiltonian structure $J$ is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $P_{c_\lambda,x}$ is of maximal dimension if and only if the following condition holds

\[ (*) \quad \text{rank} c(x) = R_0 \quad \text{for any } c \in J^C \setminus \{0\} \text{ } (J \setminus \{0\}), \]

where $R_0$ is as in 2.6.

Proof of this theorem is a consequence of the following linear algebraic fact.

2.16. Proposition ([4]) Let $V$ be a vector space over $\mathbb{R}$ ($\mathbb{C}$) and let $J$ be a two dimensional linear subspace in $\wedge^2 V$. In the real case we let $J^C \subset \wedge^2 V^C$ denote the complexification of the subspace $J$. We write $J_0 \subset J$ for the subset of bivectors of maximal rank $R_0$ and $F_0 \subset V^*$
for the subspace generated by the kernels of bivectors from $J_0$. Let $c^\sharp : \mathcal{V} \to V$ stand for the corresponding sharp map of $c \in \Lambda^2 \mathcal{V}$ (cf. 1.8).

Then, given a bivector $c_\lambda \in J_0$, the following two conditions are equivalent:

(i) $\dim(F_0|_{P_\lambda}) = 1/2 \dim P_\lambda$, where $F_0|_{P_\lambda} = \text{Span}\{f|_{P_\lambda} : f \in F_0\}$ and $P_\lambda = c^\sharp_\lambda(V^*)$;

(ii) $\text{rank } c = R_0$ for any $c \in J^C \setminus \{0\}$ ($J \setminus \{0\}$).

**Proof.** We reproduce the proof from [4] with a small completion.

Condition (i) is equivalent to the following: the skew-orthogonal complement $F_0^\perp_{\lambda} = (c^\sharp_\lambda(F_0))^\perp$ (annihilator in the sense of the dual pair $(V, V^*)$) coincides with $F_0$ (by Proposition 2.11 $F_0 \subset F_0^\perp_{\lambda}$).

Indeed, in view of Proposition 2.11 the subspace $F_0|_{P_\lambda}$ is isotropic in $(P_\lambda)^*$ with respect to $c_\lambda$. Thus (i) holds if and only if the subspace $F_0|_{P_\lambda}$ is lagrangian in $(P_\lambda)^*$ with respect to $c_\lambda$, i.e. the subspace $(F_0|_{P_\lambda})_{\perp c_\lambda} = (c^\sharp_\lambda(F_0|_{P_\lambda}))_{\perp c_\lambda}$, where $c^\sharp_\lambda : P_\lambda^* \to P_\lambda$ and the annihilator $\perp c_\lambda$ is in the sense of the dual pair $(P_\lambda, P_\lambda^*)$, coincides with $F_0|_{P_\lambda}$. But it is easy to see that there exists a subspace $F_\lambda \subset V^*$ such that $F_\lambda \cong F_0|_{P_\lambda}$ via the canonical projection $V^* \to P_\lambda^*$ and $F_0 = F_\lambda \oplus \text{ker } c_\lambda$. Hence $c^\sharp_\lambda(F_0|_{P_\lambda}) = c^\sharp_\lambda(F_0)$; denoting this subspace by $W$ and setting $2N = \dim P_\lambda, r = \dim \text{ker } c_\lambda$ one calculates dimensions. The subspace $F_0|_{P_\lambda}$ is lagrangian if and only if $\dim W = N$ if and only if $\dim W^\perp = N + r = \dim F_0$. So we proved the claim.

We continue the proof in the following three steps.

First, we observe that for any two nontrivial bivectors $a, b \in J$ one has the equality $a^\sharp(b(F_0)) = b^\sharp(F_0)$. Indeed, suppose that $a, b$ are linearly independent. The subspace $F_0$ is generated by a finite number of kernels $\text{ker } b_1, \ldots, \text{ker } b_s, b \in J_0$. Without loss of generality, we may assume that $b_i = \alpha_i a + \beta_i b$, where $\alpha_i, \beta_i \not= 0$. Since $(\alpha_i a^\sharp + \beta_i b^\sharp)(\text{ker } b_i) = 0, i = 1, \ldots, s$, then $a^\sharp(\text{ker } b_i) = b^\sharp(\text{ker } b_i)$ and, consequently, $a^\sharp(b(F_0)) = b^\sharp(F_0)$.

In the second step we consider the skew-orthogonal complement $\tilde{F}_0 = (F_0)^\perp = (b^\sharp(F_0))^\perp$ and note that: 1) it does not depend on $b \in J \setminus \{0\}$ (previous step); 2) $F_0 \subset \tilde{F}_0$ (again Proposition 2.11); 3) if $a \in J_0$, then $b^\sharp(\tilde{F}_0) \subset a^\sharp(\tilde{F}_0)$ for any $b \in J$ (this is equivalent to $(F_0)^\perp b \supset (F_0^a)^\perp$ or $F_0 + \text{ker } b \supset F_0 + \text{ker } a = F_0$).

Finally, given two linearly independent bivectors $a, b \in J$, with rank $a = R_0$, we define a "recursion" operator $\Phi : \tilde{F}_0/F_0 \to \tilde{F}_0/F_0$ by the formula $\Phi(\pi(\xi)) = \pi((a^\sharp)^{-1}b^\sharp(\xi))$, where $\xi \in \tilde{F}_0$ and $\pi : \tilde{F}_0 \to \tilde{F}_0/F_0$.
is the natural projection. The operator is correctly defined due to the conditions $a(F_0) = b(F_0)$, $b(\tilde{F}_0) \subset a(F_0)$, and $\ker a \subset F_0$. It is easy to see that the eigenvalues of $\Phi$ are precisely those $\lambda \in \mathbb{C}$ for which $\text{rank}(a - \lambda b) < R_0$. In particular (ii) holds if and only if $\Phi$ does not have eigenvalues, i.e. $F_0 = \tilde{F}_0$. q.e.d.

Theorem 2.15 shows that $J$ is complete with respect to a fixed $c_\lambda \in J_0$ at a point $x$ such that the dimension $P_{c_\lambda,x}$ is maximal if and only if $c_\lambda \in J$. This motivates the next definition.

2.17. Definition Let $(J, c_1, c_2)$ be a (complex) bihamiltonian structure. The structure $J$ (the pair $(c_1, c_2)$) is complete at a point $x \in M$ if condition $(\ast)$ of Theorem 2.15 holds at $x$. $J ((c_1, c_2))$ is called complete if it is so at any point from some open and dense subset in $M$.

2.18. Example (Method of argument translation, see [9], [4].) Let $g$ be a nonabelian Lie algebra, $g^*$ its dual space. Fix a basis $\{e_1, \ldots, e_n\}$ in $g$ with the structure constants $\{c^k_{ij}\}$; write $\{e_1, \ldots, e_n\}$ for a dual basis in $g^*$. The standard linear Poisson bivector on $g^*$ is defined as

$$c_1(x) = c^k_{ij}x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\{x_k\}$ are linear coordinates in $g^*$ corresponding to $\{e^1, \ldots, e^n\}$. In more invariant terms $c_1$ is described as dual to the Lie-multiplication map $[\cdot, \cdot] : g \wedge g \to g$. It is well-known that the symplectic leaves of $c_1$ are the coadjoint orbits in $g^*$. Now define $c_2$ as a bivector with constant coefficients $c_2 = c(a)$, where $a$ is a fixed point on any leaf of maximal dimension. It turns out that $c_1, c_2$ form a Poisson pair and it is easy to describe the set $I$ of points $x$ for which condition $(\ast)$ fails. Consider the complexification $(g^*)^\mathbb{C} \cong (\mathbb{C}g)^*$ and the sum $\text{Sing}(\mathbb{C}g^*)$ of symplectic leaves of nonmaximal dimension for the complex linear bivector $c^k_{ij}z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$, are the corresponding complex coordinates in $(g^*)^\mathbb{C}$. Then $I$ is equal to the intersection of the sets $g^* \subset (g^*)^\mathbb{C}$ and $a, \text{Sing}(\mathbb{C}g^*)$, where $a, \text{Sing}(\mathbb{C}g^*)$ denotes a cone of complex 2-dimensional subspaces passing through $a$ and $\text{Sing}(\mathbb{C}g^*)$. In particular, $(c_1, c_2)$ is complete for a semisimple $g$. Note that this gives rise to completely integrable systems, since the local Casimir functions on $g^*$ are restrictions of the global ones, i.e. the invariants of the coadjoint action.
2.19. Example  (Bihamiltonian structure of general position on an odd-dimensional manifold, see [11].) Consider a pair of bivectors \((a_1, a_2)\), \(a_i \in \Lambda^2 V, i = 1, 2\), where \(V\) is a \((2m + 1)\)-dimensional vector space; \((a_1, a_2)\) is in general position if and only if is represented by the Kronecker block of dimension \(2m + 1\), i.e.

\[
a_1 = p_1 \wedge q_1 + p_2 \wedge q_2 + \cdots + p_m \wedge q_m
\]
\[
a_2 = p_1 \wedge q_2 + p_2 \wedge q_3 + \cdots + p_m \wedge q_{m+1}
\] (2.19.1)

in an appropriate basis \(p_1, \ldots, p_m, q_1, \ldots, q_{m+1}\) of \(V\). A bihamiltonian structure \(J\) on a \((2m + 1)\)-dimensional \(M\) is in general position if and only if the pair \((c_1(x), c_2(x))\) is so for any \(x \in M\). Such \(J\) is complete: it is easy to prove that \(J = J_0 \cup \{0\}, \dim \bigcap_{c \in J} P_c(x) = n\) and then use Proposition 2.14. In general, a complete Poisson pair at a point is the direct sum of the Kronecker blocks and the zero pair as the corollary of the next theorem shows. This theorem is a reformulation of the classification result for pairs of 2-forms in a vector space ([10], [12]).

2.20. Theorem  Given a finite-dimensional vector space \(V\) over \(\mathbb{C}\) and a pair of bivectors \((c_1, c_2)\), \(c_i \in \Lambda^2 V\), there exists a direct decomposition \(V = \bigoplus_{j=1}^k V_j, c_i = \sum_{j=1}^k c_i^{(j)}, c_i^{(j)} \in \Lambda^2 V_j\), \(i = 1, 2\), such that each pair \((c_1^{(j)}, c_2^{(j)})\) is from the following list:

- **(a)** the Jordan block: \(\dim V_j = 2n_j\) and in an appropriate basis of \(V_j\) the matrix of \(c_i^{(j)}\) is equal to

\[
\begin{pmatrix}
0 & A_i \\
-A_i^T & 0
\end{pmatrix}, \ i = 1, 2,
\]

where \(A_1 = I_{n_j}\) (the unity \(n_j \times n_j\)-matrix) and \(A_2 = J_\lambda^{n_j}\) (the Jordan block with the eigenvalue \(\lambda\));

- **(b)** the Kronecker block: \(\dim V_j = 2n_j + 1\) and in an appropriate basis of \(V_j\) the matrix of \(c_i^{(j)}\) is equal to

\[
\begin{pmatrix}
0 & B_i \\
-B_i^T & 0
\end{pmatrix}, \ i = 1, 2,
\]

where \(B_1 = \begin{pmatrix}
100.00 \\
010.00 \\
\cdots \\
000.10
\end{pmatrix}\), \(B_2 = \begin{pmatrix}
010.00 \\
001.00 \\
\cdots \\
000.01
\end{pmatrix}\) \((n_j + 1) \times n_j\)-matrices).
2.21. Corollary Let $J$ be a (complex) bihamiltonian structure. It is complete at a point $x \in M$ if and only if a pair $(c_1(x), c_2(x))$, $c_i(x) \in \Lambda^2(T^*_x M), i = 1, 2$, does not contain the Jordan blocks in its decomposition.

Proof follows from the definition of completeness.

The following example of a complete Poisson pair shows that the structure of decomposition to the Kronecker blocks may change from point to point.

2.22. Example ([17]) Let $M = \mathbb{R}^6$ with coordinates $(p_1, p_2, q_1, \ldots, q_4)$, $c_1 = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}, c_2 = \frac{\partial}{\partial p_1} \wedge (\frac{\partial}{\partial q_2} + q_1 \frac{\partial}{\partial q_3}) + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_4}$. Here we have: two 3-dimensional Kronecker blocks on $M \setminus H, H = \{q_1 = 0\}$; the 5-dimensional Kronecker block and the 1-dimensional zero block on the hyperplane $H$.

2.23. Remark The decomposition of a pair of bivectors $c_1, c_2$ in a vector space $V$ to Kronecker blocks is defined noncanonically. For example, let us consider 4-dimensional $V = \text{Span}\{e, p, q_1, q_2\}$, $c_1 = p \wedge q_1, c_2 = p \wedge q_2$. Here $V = V_1 \oplus V_2$, where $V_1 = \text{Span}\{e\}, V_2 = \text{Span}\{p, q_1, q_2\}$, but instead $V_1$ one can choose any direct complement to $V_2$. However, dimensions of the direct sums for the Kronecker blocks of equal dimension are invariants (see [13],[17]). For instance, dimension of the sum of the trivial Kronecker blocks is equal to $\dim \ker c_1 \cap \ker c_2$ (see the proof of Proposition 2.24, below).

We conclude the section by a result that will be used later on.

2.24. Proposition Let $V$ be a vector space over $\mathbb{C}$ and let a pair of bivectors $c_1, c_2 \in \Lambda^2 V$ be such that there are no Jordan blocks in the decomposition of Theorem 2.20. Set

$$\mu = \dim(\ker c_1 \cap \ker c_2)$$

$$\mu_\lambda = \dim(\ker c_1 \cap \ker c_\lambda),$$

where $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2, \lambda_1, \lambda_2 \neq 0$. Then $\mu = \mu_\lambda$. 

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PROOF. Let \((V', a_1, a_2), V' \subset V, \dim V_j = 2m + 1, a_i \in \wedge^2 V',\) be a nontrivial Kronecker block. By formula 2.19.1

\[
\ker a_\lambda = \lambda_2^m q^{m+1} - \lambda_1^{m-1} \lambda_2 q^m + \cdots + (-1)^m \lambda_2^m q^1,
\]

where \(a_\lambda = \lambda_1 a_1 + \lambda_2 a_2\) and \(q^1, \ldots, q^{m+1}\) is a part of the basis \(p^1, \ldots, p^m, q^1, \ldots, q^{m+1}\) in \(V'^*\) dual to \(p_1, \ldots, p_m, q_1, \ldots, q_{m+1}\) (let us denote these bases by \(p, q\) and \(p', q'\), correspondingly, and call them adapted to the pair \(a_1, a_2\)). The above formula shows that \(\ker a_1 \cap \ker a_\lambda = \{0\}\) if \(\lambda_2 \neq 0\).

Now, let \(V = \oplus_{j=1}^k V_j, c_i = \sum_{j=1}^k c^{(j)}_i, i = 1, 2\), be the decomposition to the Kronecker blocks and let \(V_{k'+1}, \ldots, V_k\) be all trivial ones. Consider a basis of \(V\) of the following form

\[
p^{(1)}, q^{(1)}, \ldots, p^{(k'}), q^{(k'}), r_1, \ldots, r_{k-k'},
\]

where \(p^{(j)}, q^{(j)}\) is a basis of \(V_j\) adapted to \(c^{(j)}_1, c^{(j)}_2, j = 1, \ldots, k'\) and \(r_1, \ldots, r_{k-k'}\) generate \(V_{k'+1}, \ldots, V_k\), respectively. The dual basis will be of the form

\[
p^{(1)}, q^{(1)}, \ldots, p^{(k'}), q^{(k')}, r^1, \ldots, r^{k-k'}
\]

and the above considerations show that \(\ker c_1 \cap \ker c_\lambda\) is generated by \(r^1, \ldots, r^{k-k'}\) if \(\lambda_1 \neq 0\). q.e.d.

3 Reductions and realizations of bihamiltonian structures.

Our first aim is to prove that a Poisson reduction of a bihamiltonian structure is again a bihamiltonian structure. This result follows from a natural behavior of the Schouten bracket with respect to the reduction.

3.1. Consider a \(C^\infty\)-smooth surjective submersion \(p : M \rightarrow M'\) such that \(p^{-1}(x')\) is connected for any \(x' \in M'\). The foliation of its leaves will be denoted by \(\mathcal{K}\). Write \(p_* : TM \rightarrow TM' (p_*^C : T^CM \rightarrow T^CM')\) for the corresponding tangent bundle morphism, \(\wedge^k p_* : \wedge^k TM \rightarrow \wedge^k TM'\) for its exterior power extension and \(\ker \wedge^k p_*\) for a subbundle in \(\wedge^k TM\) that is a kernel of \(\wedge^k p_*\). Multivector fields on \(M\) or \(M'\) will be called multivectors for short.
If \((U, \{x^1, \ldots, x^l, y^1, \ldots, y^{m'}\})\) is a local coordinate system on \(M\) such that \(m' = \dim M'\) and \(y^1, \ldots, y^{m'}\) are constant along \(K\), then the restriction \(Z|_U\) of \(Z \in \Gamma(\bigwedge^k T M)\) belongs to \(\Gamma(\ker \bigwedge^k p_*)(U)\) if and only if each term of its decomposition with respect to \(\{\partial_{x^1}, \ldots, \partial_{x^l}, \partial_{y^1}, \ldots, \partial_{y^{m'}}\}\) contains at least one \(\partial_{x^i}\), \(1 \leq i \leq l\).

3.2. Theorem Let \(Z \in \Gamma(\bigwedge^k T M)\). The following conditions are equivalent:

(i) \(L_X Z \in \Gamma(\ker \bigwedge^k p_*) \quad \forall X \in \Gamma(\ker p_*)\), where \(L_X\) is a Lie derivation;

(ii) \(\phi^X_t Z - Z \in \Gamma(\ker \bigwedge^k p_*) \quad \forall \forall t \in \Gamma(\ker p_*)\), where \(\phi^X_t\) denotes the flow of the vector \(X\);

(iii) in any local coordinate system \((U, \{x^1, \ldots, x^l, y^1, \ldots, y^{m'}\})\) on \(M\) such that \(m' = \dim M'\) and \(y^1, \ldots, y^{m'}\) are constant on the leaves of \(p\) the multivector \(Z\) can be written as

\[
Z(x, y) = Z'(y) + \tilde{Z}(x, y),
\]

where

\[
Z'(y) = Z'^{i_1 \ldots i_k}(y)\partial_{y^{i_1}} \wedge \ldots \wedge \partial_{y^{i_k}}
\]  \hspace{1cm} (3.2.1)

and \(\tilde{Z} \in \Gamma(\ker \bigwedge^k p_*)(U)\).

If one of these conditions is satisfied for \(Z\), then \(Z'(x') = \bigwedge^k p_*(Z(x))\), \(x' \in M', x \in p^{-1}(x')\), is a correctly defined multivector on \(M'\). Moreover, if \((p(U), \{y^1, \ldots, y^{m'}\})\) is the induced local coordinate system on \(M'\), then the corresponding local expression for \(Z'\) coincides with (3.2.2).

Proof. In order to prove the last assertion it is sufficient to note that for any two points \(x_1, x_2 \in p^{-1}(x)\) there exist \(X_1, \ldots, X_s \in \Gamma(\ker p_*)\) and \(t_1, \ldots, t_s \in \mathbb{R}\) such that \(\phi^{X_1}_{t_1} \circ \cdots \circ \phi^{X_s}_{t_s}(x_1) = x_2\) and then use the second condition.

Obviously, (ii) \(\Rightarrow\) (i). To prove the converse we choose a vector bundle direct decomposition \(T M = \ker p_* \oplus C\) such that \(Z \in \Gamma(C)\) if \(Z \notin \Gamma(\ker p_*)\) and \(C\) is arbitrary otherwise. Let \(\Pi : \Gamma(TM) \to \Gamma(C)\) be a projection on \(\Gamma(C)\) along \(\Gamma(\ker p_*)\). Then

\[
\frac{d}{dt}\Pi(\phi^X_{t_*} Z - Z) = \Pi \frac{d}{dt}(\phi^X_{t_*} Z - Z) = \Pi(-\phi^X_{t_*}[X, Z]) = 0
\]

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we have used the equality \( \frac{d}{dt} \phi^*_t Z = -\phi^*_t [X, Z] \) and the fact that \([X, Z] = \mathcal{L}_X Z\), see [16]). Thus \( \Pi(\phi^*_t Z - Z) \) is a constant with respect to \( t \) multivector and, since \( \Pi(\phi^*_t Z - Z)|_{t=0} = \Pi(0) = 0 \), we deduce that \( \Pi(\phi^*_t Z - Z) \equiv 0 \).

The equivalence \((i) \iff (iii)\) follows from the local expression

\[
[X, Z]^{i_1 \ldots i_k} = \frac{1}{k!} \epsilon_{s_1 \ldots s_k} X^{r} \partial_{r} Z^{s_1 \ldots s_k} - \frac{1}{(k-1)!} \epsilon_{s_1 \ldots s_k} Z^{r s_2 \ldots s_k} \partial_{r} X^{i_1 \ldots i_k}
\]

(3.2.3)

for the Schouten bracket ([16]). Indeed, if one applies (3.2.3) to the local coordinate system from condition \((iii)\) one finds that \( L_{X} Z \in \Gamma(\ker \bigwedge^{k} p_\ast) \) if and only if (3.2.1) holds. q.e.d.

3.3. Definition We say that a multivector \( Z \in \Gamma(\bigwedge^{k} TM) \) is projectable or admits the push-forward if one of the conditions of Theorem 3.2 is satisfied. The push-forward, which will be denoted by \( Z' \), is a uniquely defined multivector from \( \Gamma(\bigwedge^{k} T M') \), see Theorem 3.2.

3.4. Definition A complex multivector \( Z \in \Gamma(\bigwedge^{k} T C M) \) admits the push-forward \( Z' \in \Gamma(\bigwedge^{k} T C M') \) if the multivectors \( \text{Re} Z, \text{Im} Z \in \Gamma(\bigwedge^{k} T M) \) do so and \( Z' = (\text{Re} Z') + i(\text{Im} Z') \).

3.5. Corollary Let \( c \) be a (complex) bivector on \( M \) admitting the push-forward \( c' \in \Gamma(\bigwedge^{2} T M') \) \( (c' \in \Gamma(\bigwedge^{2} T C M')) \). Then for any \( x' \in M' \) and any \( x \in p^{-1}(x') \) the following conditions hold:

\[(i) \] the subspace \( p_{\ast, x}(T_x \mathcal{K}^\perp) \subset T_{x'} M' \) \( (p_{\ast, x}(T_x \mathcal{K}^\perp) \subset T_{x'} C M') \), where \( \perp \) is the annihilator sign, is independent of \( x \);

\[(ii) \] the kernel of the map \( p_{\ast, x}|_{c'(T_x \mathcal{K}^\perp)} \) \( (p_{\ast, x}|_{c'(T_x \mathcal{K}^\perp)}) \) equals \( c'(T_x \mathcal{K}^\perp) \cap T_x \mathcal{K} \) \( (c'(T_x \mathcal{K}^\perp) \cap T_x C \mathcal{K}) \);

\[(iii) \]

\[
\begin{align*}
P_{c', x'} & \cong c'(T_x \mathcal{K}^\perp) / (c'(T_x \mathcal{K}^\perp) \cap T_x \mathcal{K}) \\
(P_{c', x'} & \cong c'(T_x C \mathcal{K}^\perp) / (c'(T_x \mathcal{K}^\perp) \cap T_x C \mathcal{K}))
\end{align*}
\]

Proof. \((iii)\) follows from \(i) \) and \( ii) \). These last are consequences of Theorem 3.2, condition \((iii)\). q.e.d.
3.6. Remark Although \( \dim c^2((T_xK)^\perp)/(T_xK)^\perp \cap T_xK \) is constant along \( K \), dimensions of \( c^2((T_xK)^\perp)/(T_xK)^\perp \cap T_xK \) may not be so. For example, let \( p : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) be the projection \((x_0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)\) and let \( c = x_0\partial_{x_0} \wedge \partial_{x_1} + \partial_{x_2} \wedge \partial_{x_3} \). Then \( c \) is projectable, but dimension of \( c^2((T_xK)^\perp) = \text{Span}\{\partial_{x_2}, \partial_{x_3}, x_0\partial_{x_0}\} \) jumps at 0.

3.7. Theorem Let a bivector \( Z_i \in \Gamma(\wedge^2 TM) \) admit the push-forward \( Z'_i \in \Gamma(\wedge^2 TM'), i = 1, 2 \). Then a trivector \( Z = [Z_1, Z_2] \in \Gamma(\wedge^3 TM) \) admits the push-forward \( Z' \in \Gamma(\wedge^3 TM') \) and \( Z' = [Z'_1, Z'_2] \).

Proof. In any local coordinate system as in condition (iii) of Theorem 3.2 \( Z_i \) can be written in the form
\[
Z_i(x, y) = Z'_i(y) + \tilde{Z}_i(x, y),
\]
where \( Z'_i(y) = Z^{jk}_i(y)\partial_{y^j} \wedge \partial_{y^k} \) and \( \tilde{Z}_i \in \Gamma(\ker \wedge^2 p_*)(U) \). By formula (1.2.1)
\[
[Z_1, Z_2](x, y) = [Z'_1, Z'_2](y) + \tilde{Z}(x, y),
\]
where \( \tilde{Z} \in \Gamma(\ker \wedge^3 p_*)(U) \). Thus by Theorem 3.2 \( Z \) admits the push-forward \( Z' \) and \( Z' = [Z'_1, Z'_2] \). q.e.d.

3.8. Corollary Let \((c_1, c_2)\) be a Poisson pair on \( M \) such that \( c_i \) admits the push-forward \( c'_i \in \Gamma(\wedge^2 TM'), i = 1, 2 \), and \( c'_1, c'_2 \) are linearly independent. Then \((c'_1, c'_2)\) is a Poisson pair on \( M' \).

Proof follows immediately from Proposition 2.2 and Theorem 3.2. q.e.d.

3.9. Corollary Let \((M, \omega)\) be a complex symplectic manifold and let \( G \) be a real Lie group acting on \( M \) by biholomorphic symplectomorphisms. Assume that \( M/G \) is a manifold. Then \( c_1 = \text{Re}(\omega^{-1}), c_2 = \text{Im}(\omega^{-1}) \) (see Example 2.7) admit the push-forwards \( c'_1, c'_2 \in \Gamma(\wedge^2 T(M/G)) \) and \((c'_1, c'_2)\) is a Poisson pair on \( M/G \), provided that \( c'_1, c'_2 \) are linearly independent.

Proof. It is sufficient to observe that: a) \( \mathcal{L}_{X_i} c_i = 0, i = 1, 2 \), for generators \( X_1, \ldots, X_i \in \Gamma TM \) of the \( G \)-action; b) an arbitrary vector \( X \in \Gamma(kerp_*) \), where \( p : M \rightarrow M/G \) is a natural projection, is expressed as \( X = a^j X_j \) for some \( a^j \in \mathcal{E}(M) \) and \( \mathcal{L}_X c_i = [a^j X_j, c_i] = [a^j, c_i] \wedge X_j \in \Gamma(\ker \wedge^2 p_*), i = 1, 2 \) (we have used the standard properties of the Schouten bracket, see [16],p.454). q.e.d.
3.10. Definition Let $p : M \rightarrow M'$ be as in 3.1. A bihamiltonian structure $(J, c_1, c_2)$ on $M$ is called projectable (via $p$) if the bivectors $c_1, c_2$ are so and their push-forwards $c'_1, c'_2$ are linearly independent. The bihamiltonian structure generated by $c'_1, c'_2$ on $M'$ will be denoted by $J'$ and will be called the push-forward or reduction of $J$.

3.11. Definition Let $p : M \rightarrow M'$ be as in 3.1 and let $J$ be a projectable bihamiltonian structure. We say that the triple $(M, J, K)$ is a realization of $J'$. If, moreover, $J$ is (holomorphic) symplectic (Definitions 2.6, 2.9), we call $(M, J, K)$ (holomorphic) symplectic realization.

4 From symplectic to complete

Let $p : M \rightarrow M'$ be as in 3.1 and let $J$ be a projectable symplectic bihamiltonian structure on $M$ with the push-forward $J'$. In this section we discuss some conditions on the triple $(M, J, K)$ that guarantee the completeness of $J'$.

By Corollary 3.5, iii) and by the definition of completeness (2.17) our considerations should start from linear algebra.

4.1. So let $V$ be a vector space over $\mathbb{C}$ and let $c_1, c_2 \in \bigwedge^2 V$ be such that the bihamiltonian structure $J = \{c_\lambda\}_{\lambda \in \mathbb{C}^2}, c_\lambda = \lambda_1 c_1 + \lambda_2 c_2, \lambda = (\lambda_1, \lambda_2)$, where $c_i$ is considered a constant complex bivector field, is symplectic (Definition 2.6). Also, let $K \subset V$ be a subspace such that the push-forwards $c'_1, c'_2 \in \bigwedge^2 (V/K)$ (via the canonical projection $p : V \rightarrow V/K$) are linearly independent.

Set $R_0 = \max_{\lambda \in \mathbb{C}^2} \text{rank } c_\lambda, R'_0 = \max_{\lambda \in \mathbb{C}^2} \text{rank } c'_\lambda$ and

$$d_\lambda = \dim c^\ast_\lambda(K^\perp)/(K \cap c^\ast_\lambda(K^\perp)).$$

Write $\Lambda_1 = \text{Span}_\mathbb{C}\{\tilde{\lambda}_1\}, \ldots, \Lambda_s = \text{Span}_\mathbb{C}\{\tilde{\lambda}_s\}$ for the complex lines in $\mathbb{C}^2$ on which $\text{rank } c_\lambda$ is less than maximal.

4.2. Proposition The condition of completeness

(**) $\text{rank } c'_\lambda = R'_0$ for any $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$ holds if and only if $d_\lambda$ is independent of $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$.

Proof follows from Corollary 3.5. q.e.d.
4.3. Corollary \[ \Lambda = \bigcup_{i=1}^{s} \Lambda_i \] and \( k_{\Lambda} = \dim K \cap c_{\Lambda}^{\perp}(K^{\perp}) \). Assume that \( K^{\perp} \cap \ker c_{\Lambda}^{\perp} = \{0\}, i = 1, \ldots, s \). Then the condition \((**)\) of Proposition 4.2 holds if and only if

(i) \( k_{\Lambda} = k \) is constant over \( \lambda \in \mathbb{C}^2 \setminus \Lambda \);

(ii) \( k_{\Lambda_{1}} = \cdots = k_{\Lambda_{s}} = k \).

**Proof.** Since \( K^{\perp} \cap \ker c_{\Lambda}^{\perp} = \{0\} \), then \( \dim c_{\Lambda}^{\perp}(K^{\perp}) = \dim c_{\Lambda}^{\perp}(K^{\perp}) \), \( i = 1, \ldots, s \), where \( \lambda \not\in \Lambda \). Thus conditions (i), (ii) are equivalent to the constancy of \( d_{\lambda} \) over \( \lambda \neq 0 \). q.e.d.

The following theorem gives the necessary and sufficient conditions for the completeness of the reduction \( J' \) of a holomorphic symplectic bihamiltonian structure \( J \) under an additional assumption. Namely, the foliation \( K \) of the leaves of the projection \( p \) is supposed to be a generic CR-foliation.

Let \( \tilde{\lambda}_1 = 1 + i, \tilde{\lambda}_2 = 1 - i \) and let \( \Lambda \) denote the cross \( \text{Span}_{\mathbb{C}}\{\tilde{\lambda}_1\} \cup \text{Span}_{\mathbb{C}}\{\tilde{\lambda}_2\} \subset \mathbb{C}^2 \). Given a 2-form \( \omega \in \Gamma(\wedge^2(T^\mathbb{C}M)^*) \), write \( \omega_{\lambda} \) for \( \lambda \text{Re} \omega + \lambda_2 \text{Im} \omega \).

4.4. Theorem \( (M, \omega) \) be a complex symplectic manifold with the corresponding holomorphic symplectic bihamiltonian structure \( J \) (see Definition 2.9) and let \( p : M \longrightarrow M' \) be as in 3.1. Assume that the foliation \( K \) is a generic CR-foliation on \( M \) and that \( c = \omega^{-1} \) admits the push-forward \( c' \in \Gamma(\wedge^2 T^\mathbb{C}M') \) linearly independent with \( c \). For \( x' \in M', x \in p^{-1}(x') \), and \( \lambda \in \mathbb{C}^2 \setminus \Lambda \) set

\[
\begin{align*}
 k_{\Lambda}^x &= \dim T^{\mathbb{C}}_x K \cap (T^{\mathbb{C}}_x K)^{\perp} \omega_{\lambda}(x) \\
 k^x &= \dim T^{1,0}_x K \cap (T^{1,0}_x K)^{\perp} \omega(x)
\end{align*}
\]

Assume that these numbers are constant along \( K \) (cf. Remark 3.6) and set \( k_{\Lambda}^{x'} = k_{\Lambda}^x, k^{x'} = k^x \).

Then the reduction \( J' \) of \( J \) via \( p \) is complete at a point \( x' \in M' \) if and only if

(i) \( k_{\Lambda}^{x'} = k \) is constant in \( \lambda \);

(ii) \( k^{x'} = k \)

(iii) \( k = \min_{y' \in M'} k_{\Lambda}^{y'} \).
Proof. If $W$ is a real vector space with a complex structure $\mathcal{J}$ and $Y \subset W$ a subspace, let $W_{1,0} = \{w - i\mathcal{J}w; w \in W\} \subset W^C$ and let $Y_{1,0} = Y^C \cap W_{1,0} = \{y - i\mathcal{J}y; y \in Y \cap \mathcal{J}Y\}$ (cf. 1.16).

Put $V = T^C_xM, K = T^C_x\mathcal{K}$. We claim that the assumptions of Corollary 4.3 are satisfied. Indeed, by Theorem 2.8 $\Lambda$ is appropriate since the only, up to rescaling, degenerate bivectors from family $J$ are $c$ and $\bar{c}$. On the other hand the condition $T_x\mathcal{K} + J T_x\mathcal{K} = T_x\mathcal{M}$ means equalities $(T_x\mathcal{K})^\perp \cap \mathcal{J}^*(T_x\mathcal{K})^\perp = \{0\}$, where $\mathcal{J}^* : T^* M \to T^* M$ is adjoint to the complex structure operator $\mathcal{J}$ on $T M$. This means equalities $((T_x\mathcal{K})^\perp)^{1,0} = \{0\} = ((T_x\mathcal{K})^\perp)^{0,1}$ equivalent to $K^\perp \cap T_x^{1,0}M = 0 = K^\perp \cap T_x^{0,1}M$. Recalling that $T_x^{1,0}M = \ker c$ and $T_x^{0,1}M = \ker \bar{c}$ we get the claim.

Now, put $k_\lambda = k_\lambda^x$ and $k_{\tilde{\lambda}} = k_{\tilde{\lambda}_x} = k_{\lambda'}$ and apply Corollary 4.3. Conditions (i) and (ii) are equivalent to the constancy of rank for $c_\lambda \in J'(x^\lambda), \lambda \neq 0$. Its maximality is guaranteed by (iii). q.e.d.

4.5. Corollary In the assumptions of Theorem 4.4 suppose that $\mathcal{K}$ is completely real (Definition 1.18). Then $J'$ is not complete.

Proof. Assume the contrary. By condition (ii) corank of any $c' \in J' \setminus \{0\}$ is 0. This contradicts with the definition of completeness. q.e.d.

Given a complete bihamiltonian structure $J'$ on $M'$, consider its all realizations with $\mathcal{K}$ being a generic $\mathcal{CR}$-foliation. Then the smallest realizations in this class will be characterized by the biggest intersection $T^{1,0}\mathcal{K} \cap (T^{1,0}\mathcal{K})^\perp = (T^{1,0}\mathcal{K})^\perp = T^{0,1}\mathcal{K}$.

4.6. Definition Let $J'$ be a complete bihamiltonian structure on $M'$. Its realization $(M, \omega)$ is called minimal if $T^{1,0}\mathcal{K} \cap (T^{1,0}\mathcal{K})^\perp = T^{1,0}\mathcal{K}$, i.e. $\mathcal{K}$ is a $\mathcal{CR}$-isotropic foliation (Definition 1.19).

We shall give another characterization of the minimal realizations below.

4.7. There is a natural $\mathcal{CR}$-coisotropic foliation $\mathcal{L} \supset \mathcal{K}$ associated with any realization $J$ on $(M, \omega)$ of a complete $J'$. This foliation is built as follows. Consider the "real form" $J'_R$ of $J'$, i.e. the following real bihamiltonian structure on $M'$

$$\{\lambda_1 \Re c' + \lambda_2 \Im c'\}_{\lambda = (\lambda_1, \lambda_2) \in \RR^2},$$

where $c' = p_* c, c = (\omega)^{-1}$. Now take the foliation $\mathcal{L}' = \bigcap_{\lambda \in \RR^2 \setminus \{0\}} S'_\lambda, S'_\lambda$ being the foliation of the symplectic leaves for $c'_\lambda \in J'_R$. Since $J'$
is complete, \( \mathcal{L}' \) is a lagrangian foliation (see Proposition 2.14). The equations for \( \mathcal{L}' \) are the functions from the involutive family \( \mathcal{F}_0' \) (see 2.10). We define \( \mathcal{L} \) as \( p^{-1}(\mathcal{L}') \). Note that it is CR-coisotropic due to the fact that its equations \( f \in p^{-1}(\mathcal{F}_0') \) are in involution with respect to \( c \).

4.8. Theorem A realization \( J \) of a complete bihamiltonian structure \( J' \) is minimal if and only if the foliation \( \mathcal{L} \) is a CR-lagrangian foliation.

Proof. Let \( 2n, r \) denote rank and corank of the bivector \( c' \in J' \), respectively, and let \( \dim \mathbb{C} M = 2N \). By Definiton 4.6 and Theorem \ref{thm:4.10} \( J \) is minimal if and only if \( r = d \), where \( d \) is CR-dimension of the leaves of \( \mathcal{K} \). On the other hand, since \( \mathcal{K} \) is generic, CR-codimension of the leaves is equal to their real codimension, hence \( 2N - d = 2n + r \). Thus the minimality of \( J \) is equivalent to the equality \( n + r = N \) that is necessary and sufficient for \( \mathcal{L} \) to be CR-lagrangian.

5 Canonical complex Poisson pair associated with complexification of Lie algebra

5.1. Let \( g_0 \) be a nonabelian Lie algebra over \( \mathbb{R} \) and let \( g = g_0^\mathbb{C} \) be its complexification. All our further results can be formulated and proved using \( g_0, g \) only. But we introduce the corresponding Lie groups for the convenience.

So let \( G_0 \) be a connected simply connected Lie group with the Lie algebra \( \text{Lie}(G_0) = g_0 \) and let \( G \) be a connected simply connected complex Lie group with \( \text{Lie}(G) = g \). One can consider \( G_0 \) as a real Lie subgroup in \( G \) (see \cite{5}, III.6.10).

Let \( g_0^\ast, g^\ast \) stand for the dual spaces. Fix a basis \( e_1, \ldots, e_n \) in \( g_0 \); let \( c_{ij}^k \) be the corresponding structure constants and let \( z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n \) be the complex linear coordinates in \( g^\ast \) associated to the dual basis in \( g^\ast \supset g_0^\ast \). There are the standard linear bivectors \( c = c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \) in \( g^\ast \) and \( c_0 = c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \) in \( g_0^\ast \). They can be defined intrinsically for instance as the maps

\[
\begin{align*}
g^\ast \xrightarrow{c} g^\ast \wedge g^\ast, & \quad g_0^\ast \xrightarrow{c} g_0^\ast \wedge g_0^\ast
\end{align*}
\]

dual to the Lie brackets \([,] : g \wedge g \longrightarrow g\) and \([,]_0 : g_0 \wedge g_0 \longrightarrow g_0\).
It is well-known that the symplectic leaves of \( c_0 \) (\( c \)) are the coadjoint orbits for \( G_0 \) (\( G \)). Also, there is a natural action of \( G_0 \) on \( g^* \):

\[
g(a + ib) = Ad^* g(a) + i Ad^* g(b), \quad g \in G_0, \quad a, b \in g_0.
\]

Let \( \text{Sing} \ g^*, \text{Sing} \ g^*_{0} \) be unions of symplectic leaves of nonmaximal dimension for \( c, c_0 \), respectively.

5.2. **Proposition** The sets \( \text{Sing} \ g^*, \text{Sing} \ g^*_{0} \) are algebraic. Moreover, \( \text{Sing} \ g^* \) is the complexification of \( \text{Sing} \ g^*_{0} \), i.e. the defining polynomials for \( \text{Sing} \ g^* \) have real coefficients and are obtained from the defining polynomials of \( \text{Sing} \ g^*_{0} \) by substitution \( \{x_1, \ldots, x_n\} \mapsto \{z_1, \ldots, z_n\} \).

Consequently, \( \text{codim}_R \text{Sing} \ g^* \leq \text{codim}_R \text{Sing} \ g^*_{0} \).

**Proof.** These polynomials are minors of \( m \)-th order of the \( n \times n \)-matrix \( \begin{vmatrix} c_{ij} \end{vmatrix} \), where \( m = \text{rank} \, c \). q.e.d.

5.3. **Convention** In the sequel we shall assume that the nonabelian Lie algebra \( g_0 \) satisfies condition \( \text{codim}_R \text{Sing} \ g_0 \geq 2 \).

5.4. This condition is satisfied by a wide class of Lie algebras including the semisimple ones. Indeed, in the semisimple case we can identify \( g^* \) and \( g \) by means of the Killing form. On the other hand, it is well known that the algebraic set of all nonregular (regular means semisimple contained in the unique Cartan subalgebra) elements is at least of codimension three and contains \( \text{Sing} \ g^* \).

5.5. **Definition** Let us introduce a set

\[
C = \{ z \in g^*; \exists (\lambda_1, \lambda_2) \in C^2 \setminus \{(0, 0)\}, \lambda_1 z + \lambda_2 \bar{z} \in \text{Sing} \ g^* \},
\]

where the bar stands for the complex conjugation corresponding to \( g_0 \subset g \), and call it the incompleteness set (see 5.9 for the explanation of this terminology).

5.6. **Proposition** The incompleteness set \( C \) is a real algebraic set of positive codimension.
Proof. We use the product $\Pi = g^* \times (C^2 \setminus \{(0, 0)\})$ with the coordinates $z_1, \ldots, z_n, \lambda_1, \lambda_2$ and the real algebraic map $\phi : \Pi \rightarrow g^*$ given by the formula

$$(z_1, \ldots, z_n, \lambda_1, \lambda_2) \mapsto (\lambda_1 z_1 + \lambda_2 \bar{z}_1, \ldots, \lambda_1 z_n + \lambda_2 \bar{z}_n).$$

The set $C$ can be regarded as $pr_1(\phi^{-1}(\text{Sing } g^*))$, where $pr_1$ is the projection onto $g^*$.

To prove that $C$ is of positive codimension we consider $C$ as the union of all complex 2-dimensional subspaces $S_{a,b} \subset g^*$ that are generated by pairs $a, b \in g_{0}^*$ and have a nonzero intersection with $\text{Sing } g^*$. If $S = S_{a,b}$ is such a subspace, then $S = S_0^c$ for $S_0 = \{\lambda_1 a + \lambda_2 b; (\lambda_1, \lambda_2) \in \mathbb{R}^2\}$ and $S \cap \text{Sing } g^*$ can be described by the same equations with real coefficients as $S_0^c \cap \text{Sing } g_0^*$ in $g_0^*$.

On the other hand, since $\text{codim } \text{Sing } g_0^* \geq 2$, the cone

$$\overline{a, \text{Sing } g_0^*}$$

of all $S_0 \ni a, S_0 \cap \text{Sing } g_0^* \neq \{0\}$, has codimension at lest 1 in $g_0^*$. Thus for a fixed $a \in g_0^* \setminus \text{Sing } g_0^*$ one can always find $b \in g_0^*$ such that the subspace $S_0 \subset g_0^*$ generated by $a, b$ intersects $\text{Sing } g_0^*$ only at 0, i.e. $S_0^c \cap \text{Sing } g^* = \{0\}$. q.e.d.

5.7. Example Let $g_0 = so(3), g = sl(2, \mathbb{C}) \cong \mathbb{C}^3$. Then $\text{Sing } g^* = \{0\}$, $C = \{z \in g^*; z \text{ linearly independent with } \bar{z}\}$; consequently $C$ is described by two real equations: $z_1 \bar{z}_2 - z_2 \bar{z}_1 = 0, z_1 \bar{z}_3 - z_3 \bar{z}_1 = 0$. The set $\{z \in g^*; z \text{ linearly independent with } \bar{z}\}$ is contained in $C$ for arbitrary $g$.

5.8. Now, we shall introduce a remarkable pair of complex bivectors on $g^*$ playing the crucial role in the sequel of the paper. This pair is $(c, \tilde{c})$, where $c$ is as in 5.1 and $\tilde{c}$ is given by $\tilde{c} = c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. One can define $\tilde{c}$ intrinsically by the diagram

$$
\begin{array}{ccc}
g^* & \overset{c}{\longrightarrow} & g^* \wedge g^* \\
\uparrow \tilde{c} & & \uparrow \tilde{c} \\
g^* & = & g^*,
\end{array}
$$

where $c$ is from (5.1.1) and $\tilde{c}$ stands for the complex conjugation corresponding to the real form $g_0 \subset g$. 

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5.9. Proposition  
(i) \( \tilde{c} \) is \( G_0 \)-invariant;
(ii) \( (c, \tilde{c}) \) is a complex Poisson pair;
(iii) \( (c, \tilde{c}) \) is complete at any point \( z \in g^* \setminus \mathcal{C} \) (see Definition 2.17).

Proof. (i), (ii) are obtained by direct calculations. The last assertion follows from Theorem 5.6 since the set \( \mathcal{C} \) consists precisely of the points of incompleteness for \( (c, \tilde{c}) \). Indeed, \( \text{rank}(\lambda_1 z + \lambda_2 \bar{z}) = \text{rank} \left( \frac{\partial}{\partial z_k} \left( \frac{\partial \tilde{g}}{\partial \bar{z}_j} \right) \right) \) is less than maximal if and only if \( \lambda_1 z + \lambda_2 \bar{z} \in \text{Sing} g^* \).

q.e.d.

The end of this section is devoted to the study of the first integrals (Definition 2.12) for the bihamiltonian structure \( (J, c, \tilde{c}) \).

5.10. Definition  
Let \( r = \text{corank} c \) (codimension of symplectic leaf of maximal dimension). Let us write \( \text{rank} g \) for \( r \) and call this number the rank of \( g \).

Note that for the semisimple case this notion of rank coincides with the standard one, i.e. with dimension of a Cartan subalgebra.

5.11. Definition  
Let \( Z_{hol}^c(U) \) denote the space of holomorphic Casimir functions for \( c \) over an open set \( U \subset g^* \).

An open set \( U \subset g^* \setminus \text{Sing} g^* \) is called admissible if there exist \( r = \text{rank} g \) functionally independent functions from \( Z_{hol}^c(U) \).

5.12. Proposition  
Let a set \( U \) be admissible. Given a function \( g \in Z_{hol}^c(U) \), define a function \( \tilde{g} \) over \( U \) by the formula \( \tilde{g} = (\frac{\partial g}{\partial z}) \). Then the space \( Z_{\tilde{c}}(U) \) of (smooth) Casimir functions for \( \tilde{c} \) is equal to \( \{ \tilde{g}; g \in Z_{hol}^c(U) \} \cup \mathcal{O}(U) \), where \( \mathcal{O}(U) \) is the space of antiholomorphic functions over \( U \).

Proof. The following calculation shows that \( \tilde{g} \in Z_{\tilde{c}}(U) \):
\[
\tilde{c}(\tilde{g})_j = c_{ij} \bar{z}_k \frac{\partial \tilde{g}}{\partial \bar{z}_k} = c_{ij} \zeta_k \left( \frac{\partial g}{\partial z_k} \right) = \tilde{c}(g)_j = 0
\]
(here \( v_j \) stands for \( j \)-th component of a vector field \( v = v_i \frac{\partial}{\partial z_i} \)).

Now, let \( g_1, \ldots, g_r \in Z_{hol}^c(U) \) be functionally independent. We note that the \((1,0)\)-differentials \( \partial \tilde{g}_1, \ldots, \partial \tilde{g}_r \) are linearly independent precisely at those points where \( \partial g_1, \ldots, \partial g_r \) are. Thus by the dimension arguments (rank \( \tilde{c} = \text{rank} c \)) the functions \( \tilde{g}_1, \ldots, \tilde{g}_r \) together with the antiholomorphic functions functionally generate the space \( Z_{\tilde{c}}(U) \). q.e.d.
5.13. Definition Define $\phi_\lambda: \mathfrak{g}^* \to \mathfrak{g}^*, \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ by $\phi_\lambda(z) = \lambda_1 z + \lambda_2 \bar{z}$. This is a $\mathbb{R}$-linear isomorphism if $|\lambda_1| \neq |\lambda_2|$ and an epimorphism onto an n-dimensional $(n = \dim \mathfrak{g})$ subspace otherwise.

An open set $U \subset \mathfrak{g}^* \setminus \mathcal{C}$ is called $\lambda$-admissible if the set $\phi_\lambda(U)$ has an admissible neighbourhood.

An open set $U \subset \mathfrak{g}^* \setminus \mathcal{C}$ is called strongly admissible if it is $\lambda$-admissible for any $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$.

5.14. Proposition Let a set $U \subset \mathfrak{g}^*$ be $\lambda$-admissible and let $U_\lambda$ be an admissible neighbourhood of $\phi_\lambda(U)$.

Then the space $Z_{c_\lambda}(U)$ of (smooth) Casimir functions for $c_\lambda = \lambda_1 c + \lambda_2 \bar{c}, \lambda_1 \neq 0$, is equal to $\{g \circ \phi_\lambda|_U; g \in Z_{c_\lambda}^{hol}(U_\lambda)\} \cup \overline{\mathcal{O}(U)}$.

Proof. Again, let $g_1, \ldots, g_r \in Z_{c_\lambda}^{hol}(U_\lambda)$ be functionally independent. Obviously, the functions $g_{\lambda,1} = g_1 \circ \phi_\lambda, \ldots, g_{\lambda,r} = g_r \circ \phi_\lambda$ are Casimir functions for $c_\lambda$. They are functionally independent on $U$ since the Jacobi matrices $D = \frac{\partial (g_{\lambda,1},\ldots,g_{\lambda,r})}{\partial (z_1,\ldots,z_n)}$ and $D_\lambda = \frac{\partial (g_{\lambda,1} \circ \phi_\lambda,\ldots,g_{\lambda,r} \circ \phi_\lambda)}{\partial (\bar{z}_1,\ldots,\bar{z}_n)}$ are related as follows $D_\lambda(z) = \lambda_1 D \circ \phi_\lambda(z)$.

So $g_{\lambda,1}, \ldots, g_{\lambda,r}$ and $\overline{\mathcal{O}(U)}$ generate $Z_{c_\lambda}(U)$. q.e.d.

The following proposition shows that strongly admissible sets exist and describes them in the semisimple case.

5.15. Proposition (i) Let $\| \cdot \|$ be a norm in $\mathfrak{g}^*$ Then any open set $U$ with a sufficiently small diameter $\text{diam} U = \sup_{z,z' \in U} ||z - z'||$ is strongly admissible.

(ii) Assume that $\mathfrak{g}$ is semisimple. Then any open set $U \subset \mathfrak{g}^* \setminus \mathcal{C}$ is strongly admissible.

Proof. (i) We start from the following claim: if a set $U$ is admissible, then the set $\lambda U = \{\lambda u; u \in U\}$ is so for any $\lambda \in \mathbb{C} \setminus \{0\}$. Indeed, the bivector $c$ is homogeneous with homogeneity degree 1: $h_{c,\lambda} c = \lambda c$, where $h_{c}(z) = \lambda z$. Hence, if $g_1, \ldots, g_r$ are independent Casimir functions for $c$ over $U$, then $(h_{c,\lambda}^*)^{-1} g_1, \ldots, (h_{c,\lambda}^*)^{-1} g_r$ are so over $h_{c}(U) = \lambda U$.

Now, assume that the norm is so chosen that $||z|| = ||\bar{z}||$. Then the inequality $||\lambda_1 z + \lambda_2 \bar{z} - \lambda_1 z' - \lambda_2 \bar{z}'|| \leq ||\lambda_1|| ||z - z'|| + ||\lambda_2|| ||\bar{z} - \bar{z}'||$
shows that
\[
\text{diam } \phi_\lambda(U) \leq (|\lambda_1| + |\lambda_2|) \text{diam } U,
\] (5.15.1)
where \( \phi_\lambda \) is from Definition 5.13.

Next, choose a point \( z \in U \) such that \( z \) is linearly independent with \( \bar{z} \) and consider the map \( \mathbb{C}^2 \ni \lambda \mapsto \phi_\lambda(z) \in \mathfrak{g}^* \). The image of the unit sphere \( S^1 = \{ |\lambda_1|^2 + |\lambda_2|^2 = 1 \} \) under this map can be covered by a finite number of admissible balls \( B_1, \ldots, B_m \). Inequality 5.15.1 shows that shrinking \( U \) if needed one can get the following
\[
\phi_\lambda(U) \subset \bigcup_{i=1}^m B_i \forall \lambda \in S^1.
\]
Hence for sufficiently small \( U \ni x \) the set \( \phi_\lambda(U) \), where \( \lambda \neq 0 \). Since all norms on \( \mathfrak{g}^* \) are equivalent this completes the proof.

(ii) It is enough to note that there exists a set \( g_1(z), \ldots, g_r(z), r = \text{rank } \mathfrak{g} \) of global holomorphic Casimir functions for \( c \) that are functionally independent on \( \mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^* \). One can identify \( \mathfrak{g} \) and \( \mathfrak{g}^* \) by means of the Killing form and take for \( g_1, \ldots, g_r \) an algebraic basis of the ring of \( G \)-invariant polynomials on \( \mathfrak{g} \). The functional independence of these functions on \( \mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^* \) is established in Theorem 0.1 of [15]. q.e.d.

We summarize the above results in the following Proposition.

5.16. Proposition Let \( U \subset \mathfrak{g}^* \) be a strongly admissible set and let \( U_\lambda \) be an admissible neighbourhood of \( \phi_\lambda(U) \). The set of first integrals (see Definition 2.12) \( F_0(U) \) of \( (J, c, \tilde{c}) \) over \( U \) is generated by the sets \( Z_{\text{hol}}^c(U), \{ g; g \in Z_{\text{hol}}^c(U) \}, \{ g \circ \phi_\lambda(U); g \in Z_{\text{hol}}^\lambda(U) \}, \mathcal{O}(U) \), where \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_1, \lambda_2 \neq 0 \).

Proof follows from Propositions 5.12,5.14 and from the definition of the set \( F_0(U) \). q.e.d.

The following proposition will be crucial in the proof of our main result (Theorem 6.7). As usual, given a subspace \( V \subset (T^*_z \mathfrak{g}^*)^* \), we set \( V^{1,0} = V \cap (T^*_z \mathfrak{g}^*)^* \).

5.17. Proposition Let
\[
\mu(z) = \dim(\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0}
\]
and let
\[
\mu_\lambda(z) = \dim(\ker c(z))^{1,0} \cap (\ker c_\lambda(z))^{1,0}
\]
for \( c_\lambda = \lambda_1 c + \lambda_2 \tilde{c}, \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_1, \lambda_2 \neq 0 \).

Then
(i) \( \mu(z) = \mu_\lambda(z) \) for any \( z \in g^* \setminus \mathcal{C} \);

(ii) there exists a real algebraic set \( \mathcal{R}, \mathcal{C} \subset \mathcal{R} \subset g^* \), where \( \mathcal{C} \) is the incompleteness set (see Definition 5.3), such that \( \mu(z) = \mu \) is constant and minimal on \( g^* \setminus \mathcal{R} \) and \( \mu(z) > \mu \) for \( z \in \mathcal{R} \);

(iii) if \( g \) is semisimple the set \( \mathcal{R} \setminus \mathcal{C} \) is empty and \( \mu(z) \equiv 0 \) on \( g^* \setminus \mathcal{C} \).

**Proof.** (i) We shall use the completeness of the bihamiltonian structure \((J, c, \tilde{c})\) at any \( z \in g^* \setminus \mathcal{C} \) (see proof of Proposition 5.9). By Theorem 2.20 and Corollary 2.21 the pair \( c(z), \tilde{c}(z) \in \wedge^2 T_z^{1,0} g^* \) does not have the Jordan blocks in its decomposition. Thus we can use Proposition 2.24 to deduce that \( \mu(z) = \mu_\lambda(z) \).

(ii) To prove this condition it is sufficient to note that the subspace \((\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0}\) annihilates the sum of characteristic subspaces \( P_{c,z} + P_{\tilde{c},z} \). Put \( \mathcal{R} = \{ z \in g^* : \dim(P_{c,z} + P_{\tilde{c},z}) < m \} \), where \( m = \max_z \dim(P_{c,z} + P_{\tilde{c},z}) \). The defining polynomials for \( \mathcal{R} \) are the minors of \( m \)-th order of the \( 2n \times n \)-matrix

\[
\left| \begin{array}{cc}
c_{ij} & z_k \\
k_{ij} & z_k
\end{array} \right| .
\]

If the above defined set \( \mathcal{R} \) lies in \( \mathcal{C} \), let us change the definition and put \( \mathcal{R} = \mathcal{C} \).

It remains to prove the inclusion \( \mathcal{C} \subset \mathcal{R} \) in the case \( \mathcal{C} \not\supset \mathcal{R}, \mathcal{C} \not= \mathcal{R} \).

Introduce a set \( \mathcal{R}_\lambda = \{ z \in g^* : \dim(P_{c,z} + P_{\tilde{c},z}) < m_\lambda \} \), where \( m_\lambda = \max_z \dim(P_{c,z} + P_{\tilde{c},z}) \). Then by (i) \( \mathcal{R}_\lambda, \lambda_2 \not= 0 \) coincides with \( \mathcal{R} \) outside \( \mathcal{C} \). Since \( \mathcal{R} = Cl(\mathcal{R} \setminus \mathcal{C}) \) and \( \mathcal{R}_\lambda = Cl(\mathcal{R}_\lambda \setminus \mathcal{C}) \) (Zarisski closures), one gets \( \mathcal{R} = \mathcal{R}_\lambda \).

Now, let \( z \in \mathcal{C} \) and let \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_2 \not= 0 \) be such that \( \text{rank}(\lambda_1 c + \lambda_2 \tilde{c})(z) < R_0 \) (cf. the definition of completeness, 2.17). Then \( \dim(P_{c,z} + P_{\tilde{c},z}) < m_\lambda \). Consequently, \( \mathcal{C} \subset \mathcal{R}_\lambda = \mathcal{R} \). If the only (up to the proportionality) bivector of nonmaximal rank in the family \( \{ c_\lambda(z) \} \) is \( c \), then \( \dim(P_{c,z} + P_{\tilde{c},z}) < m \) and again \( \mathcal{C} \subset \mathcal{R}_\lambda = \mathcal{R} \).

(iii) It follows from the proof of Proposition 2.24 that \( \dim(\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0} \) is equal to dimension of the sum of the trivial Kronecker blocks for the pair \( c(z), \tilde{c}(z) \). If we consider \( c, \tilde{c} \) as elements of \( \Gamma(\wedge^2 T^{1,0} g^*) \), the same arguments as for the method of argument translation ([17], Theorem 4.1) show that in the semisimple case the trivial Kronecker blocks are absent for any \( z \in g^* \setminus \mathcal{C} \). q.e.d.
5.18. Definition We call the set $\mathcal{R}$ from Proposition 5.17 the Kronecker irregularity set and the number $\mu$ the trivial Kronecker dimension of the Lie algebra $\mathfrak{g}$.

The following example shows that for nonsemisimple Lie algebras the set $\mathcal{R} \setminus \mathcal{C}$ may be nonempty and the trivial Kronecker dimension may be nonzero.

5.19. Example Let $\mathfrak{g} = \text{Span}\{p_1, p_2, q_1, \ldots, q_4, f_1, \ldots, f_4, g_1, \ldots, g_4\}$ be a fourteen-dimensional Lie algebra with the standard linear Poisson bivector $c = \frac{\partial}{\partial p_1} \wedge (f_1 \frac{\partial}{\partial q_1} + \cdots + f_4 \frac{\partial}{\partial q_4}) + \frac{\partial}{\partial p_2} \wedge (g_1 \frac{\partial}{\partial q_1} + \cdots + g_4 \frac{\partial}{\partial q_4})$. Then $\mathcal{R}$ is given by one real equation

$$\begin{vmatrix}
  f_1 & f_2 & f_3 & f_4 \\
  g_1 & g_2 & g_3 & g_4 \\
  f_1 & f_2 & f_3 & f_4 \\
  g_1 & g_2 & g_3 & g_4
\end{vmatrix} = 0.$$

Also, $\text{Sing} \mathfrak{g}^*$ consists of the points where the vectors $f_1 \frac{\partial}{\partial q_1} + \cdots + f_4 \frac{\partial}{\partial q_4}, g_1 \frac{\partial}{\partial q_1} + \cdots + g_4 \frac{\partial}{\partial q_4}$ are linearly dependent, i.e. the defining equations for $\text{Sing} \mathfrak{g}^*$ are $f_1 g_2 - f_2 g_1 = 0, f_1 g_3 - f_3 g_1 = 0, f_1 g_4 - f_4 g_1 = 0$. However, the first part of the proof of Proposition 5.6 shows that $\text{codim}_\mathbb{R} \mathcal{C} \geq \text{codim}_\mathbb{R} \text{Sing} \mathfrak{g}^* - 4$; consequently, in our example $\text{codim}_\mathbb{R} \mathcal{C} \geq 6 - 4 = 2$ and $\mathcal{C} \neq \mathcal{R}$.

Here $\mu = 0$

Adding one more dimension to $\mathfrak{g}$ and retaining $c$ as above one gets $\mu = 1$. Also, $\mu$ will be nonzero for all reductive nonsemisimple Lie algebras.

Note that the above examples agree with our Convention 5.3.

6 CR-geometry of real coadjoint orbits

We retain the notations and conventions from the previous section.

6.1. Proposition (i) The bivectors $c_1 = \text{Re} \ c, c_2 = \text{Im} \ c$ are Poisson.

(ii) The coadjoint action of $G_\mathbb{R}$, where $G_\mathbb{R}$ is $G$ considered as a real Lie group, is hamiltonian with respect to $c_1, c_2$ (see Definition 1.21).

(iii) The generalized distribution of subspaces tangent to the $G_0$-orbits is generated by the vector fields $c(z_i) + \bar{c}(\bar{z}_i), i = 1, \ldots, n$. 

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Proof. (i) The more general statement that the pair $c_1, c_2$ is Poisson is proved by the same arguments as in Example 2.7.

(ii) First, we shall prove that the holomorphic coadjoint action of $G$ on $\mathfrak{g}^*$ is hamiltonian in holomorphic sense with respect to $c$. Consider the antirepresentation $Ad^* : G \to \mathfrak{g}^*$. The corresponding Lie algebra action $\rho : \mathfrak{g} \to \text{Vect}^{hol}(\mathfrak{g}^*)$, where $\text{Vect}^{hol}(\mathfrak{g}^*)$ is the Lie algebra of holomorphic vector fields on $\mathfrak{g}^*$, can be described as follows. The vector field $\rho(v), v \in \mathfrak{g}$ is equal to

$$z \mapsto ad^*(v)z : \mathfrak{g}^* \to \mathfrak{g}^* \cong T_{z^0}^1 \mathfrak{g}^*, v \in \mathfrak{g}.$$  

On the other hand, if $e_1, \ldots, e_n, c_{ij}^k$ are as in 5.1, then

$$< ad^*(e_i)z, e_j > = < z, ad(e_i)e_j > = < z, [e_i, e_j] > = < z, c_{ij}^ke_k > = c_{ij}^kz_k,$$

where we put $z = z_1e^1 + \cdots + z_ne^n$. Hence, $\rho(e_i) = c_{ij}^kz_k \frac{\partial}{\partial z_j} = c(z_i)$ and the corresponding antihomomorphism $\psi : \mathfrak{g} \to \mathcal{O}(\mathfrak{g}^*)$ (cf. Definition 1.21) is defined by $e_i \mapsto z_i, i = 1, \ldots, n.$

Now, the Lie algebra action $\rho_{\mathbb{R}} : \mathfrak{g}_{\mathbb{R}} \cong \mathfrak{g}_0 \oplus i\mathfrak{g}_0 \to \text{Vect}(\mathfrak{g}^*)$ corresponding to the antirepresentation $Ad^* : G_{\mathbb{R}} \to \mathfrak{g}^*$ is described by the formulas

$$< ad^*(e_i)z, e_j > = c_{ij}^kx_k, < ad^*(ie_i)z, e_j > = c_{ij}^ky_k$$

$$< ad^*(e_i)z, ie_j > = c_{ij}^ky_k, < ad^*(ie_i)z, ie_j > = -c_{ij}^kx_k.$$

Consequently,

$$\rho_{\mathbb{R}}(e_i) = c_{ij}^kx_k \frac{\partial}{\partial x_j} + c_{ij}^ky_k \frac{\partial}{\partial y_j} = (c + \bar{c})(z_i + \bar{z}_i) = (c - \bar{c})(z_i - \bar{z}_i)$$

and

$$\rho_{\mathbb{R}}(ie_i) = c_{ij}^ky_k \frac{\partial}{\partial x_j} - c_{ij}^kx_k \frac{\partial}{\partial y_j} = (c + \bar{c})(z_i - \bar{z}_i) = (c - \bar{c})(z_i + \bar{z}_i).$$

(iii) This condition follows from the proof of (ii) and from the obvious equality $(c + \bar{c})(z_i + \bar{z}_i) = c(z_i) + c(\bar{z}_i)$. q.e.d.

6.2. Proposition Let $\mathcal{O}$ be a $G_0$-orbit through $z_0 \in \mathfrak{g}^*$. Then $\mathcal{O}$ is a generic CR-manifold in the $G$-orbit $G(z_0)$;
Let $\mathfrak{g}$ be the Lie algebra. Then multiplication by $i$, is equal to $\mathcal{J}(c(z_j)) = i(c(z_j))$ for the stabilizer of $\{c(z_j), j = 1, \ldots, n\}$ (Proposition 6.1, (iii)), and that $\mathcal{J}$ acts on them as follows

$$\mathcal{J}(c(z_j) + \overline{c(z_j)}) = i(c(z_j) - \overline{c(z_j)}).$$

(6.2.1)

Thus $T\mathcal{O} + \mathcal{J}T\mathcal{O}$ is generated by the real and imaginary parts of the vector fields $c(z_j), j = 1, \ldots, n$, spanning $T^{1,0}G(z_0)$. Hence $T_z\mathcal{O} + \mathcal{J}T_z\mathcal{O} = T^c_zG(z_0)$. q.e.d.

The next proposition describes the CR-dimension of a $G_0$-orbit. This result will not be used in the sequel.

6.3. Proposition Let $\mathcal{O}$ be a $G_0$-orbit through $z_0 \in \mathfrak{g}^z$. Write $G^z (G_0^z)$ for the stabilizer of $z \in \mathfrak{g}^z$ in $G$ ($G_0$) and $\mathfrak{g}^z (\mathfrak{g}_0^z)$ for the corresponding Lie algebra. Then

$$T_z^{1,0}\mathcal{O} \cong \mathfrak{g}^z/\langle \mathfrak{g}_0^z \rangle \forall z \in \mathcal{O}.$$

Proof. We study the embedding of $T_z\mathcal{O}$ in the tangent space $T_zG(z_0)$ to a $G$-orbit $G(z_0)$. At the Lie algebra level it is equal to a map

$$\iota : \mathfrak{g}_0/\mathfrak{g}_0^z \longrightarrow \mathfrak{g}/\mathfrak{g}^z$$

induced by the inclusions $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}, \mathfrak{g}_0^z \hookrightarrow \mathfrak{g}^z$. The intersection $\iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cap \mathcal{J}\iota(\mathfrak{g}_0/\mathfrak{g}_0^z)$, where $\mathcal{J}$ is the complex structure on $\mathfrak{g}/\mathfrak{g}^z$ induced by the multiplication by $i$, is equal to

$$\{s + \mathfrak{g}_0^z; s \in \mathfrak{g}_0, \exists t \in \mathfrak{g}_0, s - it \in \mathfrak{g}^z\}.$$

We define a map

$$\phi : \mathfrak{g}^z \longrightarrow \iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cap \mathcal{J}\iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cong T_z\mathcal{O} \cap \mathcal{J}T_z\mathcal{O}$$

by the formula

$$\mathfrak{g}^z \ni s - it \mapsto s + \mathfrak{g}^z,$$

where $s, t \in \mathfrak{g}_0$. The kernel of $\phi$ is equal to $\{s - it \in \mathfrak{g}^z; s \in \mathfrak{g}^z\} = \{s - it \in \mathfrak{g}^z; s \in \mathfrak{g}_0^z = \mathfrak{g}^z \cap \mathfrak{g}_0\}$. Since $ad^*(s)(x) = ad^*(s)(y) = 0$ for $s \in \mathfrak{g}_0^z, x = \text{Re } z, y = \text{Im } z$, one gets

$$0 = ad^*(s - it)(x + iy) = -iad^*(t)(x + iy) \Rightarrow$$

$$ad^*(t)x = ad^*(t)y = 0 \Rightarrow t \in \mathfrak{g}_0^z.$$

Thus $\ker \phi = (\mathfrak{g}_0^z)^\mathbb{C}$. The surjectivity of $\phi$ is obvious. q.e.d.
6.4. Theorem  Let \( \mathcal{O} \) be a \( G_0 \)-orbit in a \( G \)-orbit \( M \) of maximal dimension. Then \( T^{1,0}\mathcal{O} \) is generated by the vector fields \( c(\bar{g}_1), \ldots, c(\bar{g}_r) \).

Proof. By (6.2.1) \( \mathcal{O} \cap \mathcal{J}\mathcal{T}\mathcal{O} \) is generated by linear combinations \( \alpha^j(c(z_j) + \overline{c(z_j)}) \), \( \alpha^j \in \mathcal{E}(\mathfrak{g}^\ast) \), such that there exist \( \beta^j \in \mathcal{E}(\mathfrak{g}^\ast) \) satisfying the equality

\[
\alpha^j(c(z_j) + \overline{c(z_j)}) = \beta^j i(c(z_j) - \overline{c(z_j)}).
\]

This implies

\[
(\alpha^j + i\beta^j)c(z_j) + (\alpha^j - i\beta^j)c(z_j) = 0
\]

and

\[
\gamma^j c(z_j) = 0,
\]

where we put \( \gamma^j = \alpha^j + i\beta^j \).

In order to calculate all vector functions \( \gamma = (\gamma^1, \ldots, \gamma^n) \) satisfying (6.4.1) one observes the following two facts: First that given \( g_1, \ldots, g_r \) as in ??, the vector functions \( \gamma_m = (\frac{\partial g_m}{\partial z_1}, \ldots, \frac{\partial g_m}{\partial z_n}) \), \( i = 1, \ldots, r \), satisfy (6.4.1). Second, the dimension arguments show that any \( \gamma(z) \) for which (6.4.1) holds is a linear combination of \( \gamma(z) \) if \( z \in \mathfrak{g}^\ast \setminus \text{Sing} \mathfrak{g}^\ast \).

In other words, \( T^{CR}\mathcal{O} \) is generated by \( (\frac{\partial g_m}{\partial z_j} + \frac{\partial g_m}{\partial z_j})(c(z_j) + c(z_j)) \), \( i = 1, \ldots, r \), and \( T^{1,0}\mathcal{O} \) by

\[
(\frac{\partial g_m}{\partial z_j} + \frac{\partial g_m}{\partial z_j})(c(z_j) + c(z_j)) - i\mathcal{J}(\frac{\partial g_m}{\partial z_j} + \frac{\partial g_m}{\partial z_j})(c(z_j) + c(z_j)) = 2c(g_m) + 2(\frac{\partial g_m}{\partial z_j})c(z_j) = 2c(\bar{g}_m).
\]

q.e.d.

Our next aim is to study \( G_0 \)-orbits in \( \mathfrak{g}^\ast \) from the symplectic point of view.

6.5. Corollary  In the assumptions of Theorem 6.4 \( \mathcal{O} \) is a \( CR \)-isotropic submanifold in \( M \) (Definition 1.19).

Proof. First we shall show the \( G_0 \)-invariance of the functions \( \bar{g}_1, \ldots, \bar{g}_r \). This fact follows from the equality

\[
c_{ij} \frac{\partial g_m}{\partial z_i} + c_{ijk} \frac{\partial^2 g_m}{\partial z_i \partial z_l} = 0
\]

(6.5.1)
obtained by the differentiation of the equality $c_{kj} \frac{\partial g_m}{\partial z_i} = 0$ with respect to $z_l$. Next, conjugating (6.5.1) and multiplying by $z_l$ one gets

$$0 = c_{lj} \frac{\partial^2 g_m}{\partial z_i \partial z_l} z_l =
(c_{ij} \frac{\partial}{\partial z_i} + c_{ik} \frac{\partial}{\partial \bar{z}_k})(\frac{\partial g_m}{\partial z_i} z_l) =
-c(z_j) + c(z_j) \tilde{g}_m.
$$

Now, recall that $(T^{1,0})^\perp$ is generated by the vector fields $c(f)$, where $f$ runs through all $G_0$-invariant functions. Thus by Theorem 6.4 $T^{1,0} \subset (T^{1,0})^\perp$. q.e.d.

6.6. Let $\hat{\mathcal{R}}$ denote a sum of $G_0$-orbits of the principal orbital type on $\mathfrak{g}^*$ and let $\mathcal{R}$ be its complement. Recall ([7]) that $\mathcal{R}$ is a nowhere dense closed subset and that for any $G$-orbit $M$ a factor $(M \setminus \mathcal{R})/G_0$ is a manifold.

6.7. Theorem. Let $M \subset \mathfrak{g}^*$ be a $G$-orbit that is not contained in $C \cup \mathcal{R}$ and let $p : M \setminus \mathcal{R} \longrightarrow (M \setminus \mathcal{R})/G_0$ be the canonical projection. Write $J$ for the symplectic bihamiltonian structure on $M \setminus \mathcal{R}$ associated with the restriction of the standard holomorphic symplectic form $\omega = (c|_M)^{-1}$.

Then the reduction $J'$ of $J$ via $p$ is a complete bihamiltonian structure on $M'/\mathcal{R}/G_0$. More precisely, $J'$ is complete at any $z' \in M' \setminus \mathcal{R}$. The realization $J$ of $J'$ is minimal (see Definition 4.6).

We postpone the proof of this theorem till Subsection 6.8 and present some preliminary results.

6.8. Proof of Theorem 6.7. We are going to use Theorem 4.4 and the notations from it.

The foliation $\mathcal{K}$ of leaves of $p$ is a generic $CR$-foliation due to Proposition 6.2.

By Corollaries 6.5 and 6.5 the number $k_{z'} = \dim(\ker p_{z'}^C)^{1,0} \bigcap \text{ker} p_z^{C,1,0}_{z'} \perp \omega(z)$, $z \in p^{-1}(z')$, equals $r = \operatorname{rank} \mathfrak{g}$ for any $z' \in M'$. We now shall prove that the number $k^\lambda_{z'} = \dim(\ker p_{z'}^C)^{1,0} \perp \omega_{\lambda}(z)$, $z \in p^{-1}(z')$ satisfies the inequality

$$k^\lambda_{z'} \geq k_{z'} \quad (6.8.1)$$

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for any \( z' \in M', \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \Lambda \).

For that purpose we shall use the bivector \( c_\lambda = (\omega_\lambda)^{-1} \) and the fact that \( (\ker p^C_\lambda)^{\perp, \lambda} \) is generated by the vector fields \( c_\lambda(f) \), where \( f \) varies through the functions constant along \( \ker p \). We shall define \( r \) functions \( g_{\lambda,1}, \ldots, g_{\lambda,r} \) such that the vector fields \( c_\lambda(g_{\lambda,1}), \ldots, c_\lambda(g_{\lambda,r}) \) are tangent to \( \mathcal{K} \) and are independent at any point \( z \in M \setminus \mathcal{C} \). These functions are

\[
    g_{\lambda,1} = g_1(\lambda \tilde{\lambda}_2 z + \lambda \tilde{\lambda}_1 z), \ldots, g_{\lambda,r} = g_r(\lambda \tilde{\lambda}_2 z + \lambda \tilde{\lambda}_1 z),
\]

where \( (\lambda_1, \tilde{\lambda}_2) \in \mathbb{C}^2 \) is such that \( c_\lambda = \lambda_1 c + \tilde{\lambda}_2 \tilde{c} \). They are independent at any \( z \in M \setminus \mathcal{C} \) since the Jacobi matrices \( \text{Jac} = \frac{\partial (g_{1}, \ldots, g_{1})}{\partial(z_1, \ldots, z_n)} \) and \( \text{Jac}_\lambda = \frac{\partial (g_{1}, \ldots, g_{1})}{\partial(z_1, \ldots, z_n)} \) are related as follows

\[
    \text{Jac}_\lambda(z) = \tilde{\lambda}_2 \text{Jac}(\lambda \tilde{\lambda}_2 z + \lambda \tilde{\lambda}_1 z).
\]

The nondegeneracy of \( c_\lambda \) implies the independence of the vector fields \( c_\lambda(g_{\lambda,1}), \ldots, c_\lambda(g_{\lambda,r}) \) at \( z \in M \setminus \mathcal{C} \). The following equalities show that these vector fields are tangent to \( \mathcal{K} \) (\( \mathcal{K} \) is spanned by \( c(z_i) + \tilde{c}(z_i), i = 1, \ldots, n \))

\[
    c_\lambda(g_{\lambda,m}) = \lambda_1 c(g_{\lambda,m}) + \tilde{\lambda}_2 \tilde{c}(g_{\lambda,m}) = \lambda_1 \lambda \frac{\partial g_{\lambda,m}}{\partial z_i} c(z_i) + \tilde{\lambda}_2 \tilde{c}(g_{\lambda,m}) = \tilde{\lambda}_2 \tilde{c}(g_{\lambda,m}) = \lambda \tilde{\lambda}_2 \frac{\partial g_{\lambda,m}}{\partial z_i} \tilde{c}(z_i).
\]

Here we used the obvious identities

\[
    \frac{\partial g_{\lambda,m}}{\partial z_i} = \lambda \frac{\partial g_{\lambda,m}}{\partial z_i} \tilde{c}(z_i), \quad \frac{\partial g_{\lambda,m}}{\partial z_i} = \tilde{\lambda}_2 \frac{\partial g_{\lambda,m}}{\partial z_i} \tilde{c}(z_i).
\]

Thus we have proved (6.8.1) that is equivalent via Theorems ??, ?? to the following

\[
    \text{rank} \ c'_\lambda(z') \leq \text{rank} \ c'(z'), \ z' \in M' \setminus p(\mathcal{C})).
\]

By the lower semi-continuity of the function \( f(\lambda_1, \lambda_2) = \text{rank}(\lambda_1 c' + \lambda_2 \tilde{c}'), (\lambda_1, \lambda_2) \in \mathbb{C}^2 \), this gives

\[
    \text{rank} \ c'_\lambda(z') = \text{rank} \ c'(z'), \ z' \in M' \setminus p(\mathcal{C})).
\]

Thus we have obtained the constancy of \( k'_2 \) in \( \lambda \) and the equality \( k'_2 = k_2 \) for \( z' \in M' \setminus p(\mathcal{C}) \). Since this number is also independent of \( z' \), condition \((ii)\) of Theorem 4.4 is satisfied.

The minimality of the realization \( J \) for \( J' \) follows from Corollary 6.5.

q.e.d.
7 Some open questions

7.1. It is possible to define a notion of a Veronese web (see Introduction) for an arbitrary complete bihamiltonian structure. A manuscript with some ideas of the author concerning this subject is in preparation.

It would be desirable to generalize the result of I.M.Gelfand and I.S.Zakharevich claiming that Veronese webs are complete local invariants for bihamiltonian structures of general position ([11]).

7.2. The following question is natural (cf. [19]): given a complete bihamiltonian structure, does there exist its local realization in a complex symplectic manifold?

7.3. A question of uniqueness (cf. [19]): is it true that any two minimal realizations of a complete bihamiltonian structure are locally symplectomorphic?

7.4. The author hopes that the study of the above questions will clarify some aspects of reconstructing the bihamiltonian structure from its Veronese web (the authors of the article [11] are themselves unsatisfied by their construction, see [11], p. 166).

References


