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Bi-Hamiltonian structures with symmetries, Lie pencils and integrable systems

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Abstract

There are two classical ways of constructing integrable systems by means of bi-Hamiltonian structures. The first one supposes nondegeneracy of one of the Poisson structures generating the pencil and uses the so-called recursion operator. This situation corresponds to the absence of Kronecker blocks in the so-called Jordan–Kronecker decomposition. The second one, which corresponds to the absence of Jordan blocks in this decomposition, uses the Casimir functions of different members of the pencil. In this paper, we consider the general case of a bi-Hamiltonian structure with both Kronecker and Jordan blocks and give a criterion for the completeness of the corresponding family of functions. This result is related to a natural action of some Lie algebra which gives a symmetry of the whole pencil. The criterion is applied to bi-Hamiltonian structures related to Lie pencils, although we also discuss other possible applications.

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1. Introduction

We start with a short 'physical' motivation. Consider the n-dimensional free rigid body system on $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$. Here we describe this as a Hamiltonian system. The Poisson bracket is the canonical Lie–Poisson one. The Hamiltonian function after identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the 'trace form' becomes $H(M) = (1/2)\,\mathrm{Tr}(M \cdot L^{-1}M)$, $M \in \mathfrak{g}$. Here L is an operator on $\mathfrak{g}$ defined by $L : M \mapsto DM + MD$, D being the 'inertia' matrix of the rigid body, a diagonal matrix. We assume that D has a positive simple spectrum (cf Morosi and Pizzocchero (1996)).

There are several approaches to studying the complete integrability of this system. 'The argument translation method', which goes back to Manakov (1976) and was fully developed by Mishchenko and Fomenko (1978), uses integrals of the form $\mathrm{Tr}((M+\lambda D^2)^k)$. Note that these integrals are related to the Poisson pencil defined on $\mathfrak{sl}(n,\mathbb{R}) \supset \mathfrak{so}(n,\mathbb{R})$ that is generated by two Poisson structures: the Lie–Poisson structure $\vartheta_{\mathfrak{sl}(n,\mathbb{R})}$ and the constant one obtained by 'freezing' $\vartheta_{\mathfrak{sl}(n,\mathbb{R})}$ at the point $D^2 \in \mathfrak{sl}(n,\mathbb{R})$.

Another approach, proposed by Bolsinov (1992) (see also Morosi and Pizzocchero (1996)), exploits the another Poisson pencil. It is defined on $\mathfrak{so}(n,\mathbb{R})$ itself and is generated by two Lie–Poisson structures: one related to the standard commutator $[\,,\,]$ on $\mathfrak{g}$ and another to the modified commutator $[\,,\,]_{D^2}$, $[X,Y]_{D^2} := XD^2Y - YD^2X$. The corresponding integrals are of the form $\mathrm{Tr}(((I+\lambda D^2)^{-1/2}M(I+\lambda D^2)^{-1/2})^k)$ (cf subsection 4.3). Although looking differently from that defined above these integrals in fact define the same invariant tori (for $\mathfrak{gl}(n,\mathbb{R})$ the similar fact is proved in Panasyuk (2006, proposition 5.3).

Now assume that the matrix D has multiplicities in its spectrum. In general neither of the above series of integrals is sufficient for the Liouville integrability of the system in this case. However, the nonsimplicity of the spectrum of D is equivalent to the existence of inner symmetries of the body. In order to prove the complete integrability of the system one can add to the above-mentioned integrals the Noether integrals induced by these symmetries. One can prove the completeness of the new set of integrals consisting of the 'usual ones' (i.e., the Manakov or Bolsinov ones) and a maximal commutative subset of the set of the Noether integrals (Trofimov and Fomenko 1995, section 44).

The aim of this paper is to develop a similar method of combining 'usual integrals' with 'Noetherian integrals' in the more general setting of bi-Hamiltonian structures with 'inner symmetries' (this method is then applied to particular bi-Hamiltonian structures among which there are the above examples).

More precisely, we consider a class of Poisson pencils $\Theta := \{\vartheta^t\}_{t\in\mathbb{C}^2}$, i.e. two-dimensional linear subspaces in the set of Poisson bivectors on a given manifold M (assume for simplicity that we work in the complex analytic category), with the following additional condition: the set $E_\Theta(x) := \{t \in \mathbb{C}^2 \mid \mathrm{rank}\,\vartheta^t_x < \max_t \mathrm{rank}\,\vartheta^t_x\}$ does not depend on $x \in M$. The Poisson pencils with this condition are called admissible (see definition 2.2.5 for more details) and the set $E_\Theta := E_\Theta(x)$ is called exceptional and it turns out that it is at most a finite sum of one-dimensional subspaces $\mathrm{Span}\{t_1\} \cup \cdots \cup \mathrm{Span}\{t_n\} \subset \mathbb{C}^2$ (see subsection 2.2). The inner symmetries mentioned are provided by the Hamiltonian vector fields $\vartheta^{t_0}(f)$, where $\vartheta^{t_0}, t_0 \notin E_\Theta$, is a fixed Poisson bivector of the maximal rank and f runs through the spaces $\mathcal{Z}^{t_1},\ldots,\mathcal{Z}^{t_n}$ of the Casimir functions of the exceptional bivectors $\vartheta^{t_1},\ldots,\vartheta^{t_n}$ (see subsection 2.3).

It is well known that the functions from the space $\mathcal{Z}^\Theta := \mathrm{Span}\{\bigcup_{t\notin E_\Theta}\mathcal{Z}^t\}$ of Casimirs of nonexceptional bivectors are in involution with respect to ϑ^{t_0} and form a complete set if and only if $E_\Theta = \{0\}$. If the last condition holds the pencil is called micro-Kronecker (Zakharevich 2001). If $E_\Theta \neq \{0\}$ the set $\mathcal{Z}^\Theta$ is in general incomplete. If this is the case, one can try add to $\mathcal{Z}^\Theta$ functions from the family $\mathcal{Z}^\Theta_E := \mathcal{Z}^{t_1}+\cdots+\mathcal{Z}^{t_n}$ (they commute with those from $\mathcal{Z}^\Theta$). However, $\mathcal{Z}^\Theta_E$ is in general a noncommutative Lie subalgebra with respect to the Poisson bracket related to ϑ^{t_0}. Thus one can try to pick up an Abelian subalgebra $\mathcal{A} \subset \mathcal{Z}^\Theta_E$ and ask whether the commutative family $\mathcal{Z}^\Theta + \mathcal{A}$ is complete.

Note that in general the problem of choosing an (large enough) Abelian subalgebra $\mathcal{A}$ in an infinite-dimensional Lie algebra $\mathcal{Z}^\Theta_E$ can be very complicated.

In our approach we propose both a method of choosing an Abelian subalgebra $\mathcal{A}$ and a criterion of completeness of the family of functions $\mathcal{Z}^\Theta + \mathcal{A}$.

The method of choosing an Abelian subalgebra $\mathcal{A} \subset \mathcal{Z}_E^\Theta$ consists of fixing an appropriate finite-dimensional Lie subalgebra $\mathfrak{z} \subset \mathcal{Z}_E^\Theta$. The functions on $\mathfrak{z}^*$ can be interpreted as elements of $\mathcal{Z}_E^\Theta$, and the space of these functions $\mathcal{E}(\mathfrak{z}^*)$ forms a Lie subalgebra in $\mathcal{Z}_E^\Theta$. Thus one can pick up a maximal Abelian Lie subalgebra $\mathcal{A}(\mathfrak{z}) \subset \mathcal{E}(\mathfrak{z}^*)$ using one of the several standard methods (for instance, the above-mentioned method of argument translation). It turns out that in the main applications of our general scheme there is a natural choice of finite-dimensional subalgebra $\mathfrak{z}$ and in many examples the number of independent functions from the family $\mathcal{Z}^\Theta + \mathcal{A}(\mathfrak{z})$ is sufficient for obtaining a completely integrable system.

The first main result of this paper (theorem 2.2.10) gives necessary and sufficient conditions for the completeness of the commutative set $\mathcal{Z}^\Theta + \mathcal{A}(\mathfrak{z})$. These conditions are formulated in purely linear algebraic terms reflecting the structure of the so-called Jordan–Kronecker decomposition of a Poisson pencil at a point (see subsection 3.3).

The main result is then applied to Poisson pencils related to Lie pencils, i.e. linear 2-parameter families of Lie brackets on a given vector space. Given a Lie pencil $\Lambda := \{\mathfrak{g}^t\}_{t\in\mathbb{C}^2}, \mathfrak{g}^t := (\mathfrak{g}, [\ ,\ ]^t)$, let Θ_Λ denote the Poisson pencil on $\mathfrak{g}^*$ consisting of the corresponding Lie–Poisson structures. Then, under some additional assumptions on Λ (for instance if there is a semisimple Lie algebra among $\mathfrak{g}^t$), Θ_Λ is admissible. In such a case we have distinguished Lie algebras $\mathfrak{g}^{t_j}, j = 1, \ldots, n$, whose index is smaller than that of the generic Lie algebra of the pencil. We also have a distinguished subalgebra $\mathfrak{z} \subset \mathcal{Z}_E^{\Theta_\Lambda}$ in this case: $\mathfrak{z} := \sum_{j=1}^n \mathfrak{z}^j$, where $\mathfrak{z}^j$ is the center of $\mathfrak{g}^{t_j}$. Our second main result (theorem 4.1.3) gives necessary and sufficient conditions for the completeness of the family of functions $\mathcal{Z}^{\Theta_\Lambda}$ enlarged by a maximal involutive set of functions from $\mathcal{E}(\mathfrak{z}^*)$.

Note that some sufficient conditions for the completeness of such a family were given by Bolsinov (Trofimov and Fomenko 1995, proposition 7, section 44). They look as follows:

$$\operatorname{ind} \mathfrak{g}^{t_j} = \operatorname{ind} \mathfrak{g}^{t_0} + \dim \mathfrak{z}^j - \operatorname{ind} \mathfrak{z}^j,\ j = 1, \ldots, n,$$

where $\mathfrak{z}^j$ is regarded as a subalgebra in $\mathfrak{g}^{t_0}$. The author's interest to the subject was encouraged by finding an example of a Lie pencil for which these conditions are not satisfied but the corresponding family of functions is complete. This example (see subsection 4.3), as well as some other ones, is presented in section 4. Our necessary and sufficient conditions are more general than the conditions of Bolsinov and the latter ones follow from the former ones (see theorems 2.2.11 and 4.1.4).

The paper is organized as follows. In subsection 2.1 we recall some notions related to Poisson structures and integrable systems on Poisson manifolds. Subsection 2.2 is devoted to introduction to the geometry of bi-Hamiltonian structures and the formulation of main results. In subsection 2.3 we give an interpretation of the main results in terms of symmetries of the underlying bi-Hamiltonian structure.

Section 3 is devoted to the proof of the main results. It is divided into three subsections in which we present different aspects of the linear algebra of a pencil of bivectors on a vector space. In subsection 3.2 we formulate the linear algebraic counterpart of the main result and in subsection 3.3 we prove it with the help of the 'main tool', the so-called Jordan–Kronecker decomposition.

In section 4 we reformulate our main theorem in the context of Lie pencils and present some examples of its applications.

We conclude section 1 by mentioning that another field of applications of the main result is the above-mentioned argument translation method. In this method the following bi-Hamiltonian structure is considered. If $\mathfrak{g}$ is a Lie algebra and $a \in \mathfrak{g}^*$ is a fixed element, we put ϑ_1 for the canonical Lie–Poisson structure $\vartheta_\mathfrak{g}$ on $\mathfrak{g}^*$ and $\vartheta_2 := \vartheta_\mathfrak{g}(a)$. Consider a Poisson pencil Θ generated by ϑ_1, ϑ_2. If a is a singular element, i.e. belonging to a coadjoint orbit of nonmaximal dimension, the above-mentioned involutive set of functions $\mathcal{Z}^{\Theta}$ can be incomplete (as in the example of a free rigid body). However, we can enlarge this family by the family $\mathcal{Z}^{t_1}$ of the Casimir functions of bivector ϑ_2 (which is the only 'exceptional' bivector in the pencil) some of which do not belong to $\mathcal{Z}^{\Theta}$. Note that here we have a natural finite-dimensional Lie subalgebra $\mathfrak{z}$ in $\mathcal{Z}^{t_1}$, the stabilizer of a, and we can apply our main result for studying the completeness of the family $Z^{\Theta} + \mathcal{A}(\mathfrak{z})$.

Also, the main result can be applied to the so-called method of symmetric pairs (combining the method of Lie pencils and the argument translation) Bolsinov (1992) and to other bi-Hamiltonian structures of algebraic nature.

2. Bi-Hamiltonian structures with symmetries

2.1. Preliminaries on Poisson structures

In this subsection we give some definitions from the theory of Poisson structures. We refer the reader to the book da Silva and Weinstein (1999) for more details.

For simplicity in this paper we will work in the real analytic or complex analytic category. In the last case we will consider manifolds with sufficiently many holomorphic functions (for instance Stein manifolds). The ground field will be denoted by $\mathbb{K}$.

So M denotes a connected analytic manifold, $\mathcal{E}(M)$ stands for the space of analytic functions on M.

Definition 2.1.1. *Let (M, ϑ) be a Poisson manifold. We regard ϑ as a morphism $T^*M \to TM$. We define* rank ϑ_x *as* dim imϑ_x *and* corank ϑ_x *as* dim ker ϑ_x*. We put* $\operatorname{rank}\vartheta := \max_{x\in M}\operatorname{rank}\vartheta_x$ *and* $\operatorname{corank}\vartheta := \dim M - \operatorname{rank}\vartheta$.

A symplectic leaf S of ϑ is called regular if $\dim S = \operatorname{rank}\vartheta$*. We denote by* Sing ϑ *the union of nonregular symlectic leaves.*

We write $\{,\}^{\vartheta}$ for the Poisson bracket corresponding to a Poisson structure ϑ and $\mathcal{Z}^{\vartheta}(U)$ for the set $\{f \in \mathcal{E}(U) \mid \vartheta(df) \equiv 0\}$ of *Casimir functions* of ϑ over an open set $U \subset M$.

Definition 2.1.2. *A set $Z \subset \mathcal{Z}^{\vartheta}(U)$ of Casimir functions over an open set $U \subset M$ is called complete as a set of Casimir functions if there exist $f_1, \dots, f_k \in Z$ such that their differentials are independent on $U\backslash(U \cap \operatorname{Sing}\vartheta)$, where $k = \operatorname{corank}\vartheta$.*

In other words, Z is complete if and only if the common level sets of functions from Z coincide with the symplectic foliation on $U\backslash(U \cap \operatorname{Sing}\vartheta)$. It is clear that $\mathcal{Z}^{\vartheta}(U)$ can be incomplete if the closure of some regular symplectic leaf in M is a submanifold of dimension greater than that of the leaf. On the other hand, $\mathcal{Z}^{\vartheta}(U)$ is always complete for sufficiently small U.

Definition 2.1.3. *A set $I \subset \mathcal{E}(U)$ of functions over $U \subset M$ is called involutive with respect to ϑ if $\{f, g\}^{\vartheta} = 0$ for all $f, g \in I$. An involutive set I of functions over U is called complete as an involutive set of functions if there exists an open dense subset $V \subset U$ such that the subspace* $\operatorname{Span}\{d_x f \mid f \in I\} \subset T_x^*M$ *is of dimension* $\dim M - (1/2)\operatorname{rank}\vartheta$ *for any $x \in V$.*

If I is complete as an involutive set of functions over U, then $I|_V \supset \mathcal{Z}^{\vartheta}(V)$ and the last set is complete as a set of Casimir functions. Any such set I is a set of functions constant on a foliation of $V\backslash(V \cap \operatorname{Sing}^{\vartheta})$ of dimension $(1/2)\operatorname{rank}\vartheta$ which is Lagrangian in any regular symplectic leaf. Completeness of an involutive set I can also be interpreted in the following way: for any $x \in V\backslash(V \cap \operatorname{Sing}^{\vartheta})$ the subspace $\operatorname{Span}\{d_x f \mid f \in I\} \subset T_x^*M$ is maximal isotropic with respect to the 2-form ϑ_x (see definition 3.1.2).

Remark 2.1.4. A subset $Z \subset \mathcal{E}(M)$ will be called *functionally closed* if any finite functional combination of functions from Z belongs to Z. In particular, a set of functions constant along some generalized foliation (for instance, foliation of symplectic leaves of some Poisson structure) in M is functionally closed.

Now assume we have a Poisson structure (M, ϑ) and two Lie subalgebras $\mathfrak{z} \subset Z$ of the Lie algebra $(\mathcal{E}(M), \{,\}^\vartheta)$ such that $\mathfrak{z}$ is finite dimensional and Z is functionally closed. Then the set of functions $\mathcal{E}(\mathfrak{z}^*)$ endowed with the Lie–Poisson bracket can be regarded as a Lie subalgebra in Z. Indeed, elements of $\mathfrak{z}$ can be interpreted as linear functions on $\mathfrak{z}^*$. Functions from $\mathcal{E}(\mathfrak{z}^*)$ are functional combinations of these linear functions, thus by functional closedness of Z lie in Z. Finally, the fact that $\mathcal{E}(\mathfrak{z}^*)$ is a Lie subalgebra follows from the basic properties of the Poisson bracket.

2.2. Preliminaries on bi-Hamiltonian structures and formulation of main results

Given a Poisson bivector field (bivector for short) ϑ on M; let $\mathcal{E}^\vartheta(M)$ denote the space $\mathcal{E}(M)$ regarded as a Lie algebra with respect to the Poisson bracket $\{,\}^\vartheta$.

Definition 2.2.1. *Let a pair* $(\vartheta^{(1)}, \vartheta^{(2)})$ *of linearly independent bivectors on a manifold M be given. Assume* $\vartheta^t := t^{(1)}\vartheta^{(1)} + t^{(2)}\vartheta^{(2)}$ *is a Poisson bivector for any* $t = (t^{(1)}, t^{(2)}) \in \mathbb{K}^2$*. We say that the Poisson structures* $\vartheta^{(1)}, \vartheta^{(2)}$ *are compatible (or form a Poisson pair) and that the whole family* $\Theta := \{\vartheta^t\}_{t\in\mathbb{K}^2}$ *is a bi-Hamiltonian structure or a Poisson pencil.*

Given a bi-Hamiltonian structure $\{\vartheta^t\}$ on M, we will write $\mathcal{E}^t(M), \mathcal{Z}^t(M), \{,\}^t$, etc instead of $\mathcal{E}^{\vartheta^t}(M), \mathcal{Z}^{\vartheta^t}(M), \{,\}^{\vartheta^t}$, etc for short.

Definition 2.2.2. *Let* $\Theta = \{\vartheta^t\}$ *be a bi-Hamiltonian structure on M and* $x \in M$*. Put* $E_\Theta(x) = \{t \in \mathbb{C}^2 \mid \operatorname{rank} \vartheta^t_x < \max_{t\in\mathbb{C}^2} \operatorname{rank} \vartheta^t_x\}$ *(in the real category we regard* $\vartheta^t = t^{(1)}\vartheta^{(1)} + t^{(2)}\vartheta^{(2)}$ *as a section of the complexification of the tangent bundle on M). The set* $E_\Theta(x)$ *is called exceptional for* Θ *at x.*

Remark 2.2.3. The set $E_\Theta(x)$ is either $\{0\}$ or the union of a finite number of lines in $\mathbb{C}^2$.

Definition 2.2.4. *Let* $\Theta = \{\vartheta^t\}$ *be a bi-Hamiltonian structure on M. It is called Kronecker at a point* $x \in M$ *if* $\operatorname{rank} \vartheta^t_x$ *is constant with respect to* $t \in \mathbb{C}^2\backslash\{0\}$*, i.e.* $E_\Theta(x) = \{0\}$ *(cf definition 3.2.1). We say that* Θ *is micro-Kronecker if it is Kronecker at any point of some open dense set in M.*

Definition 2.2.5. *Assume that, given a bi-Hamiltonian structure* Θ*, there exists an open dense set* $U \subset M$ *such that* $E_\Theta(x) =: E_\Theta$ *does not depend on* $x \in U$*. We will call such a bi-Hamiltonian structure admissible.*

Remark 2.2.6. Note that micro-Kronecker bi-Hamiltonian structures are admissible. From now on we will consider only bi-Hamiltonian structures $\Theta = \{\vartheta^t\}$ that are admissible and are *not* micro-Kronecker. We will assume that $E_\Theta = \operatorname{Span}\{t_1\} \cup \cdots \cup \operatorname{Span}\{t_n\}$, where $t_i \in \mathbb{C}^2$ are pairwise nonproportional. We will say that the values of $t \in E_\Theta$ are *exceptional* and the other ones are *generic*. The same terms will be used for the corresponding bivectors ϑ^t, Poisson brackets $\{,\}^t$, etc.

Lemma 2.2.7. *Let* $\Theta = \{\vartheta^t\}$ *be an admissible bi-Hamiltonian structure on M. Fix an arbitrary* $t_0 \in \mathbb{K}^2$*. Then for any open set* $U \subset M$

(1) The set of functions $\mathcal{Z}^t(U)$ *is a Lie subalgebra in* $\mathcal{E}^{t_0}(U)$ *for any t.*

(2) The subalgebra $\mathcal{Z}^t(U)$ is Abelian for any $t \notin E_\Theta$.
(3) $\{\mathcal{Z}^{t'}(U), \mathcal{Z}^{t''}(U)\} = 0$ for any nonproportional t', t''.

Remark 2.2.8. The proof of 1 could be found in Trofimov and Fomenko (1995, proposition 4, section 44), and 2 and 3 follow from lemma 3.2.3.

Now assume we work in the complex category.

Corollary 2.2.9.

(1) The set of functions

$$\mathcal{Z}^\Theta(U) := \operatorname{Span}\left(\bigcup_{t\in\mathbb{C}^2\setminus E_\Theta} \mathcal{Z}^t(U)\right)$$

is an Abelian subalgebra in $\mathcal{E}^{t_0}(U)$ for any $t_0 \in \mathbb{C}^2$.

(2) Let $t_0 \in \mathbb{C}^2\setminus E_\Theta$ and $\mathcal{I}^i \subset \mathcal{Z}^{t_i}(U)$ be an Abelian subalgebra in $(\mathcal{Z}^{t_i}(U), \{,\}^{t_0})$. Then the set

$$\mathcal{I}^1 + \cdots + \mathcal{I}^n + \mathcal{Z}^\Theta(U)$$

is an Abelian subalgebra in $\mathcal{E}^{t_0}(U)$.

Let $t_0 \in \mathbb{C}^2\setminus E_\Theta$ and let $U \subset M$ be a sufficiently small open set. Since Θ is non-micro-Kronecker (see remark 2.2.6), by the criterion of Bolsinov (1992) (see also theorem 3.2.5) the set $\mathcal{Z}^\Theta(U)$, which is an Abelian subalgebra in $\mathcal{E}^{t_0}(U)$, is not complete as an involutive set of functions (see definition 2.1.3). Our main result (theorem 2.2.10) gives a criterion of completeness for the more general Abelian subalgebra $\mathcal{I}^1+\cdots+\mathcal{I}^n+\mathcal{Z}^\Theta(U)$, thus generalizing the criterion of Bolsinov.

In order to formulate our main result we need the following notions. Let $\mathfrak{z}^i \subset \mathcal{Z}^{t_i}(U)$, $i = 1, \ldots, n$, be a fixed finite-dimensional subspace of the Lie algebra $(\mathcal{Z}^{t_i}(U), \{,\}^{t_0})$. Given a point $x \in U$, introduce for any $i \in \{1, \ldots, n\}$ the subspaces

$$\mathfrak{z}^i_x := \{d_x z \mid z \in \mathfrak{z}^i\} \subset T^*_x M \tag{2.1}$$

$$\mathfrak{z}^{0,i}_x := \left\{v \in \mathfrak{z}^i_x \,\middle|\, \exists w \in T^*_x M\colon \vartheta^{t_0}_x(v) = \vartheta^{t_i}_x(w)\right\} \tag{2.2}$$

and the skew-symmetric forms

$$\gamma_{\mathfrak{z}^i,x} : \bigwedge\nolimits^2 \mathfrak{z}^{0,i}_x \to \mathbb{C}, \qquad \gamma_{\mathfrak{z}^i,x}(v_1, v_2) := \left\langle \vartheta^{t_i}_x(w_1), w_2\right\rangle, \tag{2.3}$$

where $w_j \in T^*_x M$ are any elements such that $\vartheta^{t_0}_x(v_j) = \vartheta^{t_i}_x(w_j)$, $j = 1, 2$ ($\langle , \rangle$ stands for the natural pairing between vectors and covectors). Note that these forms are correctly defined. Indeed, if w'_j are another elements with $\vartheta^{t_0}_x(v_j) = \vartheta^{t_i}_x(w'_j)$, we have $u_j := w_j - w'_j \in \ker \vartheta^{t_i}_x$ and $\left\langle \vartheta^{t_i}_x(w'_1), w'_2\right\rangle = \left\langle \vartheta^{t_i}_x(w_1 + u_1), w_2 + u_2\right\rangle = \left\langle \vartheta^{t_i}_x(w_1), w_2\right\rangle$.

Theorem 2.2.10. *Let Θ be an admissible bi-Hamiltonian structure on M (see definition 2.2.5 and remark 2.2.6) in the complex category. Assume $U \subset M$ is a connected open set such that $\mathcal{Z}^t(U)$ is complete as a set of Casimir functions (see definition 2.1.2) for an infinite number of pairwise nonproportional values of $t \in \mathbb{C}^2\setminus E_\Theta$.*

Let $t_0 \in \mathbb{C}^2$ be generic and let $\mathcal{Z}^{t_i}(U)$, $i = 1, \ldots, n$, be endowed with the Lie algebra structure induced from $\mathcal{E}^{t_0}(U)$.

Assume that for any $i \in \{1, \ldots, n\}$ a finite-dimensional subalgebra $\mathfrak{z}^i \subset \mathcal{Z}^{t_i}(U)$ has been chosen and that $\mathcal{I}^i \subset \mathcal{E}((\mathfrak{z}^i)^*)$ is a complete involutive set of functions with respect to *the canonical Lie–Poisson structure on* $(\mathfrak{z}^i)^*$ *(thus* $\mathcal{I}^i$ *can be regarded as an involutive set of functions in* $\mathcal{Z}^{t_i}(U) \subset \mathcal{E}^{t_0}(U)$*, see remark 2.1.4).*

Then the involutive set of functions $\mathcal{I} := \mathcal{I}^1 + \cdots + \mathcal{I}^n + \mathcal{Z}^{\Theta}(U)$ *is complete as an involutive set of functions with respect to* ϑ^{t_0} *if and only if there exists* $x \in U$ *such that*

$$\operatorname{corank} \vartheta_x^{t_0}\big|_{\mathfrak{z}_x^i} + \operatorname{corank} \vartheta_x^{t_0} = 2 \dim \mathfrak{z}_x^{0,i} - \dim \mathfrak{z}_x^i - \operatorname{rank} \gamma_{\mathfrak{z}^i,x} + \operatorname{corank} \vartheta_x^{t_i}, \quad i \in \{1, \ldots, n\}. \tag{2.4}$$

Proof. By theorem 3.2.6 condition (2.4) is equivalent to the maximality with respect to $\vartheta_x^{t_0}$ of the isotropic space generated by the differentials at x of the functions from $\mathcal{I}$. This maximality, being an open condition, in turn is equivalent to the completeness of the involutive set $\mathcal{I}$ in some neighborhood V of x, which is dense in U (since we are in analytic category). □

In the next theorem we formulate two sufficient conditions for the completeness of the above-mentioned involutive set of functions, which are less general but can be more easily checkable.

Theorem 2.2.11. *In the assumption of theorem 2.2.10 the involutive set of functions* $\mathcal{I}$ *is complete as an involutive set of functions with respect to* ϑ^{t_0} *if there exists* $x \in U$ *such that one of the following two conditions hold:*

(1) $\operatorname{corank} \vartheta_x^{t_0} = \dim \mathfrak{z}_x^{0,i} - \dim \mathfrak{z}_x^i - \operatorname{rank} \gamma_{\mathfrak{z}^i,x} + \operatorname{corank} \vartheta_x^{t_i}, i \in \{1, \ldots, n\}$*;*
(2) $\operatorname{corank} \vartheta_x^{t_0} = \operatorname{corank} \vartheta_x^{t_0}\big|_{\mathfrak{z}_x^i} - \dim \mathfrak{z}_x^i + \operatorname{corank} \vartheta_x^{t_i}, i \in \{1, \ldots, n\}$.

Proof. Proof of this theorem follows from theorem 3.2.7. □

Remark 2.2.12. In the real category theorems 2.2.10, 2.2.11 remain true if $t_0, t_1, \ldots, t_n \in \mathbb{R}^2$ (and the set $\mathcal{Z}^{\Theta}$ is substituted by $\mathcal{Z}^{\Theta}_{\mathbb{R}}(U) := \operatorname{Span}(\bigcup_{t \in \mathbb{R}^2 \setminus E_\Theta} \mathcal{Z}^t(U))$).

2.3. Interpretation of the main result from the point of view of symmetries

The following theorem will provide us with a specific interpretation of the families of functions $\mathcal{I}^i$ which appeared in the main theorem. The reader is referred to books Ortega and Ratiu (2004) and da Silva and Weinstein (1999) for notions related to Hamiltonian actions. We will write $\chi(M)$ for the space of analytic vector fields on M. If the category is real we assume $t_0, t_1, \ldots, t_n \in \mathbb{R}^2$.

Proposition 2.3.1. *Let* $\Theta = \{\vartheta^t\}$ *be an admissible bi-Hamiltonian structure on M, let* $t_0 \in \mathbb{K}^2$ *be generic and let* $\mathcal{Z}^{t_i}(M), i = 1, \ldots, n$*, be endowed with the Lie algebra structure induced from* $\mathcal{E}^{t_0}(M)$*.*

Then for any $i \in \{1, \ldots, n\}$

(1) The Lie algebra action $\rho_{0,i} : \mathcal{Z}^{t_i}(M) \to \chi(M), f \mapsto \vartheta^{t_0}(f)$*, is Hamiltonian with respect to any* ϑ^t*, where t is linearly independent with* t_i *(in particular, t can be any generic).*
(2) The momentum map of the action $\rho_{0,i}$ *with respect to* ϑ^t *is given by the formula* $J_{0,i}^t : M \to (\mathcal{Z}^{t_i}(M))^*, J_{0,i}^t(x) = (f \mapsto a_{t,i} f(x))$*, where* $x \in M$ *and* $a_{t,i}$ *is the first of two constants uniquely defined by* $\vartheta^{t_0} = a_{t,i}\vartheta^t + b_{t,i}\vartheta^{t_i}$*.*

Proof. Let $t \in \mathbb{K}^2$ be linearly independent with t_i. Then there exist $a_{t,i}, b_{t,i} \in \mathbb{K}$ such that $\vartheta^{t_0} = a_{t,i}\vartheta^t + b_{t,i}\vartheta^{t_i}, a_{t,i} \neq 0$. Thus, if $f \in \mathcal{Z}^{t_i}(M)$, we have $\vartheta^{t_0}(f) = a_{t,i}\vartheta^t(f)$ and we can put $\mathcal{J}(f) = a_{t,i} f$ concluding that $\rho_{0,i}$ is weakly Hamiltonian (i.e., there exists a momentum map of this action given by $\langle J_{0,i}^t(x), f \rangle := \mathcal{J}(f)(x) = a_{t,i} f(x)$).

The following calculation shows that $\rho_{0,i}$ is in fact Hamiltonian (i.e., the momentum map is Poisson): $\{\mathcal{J}(f), \mathcal{J}(g)\}^t = a_{t,i}^2\{f, g\}^t = a_{t,i}(a_{t,i}\{f, g\}^t + b_{t,i}\{f, g\}^{t_i}) = a_{t,i}\{f, g\}^{t_0} = \mathcal{J}(\{f, g\}^{t_0})$, $f, g \in \mathcal{Z}^{t_i}(M)$. □

Remark 2.3.2. It follows from the proposition above that the action $\rho_{0,i}$ preserves the bi-Hamiltonian structure Θ. So does the restriction $\rho_{\mathfrak{z}^i}$ of this action to a finite-dimensional subalgebra $\mathfrak{z}^i \subset \mathcal{Z}^{t_i}(M)$.

However in general the action $\rho_{\mathfrak{z}^i}$ can be non-Hamiltonian with respect to ϑ^{t_i}, for instance it can have orbits which are transversal to the symplectic leaves of ϑ^{t_i}. Moreover, if we restrict the action $\rho_{\mathfrak{z}^i}$ to a stabilizer of a particular symplectic leaf of ϑ^{t_i}, this restriction can be only weakly Hamiltonian (i.e., the corresponding momentum map can be non-Poisson). The forms $\gamma_{\mathfrak{z}^i,x}$ from the main theorem are related to the so-called nonequivariance cocycles Ortega and Ratiu (2004) and da Silva and Weinstein (1999) of this action.

Remark 2.3.3. The functions from $\mathcal{E}((\mathfrak{z}^i)^*)$ regarded as functions on M (see theorem 2.2.10) can be interpreted as the Noetherian integrals corresponding to the symmetry given by the action $\rho_{\mathfrak{z}^i}$. Note that the collection $\left(J_{0,i}^t\right)^*(\mathcal{E}((\mathfrak{z}^i)^*))$ of these Noetherian integrals does not depend on $t \in \mathbb{K}^2 \backslash E_\Theta$. This follows from the form of the momentum map (see proposition 2.3.1).

3. Auxiliary linear algebra

3.1. Linear algebra of skew-symmetric bilinear forms: some definitions and facts

Definition 3.1.1. *Let V be a vector space over $\mathbb{C}$. A 2-form on V is a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{C}$. Given a subspace $W \subset V$, we put $W^{\perp\omega} := \{v \in V \mid \forall w \in W : \omega(v, w) = 0\}$ and say that $W^{\perp\omega}$ is a skew-orthogonal complement to W. We also put* $\ker\omega := V^{\perp\omega}$, $\operatorname{corank}\omega := \dim\ker\omega$, $\operatorname{rank}\omega := \dim V - \operatorname{corank}\omega$.

A 2-form ω is called symplectic or nondegenerate if $\ker\omega = \{0\}$.

Definition 3.1.2. *Given a vector space V, a 2-form ω on V and a subspace $W \subset V$, we say that W is (co)isotropic with respect to ω if $W \subset W^{\perp\omega}$ ($W \supset W^{\perp\omega}$). An isotropic subspace $W \subset V$ is said to be maximal isotropic if it is not contained in any larger isotropic subspace.*

The following lemmas are consequences of the 'linear algebraic Darboux theorem' (saying that for any 2-form ω there exists a basis $e^1, \ldots, e^k$ of the space V^* such that $\omega = e^1 \wedge e^2 + \cdots + e^{r-1} \wedge e^r$, where $r = \operatorname{rank}\omega$). We leave them without proof.

Lemma 3.1.3. *Let V be a vector space with a 2-form ω and let $V \subset W$ be a subspace. Then the following conditions are equivalent:*

(1) W is maximal isotropic with respect to ω;
(2) $W = W^{\perp\omega}$;
(3) W is isotropic and $\dim W = \operatorname{corank}\omega + (1/2)\operatorname{rank}\omega = (1/2)(\operatorname{corank}\omega + \dim V)$.

Lemma 3.1.4. *Given a symplectic vector space (V, ω) and a subspace $W \subset V$, let $I \subset W$ be a subspace which is maximal isotropic with respect to $\omega|_W$. Then I is maximal isotropic with respect to ω itself if and only if W is co-isotropic.*

Lemma 3.1.5. *Let $V = \bigoplus_{i=1}^k V_i$ and ω be a 2-form on V such that $\omega(V_i, V_j) = 0, i \neq j$. Let $W \subset V$ is a subspace such that $W = \bigoplus_{i=1}^k W_i$, where $W_i := W \cap V_i$. Then W is maximal isotropic with respect to ω if and only if W_i is maximal isotropic with respect to $\omega|_{V_i}$ for any $i = 1, \ldots, n$.*

Lemma 3.1.6. *Given a symplectic vector space (V, ω) and a subspace $W \subset V$, the following inequality holds:*

$$\dim W + \operatorname{corank} \omega|_W \leqslant \dim V.$$

Moreover, a subspace $W \subset V$ is co-isotropic if and only if

$$\dim W + \operatorname{corank} \omega|_W = \dim V.$$

3.2. Linear algebra of pencils of bivectors: formulation of main results

We assume further on that the ground field is $\mathbb{C}$. A bivector b on a vector space V is an element of $\bigwedge^2 V$. We will view a bivector b sometimes as a skew-symmetric map $V^* \to V$ (then its value at $x \in V^*$ will be denoted by $b(x)$) and sometimes as a skew-symmetric bilinear form on V^* (then its value at $x, y \in V^*$ will be denoted by $b(x, y)$).

Definition 3.2.1. *Let V be a vector space and $b^{(1)}, b^{(2)}$ be linearly independent bivectors on V. The family of bivectors $B := \{b^t\}_{t\in\mathbb{C}^2}$, $b^t := t^{(1)}b^{(1)} + t^{(2)}b^{(2)}$, $t := (t^{(1)}, t^{(2)})$, will be called a pencil of bivectors on V. We say that a pencil B is Kronecker if* $\operatorname{rank} b^t = const$, $t \in \mathbb{C}^2\backslash\{0\}$ *(or, equivalently, there are no Jordan blocks in the JK decomposition of the pair $b^{(1)}, b^{(2)}$ see subsection 3.3).*

Definition 3.2.2. *Let $B := \{b^t\}_{t\in\mathbb{C}^2}$ be a pencil of bivectors on V. Put $E_B = \{t \in \mathbb{C}^2 \mid \operatorname{rank} b^t < \max_{t\in\mathbb{C}^2} \operatorname{rank} b^t\}$. This set is called exceptional for B. It is clear that either $E_B = \{0\}$ (Kronecker case) or $E_B = \operatorname{Span}\{t_1\} \cup \cdots \cup \operatorname{Span}\{t_n\}$, where t_i are pairwise nonproportional. We shall say that the values of t in E_B are exceptional and the other ones are generic. The same terms will be used for the corresponding bivectors b^t. We put $Z^t := \ker b^t \subset V^*$ and $Z^B := \operatorname{Span}\big(\bigcup_{t\notin E_B} Z^t\big)$.*

The 'main lemma' of the theory of bi-Hamiltonian structures is as follows.

Lemma 3.2.3. *Let $B = \{b^t\}_{t\in\mathbb{C}^2}$ be a pencil of bivectors on V. Then*

(1) for any $t \in \mathbb{C}^2$ and any linearly independent elements $t', t'' \in \mathbb{C}^2$ we have $b^t(Z^{t'}, Z^{t''}) = 0$;

(2) for any $t \in \mathbb{C}^2$ and any $t' \in \mathbb{C}^2\backslash E_B$ we have $b^t(Z^{t'}, Z^{t'}) = 0$; in particular $b^t(Z^B, Z^B) = 0$.

Proof.

(1) Obviously there exist $c', c'' \in \mathbb{C}$ such that $b^t = c'b^{t'} + c''b^{t''}$. Let $z' \in Z^{t'}$, $z'' \in Z^{t''}$. Then $b^t(z', z'') = c'b^{t'}(z', z'') + c''b^{t''}(z', z'') = 0$.

(2) Let $z', z'' \in Z^{t'}$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{C}^2\backslash E_B$, such that t_n is linearly independent with t' and $\lim_{n\to\infty} t_n = t'$, and a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in Z^{t_n}$, such that and $\lim_{n\to\infty} z_n = z''$. We have $b^t(z', z_n) = 0$ by item 1 and $b^t(z', z'') = 0$ by continuity. □

From now on let us assume that $E_B = \operatorname{Span}\{t_1\} \cup \cdots \cup \operatorname{Span}\{t_n\}$, where t_i are pairwise nonproportional and that $t_0 \in \mathbb{C}^2$ is a fixed generic element.

Remark 3.2.4. It follows from the 'main lemma' that, if $I^i \subset Z^{t_i}$ are isotropic with respect to the restriction of b^{t_0} to Z^{t_i}, $i = 1, \ldots, n$, then the subspace $I := I^1 + \cdots + I^n + Z^B$ is isotropic with respect to b^{t_0}. Indeed, $b^{t_0}(Z^B, Z^B) = 0$, $b^{t_0}(Z^B, I^i) = 0$, $b^{t_0}(I^i, I^j) = 0$ for $i \neq j$ and $b^{t_0}(I^i, I^i) = 0$ for any i.

The linear algebraic counterpart (theorem 3.2.6) of our main theorem deals with the isotropic subspace I. It generalizes the following classical result.

Theorem 3.2.5. *Bolsinov (1992) The isotropic subspace Z^B is maximal isotropic with respect to b^{t_0} if and only if the pencil B is Kronecker.*

Let a pencil B of bivectors on V be given and let a subspace $\mathfrak{z}^i \subset Z^{t_i}$ be chosen for any $i \in \{1, \ldots, n\}$. Introduce the subspaces

$$\mathfrak{z}^{0,i} := \{z \in \mathfrak{z}^i \mid \exists w \in V^* : b^{t_0}(z) = b^{t_i}(w)\}, \qquad i = 1, \ldots, n, \tag{3.1}$$

and the skew-symmetric forms

$$\gamma_{\mathfrak{z}^i} : \bigwedge^2 \mathfrak{z}^{0,i} \to \mathbb{C}, \qquad \gamma_{\mathfrak{z}^i}(z_1, z_2) := b^{t_i}(w_1, w_2), \tag{3.2}$$

where $w_j \in V^*$ are any elements such that $b^{t_0}(z_j) = b^{t_i}(w_j)$, $j = 1, 2$. Note that these forms are correctly defined. Indeed, if w_j' are another elements with $b^{t_0}(z_j) = b^{t_i}(w_j')$, we have $v_j := w_j - w_j' \in Z^{t_i}$ and $b^{t_i}(w_1', w_2') = b^{t_i}(w_1 + v_1, w_2 + v_2) = b^{t_i}(w_1, w_2)$.

Theorem 3.2.6. *Let a pencil B of bivectors on V be given and $E_B = \mathrm{Span}\{t_1\} \cup \cdots \cup \mathrm{Span}\{t_n\}$, t_i being pairwise nonproportional. Fix $t_0 \in \mathbb{C}^2$, a generic element, and assume that $\mathfrak{i}^i \subset \mathfrak{z}^i, i = 1, \ldots, n$, is a maximal isotropic subspace with respect to $b^{t_0}|_{\mathfrak{z}^i}$. Then the following conditions are equivalent.*

(1) The isotropic subspace $I := \mathfrak{i}^1 + \cdots + \mathfrak{i}^n + Z^B \subset V^*$ is maximal isotropic with respect to b^{t_0}.
(2) $\mathrm{corank}\, b^{t_0}|_{\mathfrak{z}^i} + \mathrm{corank}\, b^{t_0} = 2 \dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i - \mathrm{rank}\, \gamma_{\mathfrak{z}^i} + \mathrm{corank}\, b^{t_i}$, $i \in \{1, \ldots, n\}$.

The proof of this theorem as well as of the next one is postponed to subsection 3.3.

Theorem 3.2.7. *In the hypotheses of theorem 3.2.6, the isotropic subspace $I$is maximal isotropic with respect to b^{t_0} if one of the following conditions holds:*

(1) $\mathrm{corank}\, b^{t_0} = \dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i - \mathrm{rank}\, \gamma_{\mathfrak{z}^i} + \mathrm{corank}\, b^{t_i}$, $i \in \{1, \ldots, n\}$.
(2) $\mathrm{corank}\, b^{t_0} = \mathrm{corank}\, b^{t_0}|_{\mathfrak{z}^i} - \dim \mathfrak{z}^i + \mathrm{corank}\, b^{t_i}$, $i \in \{1, \ldots, n\}$.

3.3. Linear algebra of pencils of bivectors: the Jordan–Kronecker decomposition and the proof of the main results

In this section we will exploit the following notation: given a $l \times m$-matrix M, we will write $\widetilde{M}$ for the skew-symmetric $(l+m) \times (l+m)$-matrix $\left(\begin{smallmatrix} \mathbf{0} & M \\ -M^T & \mathbf{0} \end{smallmatrix}\right)$.

The following basic theorem (Gelfand and Zakharevich 1989, Gelfand and Zakharevich 1993) describes the algebraic structure of a pair of bivectors on a vector space.

Theorem 3.3.1. *Given a finite-dimensional vector space V over $\mathbb{C}$ and a pair of bivectors $(b^{(1)}, b^{(2)})$, $b^{(i)} : \bigwedge^2 V^* \to \mathbb{C}$, there exists a direct decomposition $V^* = \oplus_{m=1}^k V_m^*$ such that $b^{(i)}(V_l^*, V_m^*) = 0$ for $i = 1, 2, l \neq m$, and the triples $\left(V_m^*, b_m^{(1)}, b_m^{(2)}\right)$, where $b_m^{(i)} := b^{(i)}|_{V_m^*}$ are from the following list.*

(1) The Jordan block $\underbrace{\mathbf{j}}_{2j_m}(\lambda)$. $\dim V_m^* = 2j_m$ and in an appropriate basis of V_m^* the matrix of $b_m^{(i)}$ is equal to $\widetilde{A_{j_m}^{(i)}}$, $i = 1, 2$, where $A_{j_m}^{(1)} = I_{j_m}$ (the unity $j_m \times j_m$-matrix) and $A_{j_m}^{(2)} = J_{j_m}^{\lambda}$ (the Jordan $j_m \times j_m$-block with the eigenvalue λ).
(2) The Jordan block $\underbrace{\mathbf{j}_{2j_m}}(\infty)$. $\dim V_m^* = 2j_m$ and in an appropriate basis of V_m^* the matrix of $b_m^{(i)}$ is equal to $\widetilde{A_{j_m}^{(i)}}$, $i = 1, 2$, where $A_{j_m}^{(1)} = J_{j_m}^0$ and $A_{j_m}^{(2)} = I_{j_m}$.

(3) *The Kronecker block* $\underbrace{\mathbf{k}_{2k_m+1}}$. $\dim V_m^* = 2k_m + 1$ *and in an appropriate basis of* V_m^* *the matrix of* $b_m^{(i)}$ *is equal to* B_{i,k_m}, $i = 1, 2$, *where*

$$B_{1,k_m} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$B_{2,k_m} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (k_m \times (k_m + 1)\text{-matrices}).$$

We will refer to the decomposition appeared in this theorem as to the Jordan–Kronecker (JK for short) decomposition. Now we will use this theorem for studying the pencil of bivectors generated by $b^{(1)}, b^{(2)}$.

Let $B := \{b^t\}_{t\in\mathbb{C}^2}, b^t := t^{(1)}b^{(1)} + t^{(2)}b^{(2)}$, be a pencil of bivectors on V and let $E_B = \mathrm{Span}\{t_1\} \cup \dots \cup \mathrm{Span}\{t_n\}$, where $t_i \neq 0$ are pairwise nonproportional.

It follows from theorem 3.3.1 that there is a direct biorthogonal (i.e., orthogonal with respect to both $b^{(1)}$ and $b^{(2)}$) decomposition

$$V^* = J_2 \oplus J_4 \oplus \dots \oplus K,$$

where J_m stands for the sum of the Jordan blocks of dimension m and K stands for the sum of the Kronecker blocks involved in the JK decomposition of the pair $b^{(1)}, b^{(2)}$ (we will also denote by $J_{>m}$ the sum of Jordan blocks of dimension greater than m in this decomposition). Note that either of the components of this decomposition can be trivial (zero dimensional).

Introduce the following notations:

(1) $Z^i := Z^{t_i} = \ker b^{t_i}, i = 1, \dots, n, Z := Z^1 + \dots + Z^n + Z^B$.
(2) Given t_0, a generic value of the parameter, put $Z^{0,i} := \{z \in Z^i \mid \exists w \in V^*: b^{t_0}(z) = b^{t_i}(w)\}, i = 1, \dots, n$, (in particular $Z^{0,i} = \mathfrak{z}^{0,i}$ for $\mathfrak{z}^i := Z^i$, cf formula (3.1)).
(3) Put $\Gamma_i := \gamma_{Z^i}$ (see formula (3.2)).

Lemma 3.3.2. *Let* $t_0 \in \mathbb{C}^2$ *be a fixed generic element and let* $i \in \{1, \dots, n\}$ *be fixed. Then*

(1) *There is a direct decomposition*

$$Z = J_2 \oplus Z_{>2} \oplus Z^B; \tag{3.3}$$

here $Z_{>2} := Z \cap J_{>2}, \dim Z_{>2} = 2j_{>2}$, *where* $j_{>2}$ *stands for the number of Jordan blocks in* $J_{>2}$. *Moreover* $\dim Z^B = 0$ *if and only if* $\dim K = 0$.

(2) *There is a direct decomposition*

$$Z^i = Z_2^i \oplus Z_{>2}^i \oplus Z_K^i;$$

here $Z_2^i := Z^i \cap J_2, \dim Z_2^i = 2j_2(\lambda_i)$, *where* $j_2(\lambda_i)$ *is the number of Jordan blocks* $\mathbf{j}_2(\lambda_i)$ *with the eigenvalue* $\lambda_i = -t_i^{(1)}/t_i^{(2)}$ *(or blocks* $\mathbf{j}_2(\infty)$ *in case* $t_i^{(2)} = 0$*) in the JK decomposition;* $Z_{>2}^i := Z^i \cap J_{>2}, \dim Z_{>2}^i = 2j_{>2}(\lambda_i)$ *(with the self-explaining notations);* $Z_K^i := Z^i \cap K$. *Moreover,* $Z_K^i \subset Z^B$ *and* $\dim Z_K^i = k$, *where k is the number of Kronecker blocks in the JK decomposition.*

(3) *There is a direct decomposition*

$$Z^{0,i} = Z_{>2}^i \oplus Z_K^i = Z_4^i \oplus Z_{>4}^i \oplus Z_K^i;$$

here $Z_4^i := Z^i \cap J_4, Z_{>4}^i := Z^i \cap J_{>4}$. *In particular, the subspace* $Z^{0,i}$ *does not depend on the choice of the generic element* t_0.

(4) The restriction of the 2-form b^{t_0} *to any Jordan block is symplectic. Moreover,* $\operatorname{corank} b^{t_0} = k$, $\ker b^{t_0}|_{Z^i} = Z^{0,i}$ *and* $\ker b^{t_0}|_Z = Z_{>2} \oplus Z^B$.

(5) The form $\Gamma_i|_{Z^i_4}$ *is nondegenerate. Moreover,* $\operatorname{rank} \Gamma_i = \dim Z^i_4$, $\ker \Gamma_i = Z^i_{>4} \oplus Z^i_K$.

Proof. This lemma is proven by direct inspection. □

Lemma 3.3.3. *Let* $I \subset V^*$ *be an isotropic subspace with respect to* b^{t_0} *such that* $Z^B \subset I \subset Z$. *Then I is maximal isotropic with respect to* b^{t_0} *if and only if the following conditions hold.*

(1) $I = I_2 \oplus Z_{>2} \oplus Z^B$, *where* $I_2 := I \cap J_2$.

(2) $J_{>2} = J_4$, *i.e. there are no Jordan blocks of dimension greater than 4 in the JK decomposition.*

(3) I_2 *is maximal isotropic with respect to* $b^{t_0}|_{J_2}$.

(Note that conditions 1 and 2 imply equality $I = I_2 \oplus Z_4 \oplus Z^B$, *where* $Z_4 := Z \cap J_4$.*)*

Proof. Assume I is maximal isotropic with respect to b^{t_0}. Then the relations $I \subset Z$ and $I = I^{\perp b^{t_0}}$ imply $Z \supset I = I^{\perp b^{t_0}} \supset Z^{\perp b^{t_0}}$. From this and from the formula $\ker b^{t_0}|_Z = Z \cap Z^{\perp b^{t_0}}$ we have $\ker b^{t_0}|_Z = Z^{\perp b^{t_0}}$. Since by lemma 3.3.2(4) $\ker b^{t_0}|_Z = Z_{>2} \oplus Z^B$, we get $I \supset Z_{>2} \oplus Z^B$. This inclusion and decomposition (3.3) imply condition 1.

Now we notice that decomposition (3.3) is orthogonal with respect to b^{t_0}. By lemma 3.1.5 the fact that I is maximal isotropic with respect to b^{t_0} implies: (a) I_2 is maximal isotropic with respect to $b^{t_0}|_{J_2}$; (b) $Z_{>2}$ is maximal isotropic with respect to $b^{t_0}|_{J_{>2}}$; (c) Z^B is maximal isotropic with respect to $b^{t_0}|_K$. Condition (a) coincides with condition 3 and condition (c) is satisfied tautologically (this can be seen from the JK decomposition but this also follows from theorem 3.2.5).

Now we will show that condition (b) implies condition 2. Indeed, assume the subspace $Z_{>2}$ is maximal isotropic with respect to $b^{t_0}|_{J_{>2}}$. Then in view of lemma 3.1.5 the intersection of Z with any Jordan block $\mathbf{j}$, $\dim \mathbf{j} > 2$, is maximal isotropic with respect to $b^{t_0}|_{\mathbf{j}}$. By lemma 3.3.2 this last form is symplectic and $Z \cap \mathbf{j}$ is two dimensional. Thus $\dim \mathbf{j} = 4$.

Conversely, assume conditions 1, 2, 3 hold. Inverting the considerations above, we can show that conditions (a), (b), (c) are satisfied and apply lemma 3.1.5 to deduce that I is maximal isotropic with respect to b^{t_0}. □

Lemma 3.3.4. *Let* $\mathfrak{z}^i \subset Z^i$ *be any subspace,* $i = 1, \ldots, n$. *Then*

(1) The following inequalities are satisfied for any $i \in \{1, \ldots, n\}$*:*

- *(1a)* $\operatorname{rank} \Gamma_i \leqslant \dim Z^{0,i} - \operatorname{corank} b^{t_0}$;
- *(1b)* $\operatorname{rank} \gamma_{\mathfrak{z}^i} \leqslant \operatorname{rank} \Gamma_i$;
- *(1c)* $\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} - 2\dim \mathfrak{z}^{0,i} + \dim \mathfrak{z}^i \leqslant \operatorname{corank} b^{t_i} - \dim Z^{0,i}$.

(2) Condition 2 of theorem 3.2.6 is satisfied if and only if the following three conditions are satisfied simultaneously:

- *(2a)* $\operatorname{rank} \Gamma_i = \dim Z^{0,i} - \operatorname{corank} b^{t_0}$, $i \in \{1, \ldots, n\}$;
- *(2b)* $\operatorname{rank} \gamma_{\mathfrak{z}^i} = \operatorname{rank} \Gamma_i$, $i \in \{1, \ldots, n\}$;
- *(2c)* $\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} - 2\dim \mathfrak{z}^{0,i} + \dim \mathfrak{z}^i = \operatorname{corank} b^{t_i} - \dim Z^{0,i}$, $i \in \{1, \ldots, n\}$.

Proof.

(1) Inequality $(1a)$ is a consequence of lemma 3.3.2. Indeed, it follows from 3.3.2(4) that $\operatorname{corank} b^{t_0}$ equals the number of Kronecker blocks in the JK decomposition. On the other hand, by 3.3.2(2) this number is equal to dimension of $Z^i_K = Z^i \cap K$ for any $i \in \{1, \ldots, n\}$. Hence, in view of 3.3.2(3,5) $\dim Z^{0,i} - \operatorname{corank} b^{t_0} = \dim Z^i_4 + \dim Z^i_{>4} = \operatorname{rank} \Gamma_i + \dim Z^i_{>4}$.

Inequality (1b) is obvious due to the equality $\gamma_{\mathfrak{z}^i} = \Gamma_i|_{\mathfrak{z}^{0,i}}$.

Now we will prove inequality (1c). We have $\ker b^{t_0}|_{Z^i} = Z^{0,i}$ by lemma 3.3.2(4). Thus the form b^{t_0} induces a symplectic form ω_i on the space $Z^i/Z^{0,i}$, $i = 1, \ldots, n$. The subspace $\mathfrak{z}^i \subset Z^i$ induces a subspace $\mathfrak{z}^i := \mathfrak{z}^i/(\mathfrak{z}^i \cap Z^{0,i}) = \mathfrak{z}^i/\mathfrak{z}^{0,i}$ of the space $Z^i/Z^{0,i}$. By lemma 3.1.6 applied to the symplectic space $(Z^i/Z^{0,i}, \omega_i)$ and the subspace $\mathfrak{z}^i \subset Z^i/Z^{0,i}$ we conclude that

$$\dim \mathfrak{z}^i + \operatorname{corank} \omega_i|_{\mathfrak{z}^i} \leqslant \dim Z^i/Z^{0,i}. \tag{3.4}$$

The left-hand side of this inequality equals $\dim \mathfrak{z}^i + \operatorname{corank} \omega_i|_{\mathfrak{z}^i} = 2\dim \mathfrak{z}^i - \operatorname{rank} \omega_i|_{\mathfrak{z}^i} = 2\dim \mathfrak{z}^i - \operatorname{rank} b^{t_0}|_{\mathfrak{z}^i} = 2(\dim \mathfrak{z}^i - \dim \mathfrak{z}^{0,i}) - \operatorname{rank} b^{t_0}|_{\mathfrak{z}^i} = \operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} + \dim \mathfrak{z}^i - 2\dim \mathfrak{z}^{0,i}$. For the right-hand side we obviously have $\dim Z^i/Z^{0,i} = \dim Z^i - \dim Z^{0,i} = \operatorname{corank} b^{t_i} - \dim Z^{0,i}$, which proves (1$c$).

(2) Assume first that conditions (2a), (2b), (2c) hold. Adding the corresponding equalities we get the equality from condition 2 of theorem 3.2.6.

Conversely, assume condition 2 of theorem 3.2.6 holds. It follows from inequalities (1a), (1b), (1c) that $P := \dim Z^{0,i} - \operatorname{corank} b^{t_0} - \operatorname{rank} \Gamma_i \geqslant 0$, $Q := \operatorname{rank} \Gamma_i - \operatorname{rank} \gamma_{\mathfrak{z}^i} \geqslant 0$, $R := \operatorname{corank} b^{t_i} - \dim Z^{0,i} - \operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} + 2\dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i \geqslant 0$. By condition 2 of theorem 3.2.6 $P + Q + R = 0$. Hence $P = 0$, $Q = 0$, $R = 0$. □

Proof of theorem 3.2.6. (1 ⇒ 2) Let $I = \mathfrak{i}^1 + \cdots + \mathfrak{i}^n + Z^B$ be maximal isotropic. Then, by lemma 3.3.3 we have (1) $I = I_2 \oplus Z_{>2} \oplus Z^B$; (2) $J_{>2} = J_4$; (3) I_2 is maximal isotropic with respect to $b^{t_0}|_{J_2}$. Condition (2) implies condition (2a) of lemma 3.3.4. Indeed, we have mentioned in the proof of lemma 3.3.4 that $\dim Z^{0,i} - \operatorname{corank} b^{t_0} = \operatorname{rank} \Gamma_i + \dim Z^i_{>4}$. Condition 2) is equivalent to $\dim Z^i_{>4} = 0$, hence also equivalent to (2a).

Now we will prove that condition (2b) is satisfied. Consider the space $\mathfrak{z} := \mathfrak{z}^1 + \cdots + \mathfrak{z}^n + Z^B$. We have $J_2 \oplus Z_4 \oplus Z^B = Z \supset \mathfrak{z} \supset I \supset Z_4 \oplus Z^B$ (the first equality follows from lemma 3.3.2(1) and from condition (2)). Hence $\mathfrak{z} = \mathfrak{z}_2 \oplus Z_4 \oplus Z^B$, where $\mathfrak{z}_2 := \mathfrak{z} \cap J_2$. This yields decompositions $\mathfrak{z}^i := \mathfrak{z}^i_2 \oplus Z^i_4 \oplus \mathfrak{z}^i_K$, $i = 1, \ldots, n$, where $\mathfrak{z}^i_2 := \mathfrak{z}^i \cap J_2$, $\mathfrak{z}^i_K := \mathfrak{z}^i \cap K$, and $\mathfrak{z}^{0,i} = Z^i_4 \oplus \mathfrak{z}^i_K$ (cf lemma 3.3.2(3)). Thus, by lemma 3.3.2(5) we have $\operatorname{rank} \Gamma_i = \dim Z^i_4 = \operatorname{rank} \Gamma_i|_{\mathfrak{z}^{0,i}} = \operatorname{rank} \gamma_{\mathfrak{z}^i}$.

Finally, we will use condition (3) to show that condition (2c) is satisfied. Moreover, we will show that they are equivalent (provided that conditions (1) and (2) are satisfied). The 2-form $b^{t_0}|_{Z^i_2}$ is symplectic by lemma 3.3.2(4). Condition (3) is equivalent via lemma 3.1.5 to the following: the subspace $\mathfrak{i}^i_2 \subset \mathfrak{z}^i_2 \subset Z^i_2$, where $\mathfrak{i}^i_2 := \mathfrak{i}^i \cap J_2$, is maximal isotropic with respect to $b^{t_0}|_{Z^i_2}$ for any i. Lemma 3.1.4 implies (recall the assumption that $\mathfrak{i}^i$ is maximal isotropic with respect to $b^{t_0}|_{\mathfrak{z}^i}$) that this last holds if and only if $\mathfrak{z}^i_2$ is coisotropic with respect to $b^{t_0}|_{Z^i_2}$ for any i. This in turn is equivalent, in view of lemma 3.1.6, to the following condition:

$$\dim \mathfrak{z}^i_2 + \operatorname{corank} b^{t_0}|_{\mathfrak{z}^i_2} = \dim Z^i_2, \qquad i = 1, \ldots, n. \tag{3.5}$$

Using the notations from the proof of lemma 3.3.4 and natural identifications $\mathfrak{z}^i_2 = \mathfrak{z}^i$, $Z^i_2 = Z^i/Z^{0,i}$, $\omega_i = b^{t_0}|_{Z^i_2}$, we can proceed as in this proof to conclude that condition (3.5) is equivalent to condition (2c).

(2 ⇒ 1) Let condition 2 of theorem 3.2.6 hold. Then conditions (2a), (2b), (2c) are satisfied by lemma 3.3.4.

We have seen in the first part of the proof that condition (2a) is equivalent to the equality $J_{>2} = J_4$.

Now we will show that the assumption that $\mathfrak{i}^i$ is maximal isotropic with respect to $b^{t_0}|_{\mathfrak{z}^i}$ for any i together with condition (2b) imply the direct decompositions $\mathfrak{z} = \mathfrak{z}_2 \oplus Z_4 \oplus Z^B$ and $I = I_2 \oplus Z_4 \oplus Z^B$.

Since the set $\mathfrak{i}^i$ is maximal isotropic with respect to $b^{t_0}|_{\mathfrak{z}^i}$, it contains $\ker b^{t_0}|_{\mathfrak{z}^i}$. Lemma 3.3.2(4) implies $\ker b^{t_0}|_{\mathfrak{z}^i} \supset \mathfrak{z}^i \cap Z^{0,i} = \mathfrak{z}^{0,i}$, whence $\mathfrak{i}^i \supset \mathfrak{z}^{0,i}$. On the other hand, since in view of lemma 3.3.2 (and equality $J_{>2} = J_2$) we have $Z^{0,i} = Z_4^i \oplus Z_K^i$ and $Z_K^i = \ker \Gamma_i$, the equality from condition (2b) implies the equality $\mathfrak{z}^{0,i} + Z_K^i = Z^{0,i}$. This equality together with the inclusion $\mathfrak{i}^i \supset \mathfrak{z}^{0,i}$ imply the relation $\mathfrak{z} \supset I \supset Z_4 + Z^B$, whence $\mathfrak{z} = \mathfrak{z}_2 \oplus Z_4 \oplus Z^B$ and $I = I_2 \oplus Z_4 \oplus Z^B$.

We have already proven that conditions (3) and (2c) are equivalent. Finally, I is an isotropic subspace such that its components I_2, Z_4, Z^B are maximal isotropic with respect to $b^{t_0}|_{J_2}, b^{t_0}|_{J_{>2}}, b^{t_0}|_K$, respectively. Now it remains to use lemma 3.1.5 to conclude that I is maximal isotropic with respect to b^{t_0} itself. □

Proof of theorem 3.2.7. We first notice that adding inequalities (1a), (1b) and (1c) from lemma 3.3.4 we get the following inequality:

$$\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} + \operatorname{corank} b^{t_0} \leqslant 2\dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i - \operatorname{rank} \gamma_{\mathfrak{z}^i} + \operatorname{corank} b^{t_i}, \qquad i = 1, \dots, n. \tag{3.6}$$

Assume condition 1 of theorem 3.2.7 holds. Besides, we have $\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} = \dim\ker b^{t_0}|_{\mathfrak{z}^i} \geqslant \dim \mathfrak{z}^{0,i}$ (since we have already shown the inclusion $\ker b^{t_0}|_{\mathfrak{z}^i} \supset \mathfrak{z}^{0,i}$). Thus we deduce from this and from condition 1 the inequality

$$\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} + \operatorname{corank} b^{t_0} \geqslant 2\dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i - \operatorname{rank} \gamma_{\mathfrak{z}^i} + \operatorname{corank} b^{t_i}, \qquad i = 1, \dots, n. \tag{3.7}$$

Combining this inequality with inequality (3.6) we see that condition 2 of theorem 3.2.6 is satisfied, and hence we get the maximality of the isotropic subspace I.

Now assume condition 2 of theorem 3.2.7 is satisfied. Again combining it with the inequality $\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} \geqslant \dim \mathfrak{z}^{0,i}$ we get $\operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} + \operatorname{corank} b^{t_0} \geqslant \dim \mathfrak{z}^{0,i} + \operatorname{corank} b^{t_0}|_{\mathfrak{z}^i} - \dim \mathfrak{z}^i + \operatorname{corank} b^{t_i} \geqslant 2\dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i + \operatorname{corank} b^{t_i} \geqslant 2\dim \mathfrak{z}^{0,i} - \dim \mathfrak{z}^i - \operatorname{rank} \gamma_{\mathfrak{z}^i} + \operatorname{corank} b^{t_i}$, $i = 1, \dots, n$. Thus we have come to inequality (3.7) once more and we can proceed as above. □

4. Application to Lie pencils

4.1. Lie pencils with symmetries

In this section we apply the results of section 2 to pencils consisting of linear Poisson structures.

All Lie algebras considered below will be finite dimensional and defined over $\mathbb{C}$. Let $\mathfrak{g}$ be a vector space and $[\,,\,]^{(i)} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $i = 1, 2$, a bilinear operation.

Definition 12. *Assume* $[\,,\,]^t := t^{(1)}[\,,\,]^{(1)} + t^{(2)}[\,,\,]^{(2)}$ *is a Lie algebra structure on* $\mathfrak{g}$ *for any* $t = (t^{(1)}, t^{(2)}) \in \mathbb{C}^2$. *We say that the Lie brackets* $[\,,\,]^{(1)}, [\,,\,]^{(2)}$ *are compatible and that* $\Lambda := (\mathfrak{g}, \{[\,,\,]^t\}_{t\in\mathbb{C}^2})$ *is a Lie pencil. We put* $\mathfrak{g}^t := (\mathfrak{g}, [\,,\,]^t)$ *for short.*

Given a Lie pencil Λ, we have the pencil $\Theta_\Lambda = \{\vartheta_{\mathfrak{g}^t}\}$ of the Lie–Poisson structures on the dual space $\mathfrak{g}^*$. We will write ϑ^t for $\vartheta_{\mathfrak{g}^t}$ for short.

Definition 4.1.2. *A Lie pencil will be called admissible if the corresponding Poisson pencil* Θ_Λ *is admissible (see definition 2.2.5). From now on we will consider only admissible Lie pencils such that the corresponding exceptional set* $E_\Lambda := E_{\Theta_\Lambda}$ *is equal to* $\operatorname{Span}\{t_1\} \cup \cdots \cup \operatorname{Span}\{t_n\}$. *We assume that the exceptional elements* $t_1, \dots, t_n$ *are pairwise nonproportional and we call exceptional also the corresponding Lie algebras* $\mathfrak{g}^{t_i}$, $i = 1, \dots, n$.

Put $\mathfrak{z}^i$ for the center of the exceptional algebra $\mathfrak{g}^{t_i}$. It is shown in Trofimov and Fomenko (1995, proposition 4, section 44) that $\mathfrak{z}^i$ is a Lie subalgebra in $\mathfrak{g}^t$ for any t. Now we are ready to formulate our main theorem about Lie pencils. We recall that the *index* $\operatorname{ind}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the codimension of a regular coadjoint orbit.

Theorem 4.1.3. *Let $\Lambda := \{\mathfrak{g}^t\}_{t\in\mathbb{C}^2}$ be an admissible Lie pencil and let $\Theta := \Theta_\Lambda = \{\vartheta^t\}$ stand for the corresponding bi-Hamiltonian structure on $\mathfrak{g}^*$. We assume that the set of Casimir functions $\mathcal{Z}^t(\mathfrak{g}^*)$ is complete as a set of Casimir functions (see definition 2.1.2) for an infinite number of pairwise nonproportional generic values of t. Fix a generic $t_0 \in \mathbb{C}^2$.*

Let $\mathcal{I}_i \subset \mathcal{E}((\mathfrak{z}^i)^*), i = 1, \dots, n$, be an involutive set of functions which is complete as an involutive set of functions (see definition 2.1.3) with respect to $\vartheta_{\mathfrak{z}^i}$, where $\mathfrak{z}^i$ is endowed with the Lie algebra structure induced from $\mathfrak{g}^{t_0}$. Let $\mathcal{Z}^\Theta(\mathfrak{g}^*)$ be the involutive set of functions defined in corollary 2.2.9 and let $\pi_j : \mathfrak{g}^* \to (\mathfrak{z}^j)^*$ be the canonical projection.

Then the following conditions are equivalent.

(1) The involutive set of functions $\mathcal{I} := \pi_1^*(\mathcal{I}_1) + \dots + \pi_n^*(\mathcal{I}_n) + \mathcal{Z}^\Theta(\mathfrak{g}^*)$ is complete as an involutive set of functions with respect to ϑ^{t_0}.

(2) There exists a point $x \in \mathfrak{g}^*$ such that for any $i = 1, \dots, n$,

$$\operatorname{ind}\mathfrak{z}^i + \operatorname{ind}\mathfrak{g}^{t_0} = 2\dim\mathfrak{z}_x^{0,i} - \dim\mathfrak{z}^i - \operatorname{rank}\gamma_{\mathfrak{z}^i,x} + \operatorname{ind}\mathfrak{g}^{t_i}, \tag{4.1}$$

where the vector space $\mathfrak{z}_x^{0,i}$ is defined as $\mathfrak{z}_x^{0,i} := \left\{v \in \mathfrak{z}^i \mid \exists w \in \mathfrak{g}\colon \left(\mathrm{ad}_v^{t_0}\right)^* x = \left(\mathrm{ad}_w^{t_i}\right)^* x\right\}$, $\mathrm{ad}_w^{t_i} u := [w, u]^{t_i}, u \in \mathfrak{g}$, the 2-form $\gamma_{\mathfrak{z}^i,x}$ on $\mathfrak{z}_x^{0,i}$ is defined as $\gamma_{\mathfrak{z}^i,x}(v_1, v_2) := \langle [w_1, w_2]^{t_i}, x\rangle$, $w_1, w_2 \in \mathfrak{g}$ being any elements such that $\left(\mathrm{ad}_{v_j}^{t_0}\right)^* x = (\mathrm{ad}_{w_j}^{t_i})^* x$, $j = 1, 2$.

Proof. We first note that equalities (2.4) are satisfied at some point $x \in \mathfrak{g}^*$ if and only if they are satisfied at all points from some Zariski open set U in $\mathfrak{g}^*$. Shrinking U if needed we may assume that U does not intersect the singular sets $\operatorname{Sing}\vartheta^{t_0}$ and $\operatorname{Sing}\vartheta^{t_i}, \pi_i^{-1}(\operatorname{Sing}\vartheta_{\mathfrak{z}^i}), i = 1, \dots, n$. Thus for any $x \in U$ we have $\operatorname{corank}\vartheta_x^{t_0} = \operatorname{ind}\mathfrak{g}^{t_0}$, $\operatorname{corank}\vartheta_x^{t_0}|_{\mathfrak{z}_x^i} = \operatorname{ind}\mathfrak{z}^i$, $\operatorname{corank}\vartheta_x^{t_i} = \operatorname{ind}\mathfrak{g}^{t_i}$.

Since $\mathfrak{z}^i$ consist of elements of $\mathfrak{g}$, i.e. linear functions on $\mathfrak{g}^*$, we have $\dim\mathfrak{z}_x^i = \dim\mathfrak{z}^i$ for any x. Finally, we recall that by definition $\vartheta_x^{t_i}(v) = \left(\mathrm{ad}_v^{t_i}\right)^* x, i = 0, \dots, n$, for any $x \in \mathfrak{g}^*, v \in \mathfrak{g} \cong T_x^*\mathfrak{g}^*$.

With these remarks the proof is a direct consequence of theorem 2.2.10. □

Theorem 4.1.4. *In the assumptions of theorem 4.1.3, the involutive set of functions $\mathcal{I}$ is complete as an involive set of functions with respect to ϑ^{t_0} if there exists $x \in \mathfrak{g}^*$ such that one of the following two conditions hold:*

$$(1)\quad \operatorname{ind}\mathfrak{g}^{t_0} = \dim\mathfrak{z}_x^{0,i} - \dim\mathfrak{z}^i - \operatorname{rank}\gamma_{\mathfrak{z}^i,x} + \operatorname{ind}\mathfrak{g}^{t_i}, \qquad i \in \{1, \dots, n\}; \tag{4.2}$$

$$(2)\quad \operatorname{ind}\mathfrak{g}^{t_0} = \operatorname{ind}\mathfrak{z}^i - \dim\mathfrak{z}^i + \operatorname{ind}\mathfrak{g}^{t_i}, \qquad i \in \{1, \dots, n\}. \tag{4.3}$$

Proof. The proof follows from theorem 2.2.11. □

Remark 4.1.5. Condition (4.3) (which is independent of $x \in \mathfrak{g}^*$) coincides with the sufficient condition of Bolsinov (Trofimov and Fomenko 1995, proposition 7, section 44) (see also Introduction).

Now we will indicate some class of Lie pencils which satisfy assumptions of theorem 4.1.3.

Lemma 4.1.6. *Let a Lie pencil $\Lambda = \{\mathfrak{g}^t\}_{t\in\mathbb{C}^2}$ on a vector space $\mathfrak{g}$ be such that for some $t_0 \in \mathbb{C}^2$ the Lie algebra $\mathfrak{g}^{t_0}$ is semisimple. Then Λ satisfies the assumptions of theorem 2.2.10.*

Proof. Since $\mathfrak{g}^{t_0}$ is semisimple we have $H^2(\mathfrak{g}^{t_0}, \mathfrak{g}^{t_0}) = 0$, where we denote by $H^2(\mathfrak{g}^{t_0}, \mathfrak{g}^{t_0})$ the second cohomology with coefficients in the adjoint module. Then it follows from the general theory of deformations of Lie algebras (Nijenhuis and Richardson 1967, theorem 7.2) that the orbit of $\mathfrak{g}^{t_0}$ in the space of all Lie algebra structures on $\mathfrak{g}$ under the action of $GL(\mathfrak{g})$ is Zariski open. In particular, the intersection of this open set with the two-dimensional vector space Λ is Zariski open in Λ. Thus the complement C to this set in Λ is closed, and, being homogeneous, is either zero or a finite union of lines.

It is known that for a semisimple Lie algebra $\mathfrak{g}$ the algebraic set Sing $\mathfrak{g}$ of all adjoint orbits of dimension less than the maximal one is of codimension at least 3. Thus we have codim Sing $\mathfrak{g}^t \geqslant 3$ for any $t \in \mathbb{C}^2 \backslash C$ (since by the considerations above $\mathfrak{g}^t \cong \mathfrak{g}^{t_0}$) and the codimension of the variety $S := \bigcup_{t \in \mathbb{C}^2 \backslash C} \operatorname{Sing} \mathfrak{g}^t$ is greater than or equal to 2.

We conclude that for x from the open dense set $\mathfrak{g} \backslash S$ the exceptional set $E_{\Theta_\Lambda}(x)$ does not depend on x (indeed, corank $\vartheta_x^t = \operatorname{rank} \mathfrak{g}^{t_0}$, in particular $E_{\Theta_\Lambda}(x) \subset C$). Hence Θ_Λ and Λ itself are admissible.

The assumption of theorem 4.1.3 related to Casimir functions is also satisfied. Indeed, the considerations above show that for $t \in \mathbb{C}^2 \backslash C$ the Lie algebra $\mathfrak{g}^t$ is semisimple. Now the completeness of the set of global Casimir functions for the semisimple Lie algebra $\mathfrak{g}^t$ is the standard fact (they are invariant polynomials and there are rank $\mathfrak{g}^t$ functionally independent invariant polynomials on $\mathfrak{g}^t$). □

4.2. Applications of the main theorem to Lie pencils of the Kantor–Persits type

The aim of this section indicates a class of Lie pencils for which condition 2 of theorem 4.1.3 is satisfied.

Introduce the following notation: if $X, A, Y \in \mathfrak{gl}(n, \mathbb{C})$, put $[X, Y]_A := XAY - YAX$. Write I_n for the identity $n \times n$-matrix. It is easy to see that the following families are Lie pencils

$$(\mathfrak{gl}(n, \mathbb{C}), [\, ,\,]_{t^{(1)} I_n + t^{(2)} A})_{(t^{(1)}, t^{(2)}) \in \mathbb{C}^2}, \qquad A \in \mathfrak{gl}(n, \mathbb{C}), \tag{4.4}$$

$$(\mathfrak{so}(n, \mathbb{C}), [\, ,\,]_{t^{(1)} I_n + t^{(2)} A})_{(t^{(1)}, t^{(2)}) \in \mathbb{C}^2}, \qquad A \in \mathfrak{symm}(n, \mathbb{C}), \tag{4.5}$$

$$(\mathfrak{sp}(n, \mathbb{C}), [\, ,\,]_{t^{(1)} I_{2n} + t^{(2)} A})_{(t^{(1)}, t^{(2)}) \in \mathbb{C}^2}, \qquad A \in \mathfrak{m}(n, \mathbb{C}). \tag{4.6}$$

Here $\mathfrak{symm}(n, \mathbb{C})$ is the space of symmetric $n \times n$-matrices, $\mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid XJ + JX^T = 0\}$ is the symplectic Lie algebra, $\mathfrak{m}(n, \mathbb{C}) := \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid XJ - JX^T = 0\}$ is its orthogonal complement in $\mathfrak{gl}(2n, \mathbb{C})$ with respect to the 'trace form', the matrix A is fixed.

Definition 4.2.1. *The Lie pencils given by formulae (4.4), (4.5) and (4.6) will be called of Kantor–Persits (KP for short) type.*

Remark 4.2.2. These Lie pencils appeared in the paper Kantor and Persits (1988) (see also Trofimov and Fomenko 1995, section 44).

We put $N := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$ to be the standard nilpotent $(n \times n)$-matrix. It is known from linear algebra that the operator in the complex Euclidean space of dimension n which has such a matrix in the so-called normal base, i.e. the base with the Gramm matrix $G_n := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \cdots & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$, is symmetric. In particular, if Q_n is the transition matrix from the normal to the orthonormal base, that is $Q_nG_nQ_n^T = I_n$, then the matrix $\widetilde{N} := Q_nNQ_n^{-1}$ is a symmetric nilpotent matrix.

Given a $n \times n$-matrix X we put $\widehat{X} := \begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & X^T \end{bmatrix}$ ($(2n \times 2n)$-matrix). We will also exploit the notation $\operatorname{diag}(D_1, D_2)$ for the block-diagonal matrix with the square blocks D_1, D_2 on the diagonal.

Proposition 4.2.3. *Let a Lie algebra* $\mathfrak{g}$ *and a matrix A be from the following list:*

(a) $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ *and* $A = N^k$, $k \leqslant n$;
(b) $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ *and* $A = \widetilde{N}^k$, $k \leqslant n$;
(c) $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ *and* $A = \widehat{N}^k$, $k \leqslant n$.

Consider the Lie pencil $\Lambda := \{(\mathfrak{g}, [\,,\,]_{t^{(1)}I_n + t^{(2)}A})\}$. *Then*

(1) The only (up to proportionality) exceptional Lie algebra in this pencil is $\mathfrak{g}^{t_1} := (\mathfrak{g}, [\,,\,]_A)$ *(here* $t_1 = (0, 1)$*).*
(2) The Lie algebra $\mathfrak{g}^{t_1}$ *is isomorphic to the Lie algebra* $(\mathfrak{g}, [\,,\,]_P)$, *where* $P := \operatorname{diag}(I_{n-k}, \mathbf{0}_k)$ *in cases a, b, and to the Lie algebra* $(\mathfrak{g}, [\,,\,]_{\widehat{P}})$ *in case c. In particular,* $\operatorname{ind}\mathfrak{g}^{t_1} = \operatorname{ind}\mathfrak{gl}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{gl}(k, \mathbb{C}) + \dim\mathfrak{gl}(k, \mathbb{C})$ *in case a,* $\operatorname{ind}\mathfrak{g}^{t_1} = \operatorname{ind}\mathfrak{so}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{so}(k, \mathbb{C}) + \dim\mathfrak{so}(k, \mathbb{C})$ *in case b, and* $\operatorname{ind}\mathfrak{g}^{t_1} = \operatorname{ind}\mathfrak{sp}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{sp}(k, \mathbb{C}) + \dim\mathfrak{sp}(k, \mathbb{C})$ *in case c.*
(3) The center $\mathfrak{z}^1$ *of the Lie algebra* $\mathfrak{g}^{t_1}$ *has dimension equal to* $\dim\mathfrak{gl}(n, \mathbb{C}) = k^2$ *in case a, to* $\dim\mathfrak{so}(k, \mathbb{C}) = k(k-1)/2$ *in case b, and to* $\dim\mathfrak{sp}(k, \mathbb{C}) = 2k^2 + k$ *in case c. If* $k \leqslant n/2$, $\mathfrak{z}^1$ *is an Abelian subalgebra of* $(\mathfrak{g}, [\,,\,])$.
(4) If $k \leqslant n/2$ *and* $t_0 = (1, 0)$, *the subspace* $\mathfrak{z}_x^{0,1} = \left\{v \in \mathfrak{z}^1 \mid \exists w \in \mathfrak{g}\colon \left(\operatorname{ad}_v^{t_0}\right)^* x = \left(\operatorname{ad}_w^{t_1}\right)^* x\right\} \subset \mathfrak{z}^1$ *coincides with the whole* $\mathfrak{z}^1$.
(5) If $k = n/2$, *the rank of the skew-symmetric form* $\gamma_{\mathfrak{z}^1,x}$ *for generic* $x \in \mathfrak{g}^*$ *is equal to* $\dim\mathfrak{gl}(k, \mathbb{C}) - \operatorname{ind}\mathfrak{gl}(k, \mathbb{C})$ *in case a, to* $\dim\mathfrak{so}(k, \mathbb{C}) - \operatorname{ind}\mathfrak{so}(k, \mathbb{C})$ *in case b, and to* $\dim\mathfrak{sp}(k, \mathbb{C}) - \operatorname{ind}\mathfrak{sp}(k, \mathbb{C})$ *in case c.*
(6) If $k = n/2$, *the Lie pencil* Λ *satisfies condition (4.1) and the weaker condition (4.2). It does not satisfy condition (4.3) in case c, and if* $1 < k = n/2$ *(respectively* $2 < k = n/2$*) also in case a (respectively b).*

Proof.

(1) If t is nonproportional to $(0, 1)$, the map $X \mapsto \sqrt{t^{(1)}I_n + t^{(2)}A}\,X\sqrt{t^{(1)}I_n + t^{(2)}A}$, $X \in \mathfrak{g}$, is an isomorphism of the Lie algebras $(\mathfrak{g}, [\,,\,])$ and $(\mathfrak{g}, [\,,\,]_{t^{(1)}I_n + t^{(2)}A})$.
(2) *Case a.* The isomorphism is realized by $[X, Y]_{N^k} = L^{-1}[LX, LY]_P$, where $L : \mathfrak{g} \to \mathfrak{g}$ is an invertible map given by $LX := CX$, $C := \begin{bmatrix} \mathbf{0} & I_{n-k} \\ I_k & \mathbf{0} \end{bmatrix}$. The index of the Lie algebra $(\mathfrak{g}, [\,,\,]_P)$ is equal to $\operatorname{ind}\mathfrak{gl}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{gl}(k, \mathbb{C}) + \dim\mathfrak{gl}(k, \mathbb{C}) = k^2 + (n-k)$ (this can be proven, for instance, using the generalized Raïs formula (Panasyuk 2008).

Case b. The isomorphism is realized by $[X, Y]_P = L^{-1}[LX, LY]_{\widetilde{N}^k}$, where $L : \mathfrak{g} \to \mathfrak{g}$ is an invertible linear map given by $LX := Q_nG_nR_{n-k}^TX(Q_nG_nR_{n-k}^T)^T$, $R_{n-k} := \begin{bmatrix} Q_{n-k} & \mathbf{0} \\ \mathbf{0} & I_k \end{bmatrix}$. Indeed, $L^{-1}((LX)\widetilde{N}^k(LY)) = XR_{n-k}G_nQ_n^T\widetilde{N}^kQ_nG_nR_{n-k}^TY = XR_{n-k}N^kG_nR_{n-k}^TY$ (we used the obvious equalities $G_n^T = G_n$ and $G_nQ_n^TQ_n = I_n$). Hence $L^{-1}[LX, LY]_{\widetilde{N}^k} = [X, Y]_{R_{n-k}N^kG_nR_{n-k}^T}$. On the other hand, it is easy to see that $N^kG_n = \begin{bmatrix} G_{n-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and, since $Q_{n-k}G_{n-k}Q_{n-k}^T = I_{n-k}$, we get $R_{n-k}N^kG_nR_{n-k}^T = P$.

The structure of the Lie algebra $\mathfrak{g}_P$ was studied by Bolsinov (Trofimov and Fomenko 1995, section 44). In particular, (1) its center is isomorphic to the subalgebra $\mathfrak{so}(k, \mathbb{C})$ and is of dimension $k(k-1)/2$; (2) its index is equal to $\operatorname{ind}\mathfrak{so}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{so}(k, \mathbb{C}) + \dim\mathfrak{so}(k, \mathbb{C})$.

Case c. The isomorphism is realized by $[X, Y]_{\widehat{N}^k} = L^{-1}[LX, LY]_{\widehat{P}}$, where $L : \mathfrak{g} \to \mathfrak{g}$ is an invertible linear map given by $LX := \widehat{C}X$, $C := \begin{bmatrix} \mathbf{0} & I_{n-k} \\ I_k & \mathbf{0} \end{bmatrix}$.

The Lie algebra $\mathfrak{g}_P$ was also studied by Bolsinov (Trofimov and Fomenko 1995, section 44). In particular, (1) its center is isomorphic to the subalgebra $\mathfrak{sp}(k, \mathbb{C})$ and is of dimension $2k^2 + k$; (2) its index is equal to $\operatorname{ind}\mathfrak{sp}(n, \mathbb{C}) - \operatorname{ind}\mathfrak{sp}(k, \mathbb{C}) + \dim\mathfrak{sp}(k, \mathbb{C})$.

(3) The dimension of $\mathfrak{z}^1$ is clear from the proof of the preceding item. The fact that this is an Abelian subalgebra for $k \leqslant n/2$ follows from the representation $\mathfrak{z}^1 = \{Q^{n-k}vQ^{n-k} \mid v \in \mathfrak{g}\}$, where $Q := N$ in case a, $Q := \widetilde{N}$ in case b, $Q := \widehat{N}$ in case c.

(4) We have mentioned that $\mathfrak{z}^1 = \{Q^{n-k}vQ^{n-k} \mid v \in \mathfrak{g}\}$. On the other hand, direct calculations show that $\operatorname{ad}^{t_1}_{Q^{n-k}vQ^{n-2k}+Q^{n-2k}vQ^{n-k}} = \operatorname{ad}^{t_0}_{Q^{n-k}vQ^{n-k}}$ for any $v \in \mathfrak{g}$.

(5) By definition, given $x \in \mathfrak{g}^*$, we have: $\gamma_{\mathfrak{z}^1,x}(Q^{n-k}vQ^{n-k}, Q^{n-k}wQ^{n-k}) = \langle [v', w']_{Q^k}, x\rangle$, where $v' := Q^{n-k}vQ^{n-2k} + Q^{n-2k}vQ^{n-k}$, $w' := Q^{n-k}wQ^{n-2k} + Q^{n-2k}wQ^{n-k}$.

Case a. Introduce the following block-matrices: $v := \begin{bmatrix} * & * \\ V & * \end{bmatrix}$, $w := \begin{bmatrix} * & * \\ W & * \end{bmatrix}$, where V, W are arbitrary $(k \times k)$-matrices. It is easy to see that $v' = \begin{bmatrix} V & * \\ \mathbf{0} & V \end{bmatrix}$, $w' = \begin{bmatrix} W & * \\ \mathbf{0} & W \end{bmatrix}$ and $[v', w']_{N^k} = \begin{bmatrix} \mathbf{0} & [V, W] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Thus$\gamma_{\mathfrak{z}^1,x}(N^{n-k}vN^{n-k}, N^{n-k}wN^{n-k}) = \left\langle \begin{bmatrix} \mathbf{0} & [V, W] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, x \right\rangle$. In other words, for generic x, the rank of the form $\gamma_{\mathfrak{z}^1,x}$ coincides with the rank of the Kirillov form $\langle [V, W], \tilde{x}\rangle$ on the Lie algebra $\mathfrak{gl}(k, \mathbb{C})$, where $\tilde{x} \in \mathfrak{gl}(k, \mathbb{C})^*$ is also generic, i.e. $\operatorname{rank}\gamma_{\mathfrak{z}^1,x} = \dim\mathfrak{gl}(k, \mathbb{C}) - \operatorname{ind}\mathfrak{gl}(k, \mathbb{C})$.

Cases b and c. The proofs are similar to that in case a but more cumbersome, so we skip them.

(6) Direct calculation.

□

Remark 4.2.4. Retain the notations of proposition 4.2.3. Let $\Theta := \Theta_\Lambda$ be the Poisson pencil corresponding to the Lie pencil Λ and let ϑ^{t_0} be the standard Lie–Poisson structure on $\mathfrak{g}^*$. Let $\mathcal{I}_1 \subset \mathcal{E}((\mathfrak{z}^1)^*)$ be an involutive set of functions which is complete as an involutive set of functions (see definition 2.1.3) with respect to $\vartheta_{\mathfrak{z}^1}$, where $\mathfrak{z}^1$ is endowed with the Lie algebra structure induced from $\mathfrak{g}^{t_0}$.

Then it follows from this proposition and from theorem 4.1.3 that, if n is even and $k = n/2$, the involutive set of functions $\pi_1^*(\mathcal{I}_1) + \mathcal{Z}^\Theta(\mathfrak{g}^*)$ is complete as an involutive set of functions with respect to ϑ^{t_0} in all the three cases.

4.3. One explicit example

In this example we indicate explicitly the commuting functions generating the family $\pi_1^*(\mathcal{I}^1) + \mathcal{Z}^\Theta(\mathfrak{g}^*)$ in the case of the Lie pencil of the KP type on

$$\mathfrak{g} := \mathfrak{sp}(4, \mathbb{C}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{12} & b_{22} \\ c_{11} & c_{12} & -a_{11} & -a_{21} \\ c_{12} & c_{22} & -a_{12} & -a_{22} \end{bmatrix} \right\},$$

which was considered in the previous subsection. This is the lowest dimensional case of a KP pencil for which the Bolsinov sufficient condition (4.3) is not satisfied but the more general sufficient condition (4.2) is satisfied.

The Lie algebra $\mathfrak{z}^1$ consists of matrices of the form $\begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & -a & 0 \end{bmatrix}$, i.e. an Abelian three-dimensional Lie algebra. Thus the set $\mathcal{I}^1$ is generated by some linear coordinates g_1, g_2, g_3 on $\mathfrak{z}^*$.

The Lie algebra $\mathfrak{g}$ is ten dimensional and has index 2, and hence the dimension of a generic coadjoint orbit is 8. Therefore a complete involutive set of functions on $\mathfrak{g}^*$ will be generated by six independent functions and we need to indicate only three independent functions from the set $\mathcal{Z}^{\Theta}$. Two of them, f_1, f_2, are independent Casimir functions of the Lie–Poisson structure, the third one, H, the 'Hamiltonian', will be indicated below.

Put $L(\lambda)X := \sqrt{I_4 + \lambda\widehat{N}} \cdot X \cdot \sqrt{I_4 + \lambda\widehat{N}}$, where $N := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ (see the notation introduced before proposition 4.2.3). It is easy to see that $\sqrt{I_4 + \lambda\widehat{N}} = \widehat{Q}$, $Q := \left[\begin{smallmatrix} 1 & \lambda/2 \\ 0 & 1 \end{smallmatrix}\right]$ and that $(L(\lambda))^{-1}[L(\lambda)X, L(\lambda)Y] = [X, Y]_{I_4+\lambda\widehat{N}}, \lambda \in \mathbb{C}$.

The set $\mathcal{Z}^{\Theta}$ is generated by the Casimir functions of the generic brackets of the pencil, in other words, by the functions of the form $f_i(((L(\lambda))^*)^{-1}x), i = 1, 2$, or by the coefficients of their Taylor expansions with respect to λ. Since we need only one function, we can put $H(x) := -\frac{\partial}{\partial\lambda}\big|_{\lambda=0} f_1(((L(\lambda))^{-1})^*x), x \in \mathfrak{g}^*$, where f_1 is a quadratic Casimir function.

Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the 'trace form' $(X, Y) := \mathrm{Tr}(XY)$ we can choose $f_1(X) = \mathrm{Tr}(X^2)$, $f_2(X) = \mathrm{Tr}(X^4)$, $g_1 = a_{21}, g_2 = c_{11}, g_3 = b_{22}$ and we have $H = \mathrm{Tr}(L(X) \cdot X)$, where $L = \frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=0} L(\lambda)$ is given by $L(X) = \frac{1}{2}(\widehat{N}X + X\widehat{N})$. It is easy to calculate that $H = 2(a_{11}a_{21} + b_{12}c_{11} + a_{22}a_{21} + b_{22}c_{12})$.

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References

Bolsinov A 1992 Compatible Poisson brackets on Lie algebras and completeness of families of functions in involution *Math. USSR Izvestiya* **38** 69–90 (Translated from Russian)

da Silva A C and Weinstein A 1999 *Geometric Models for Noncommutative Algebras* (Providence, RI: American Mathematical Society)

Gelfand I and Zakharevich I 1989 Spectral theory for a pair of skew-symmetrical operators on S^1 *Funct. Anal. Appl.* **23** 85–93 (Translation from Russian)

Gelfand I and Zakharevich I 1993 *'The Gelfand Mathematical Seminars 1990-1992'* (Basle: Birkhauser) pp 51–112

Kantor I L and Persits D B 1988 *'IX All-Union Geometric Conference'* (Kishinev: Shtiintsa) p 141 (in Russian)

Manakov S V 1976 A remark on the integration of the Euler equation of the dynamics of an n-dimensional rigid body *Funct. Anal. Appl.* **10** 328–9

Mishchenko A S and Fomenko A T 1978 Euler equations on finite dimensional Lie groups *Math. USSR Izvestija* **12** 371–89 (Translation from Russian)

Morosi C and Pizzocchero L 1996 On the Euler equation: Bi-Hamiltonian structure and integrals in involution *Lett. Math. Phys.* **37** 117–35

Nijenhuis A and Richardson R W J 1967 Deformations of Lie algebra structures *J. Math. Mech.* **17** 89–106

Ortega J P and Ratiu T S 2004 *Momentum Maps and Hamiltonian Reduction* (Basle: Birkhauser)

Panasyuk A 2006 Algebraic Nijenhuis operators and Kronecker Poisson pencils *Diff. Geom. Appl.* **24** 482–91

Panasyuk A 2008 Reduction by stages and the Rais-type formula for the index of a Lie algebra with an ideal *Ann. Global Anal. Geom.* **33** 1–10

Trofimov V V and Fomenko A T 1995 *Algebra and Geometry of Integrable Hamiltonian Differential Equations* (Moscow: Factorial) (in Russian)

Zakharevich I 2001 Kronecker webs, bi-Hamiltonian structures, and the method of argument translation *Transform. Groups* **6** 267–300