ON INTEGRABILITY OF GENERALIZED VERONESE CURVES OF DISTRIBUTIONS

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Given a 1-parameter family of 1-forms $\gamma(t) = \gamma_0 + t\gamma_1 + \cdots + t^n\gamma_n$, consider the condition $d\gamma(t) \wedge \gamma(t) = 0$ (of integrability for the annihilated by $\gamma(t)$ distribution $w(t)$). We prove that in order that this condition is satisfied for any t it is sufficient that it is satisfied for $N = n+3$ different values of t (the corresponding implication for $N = 2n+1$ is obvious). In fact we give a stronger result dealing with distributions of higher codimension. This result is related to the so-called Veronese webs and can be applied in the theory of bihamiltonian structures.

Keywords: webs, Veronese webs, integrability.

Introduction

The notion of a Veronese web was introduced by I. M. Gelfand and I. S. Zakharevich [2] as a natural invariant of bihamiltonian structures of corank 1. They conjectured that locally this invariant is complete, i.e. determines the bihamiltonian structure up to an isomorphism. This conjecture was proved by F. J. Turiel [6].

Let us briefly recall relevant definitions. Assume that we have a 1-parameter family $\{w(t)\}_{t\in\mathbb{RP}^1}, w(t) \subset TM$, of distributions of codimension 1 on a smooth manifold M such that in a neighbourhood of any point there exist an annihilating $w(t)$ 1-form $\gamma(t) \in \Gamma(w(t))^\perp$ and a coframe $\{\gamma_0, \ldots, \gamma_n\}$ with the property $\gamma(t) = \gamma_0 + t\gamma_1 + \cdots + t^n\gamma_n$ (we assume that $\gamma(\infty) = \gamma_n$ annihilates $w(\infty)$ and write Γ for the space of sections of a vector bundle). We call this family of distributions a Veronese curve of distributions. We say that it is integrable or it is a Veronese web if each distribution $w(t)$ is integrable, i.e.

$$d\gamma(t) \wedge \gamma(t) = 0, \qquad t \in \mathbb{R}, \tag{1}$$

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and

$$d\gamma(\infty) \wedge \gamma(\infty) = 0 \tag{2}$$

(it can be shown that (1) $\Rightarrow$ (2)).

Eq. (1) is polynomial of degree $2n$ in t, hence it is sufficient that it is satisfied for $2n+1$ different values of t in order that it is satisfied for any $t \in \mathbb{R}$. In other words, integrability of $w(t)$ at $2n+1$ different points implies integrability of a Veronese curve of distributions.

It is remarkable that this number can be essentially reduced. Zakharevich [8] conjectured that the integrability of $w(t)$ at $n+3$ different points implies the integrability of the Veronese curve of distributions.

The aim of this paper is to prove this result in a more general setting. We give it for generalized Veronese curves of distributions. The precise definition is given in Section 1. Here we shall only mention that integrable generalized Veronese curves of distributions coincide with a particular case of the so-called Kronecker webs introduced in [9]. The last notion serves as an invariant of bihamiltonian structures of higher corank in the same manner as the notion of a Veronese web does this for bihamiltonian structures of corank 1 (see also [7, 5]).

One of possible applications of our result is that it allows to generalize theory of nonlinear wave equations [8] to higher dimensional and codimensional cases. Also it may help to establish new relations between Veronese webs and classical webs. In [4] such relations are studied in the case $n = 1$.

Briefly, our method can be described as follows. Given $n+3$ foliations corresponding to the integrable points of the curve of distributions $w(t)$, we construct locally a curve of foliations $\widetilde{w}(t)$ in a larger space such that $\widetilde{w}(t)$ projects onto $w(t)$, so proving the integrability of $w(t)$. Our proof works only in the real-analytic category, since it uses the complexifications of the objects. This, of course, is one of the disadvantages of the method.

All objects in this paper are from C^ω-category.

1. Basic definitions

DEFINITION 1. Let $M^{k(n+1)}$ be a manifold of dimension $m = k(n+1)$ and let $\{w(t)\}_{t\in\mathbb{RP}^1}$, $w(t) \subset TM$, be a family of distributions of codimension k (as subbundles they have rank kn). Assume that in a neighbourhood of any point there exist k independent annihilating $w(t)$ 1-forms $\gamma^1(t), \dots, \gamma^k(t) \in \Gamma(w(t))^\perp$ and a coframe $\{\gamma_0^1, \dots, \gamma_n^1, \gamma_0^2, \dots, \gamma_n^2, \dots, \gamma_0^k, \dots, \gamma_n^k\}$ such that $\gamma^i(t) = \gamma_0^i + t\gamma_1^i + \cdots + t^n\gamma_n^i$, $i \in \overline{1,k}$. Then we call $\{w(t)\}_{t\in\mathbb{RP}^1}$ a *generalized Veronese curve of distributions.*

DEFINITION 2. A generalized Veronese curve of distributions $\{w(t)\}_{t\in\mathbb{RP}^1}$ is called *integrable* or a *generalized Veronese web* if each distribution $w(t)$ is integrable, i.e.

$$d\gamma^i(t) = \sum_s \beta_s^i(t) \wedge \gamma^s(t), \qquad i \in \overline{1,k}, \tag{3}$$

for some depending on t 1-forms $\beta_s^i(t), i, s \in \overline{1,k}$ (we put $\gamma^i(\infty) := \gamma_n^i, i \in \overline{1,k}$). The corresponding foliations will be denoted by $\mathcal{W}(t)$.

REMARK 1. It can be proved that if Eqs. (3) are satisfied for each $t \in \mathbb{R}$, then $w(\infty)$ is automatically integrable.

REMARK 2. In case $k = 1$ one gets the standard definition of a Veronese web. For $k > 1$ we obtain a particular case of Kronecker webs [9] with Kronecker blocks of equal dimension. In [5] the terminology "generalized Veronese webs" was used for different objects, namely for Kronecker webs without Kronecker blocks of equal dimension.

REMARK 3. If $\{w(t)\}_{t\in\mathbb{RP}^1}$ is a generalized Veronese web and $a_1, \dots, a_l \in \mathbb{RP}^1$, $l \geq m$, are different, then $\{\mathcal{W}(a_1), \dots, \mathcal{W}(a_l)\}$ is a "classical" l-web of codimension k (see [3]), i.e. the foliations $\mathcal{W}(a_1), \dots, \mathcal{W}(a_l)$ are in general position.

2. Main theorem

LEMMA. *Let M be a complex manifold, $J : TM \to TM$ be the complex structure operator on the real tangent bundle TM. Assume that $F \subset TM$ is an integrable distribution such that the distribution $JF \subset TM$ is also integrable. Then the distribution $(\mathrm{Id} + tJ)F$ is integrable for any $t \in \mathbb{R}$.*

Proof: Let $\{v_1, \dots, v_s\}$ be a local system of generating F vector fields. Then

$$[v_i, v_j] = \sum_l \alpha_{ij}^l v_l, \qquad [Jv_i, Jv_j] = \sum_l \beta_{ij}^l Jv_l$$

for some functions $\alpha_{ij}^l, \beta_{ij}^l$. The following calculations use the integrability condition for the complex structure

$$J[v_i, v_j] - J[Jv_i, Jv_j] = [Jv_i, v_j] + [v_i, Jv_j]$$

and complete the proof:

$$\begin{aligned}
&[v_i + tJv_i, v_j + tJv_j] \\
&\qquad = [v_i, v_j] + t^2[Jv_i, Jv_j] + t([v_i, Jv_j] + [Jv_i, v_j]) \\
&\qquad = [v_i, v_j] + t^2[Jv_i, Jv_j] + t(J[v_i, v_j] - J[Jv_i, Jv_j]) \\
&\qquad = [v_i, v_j] + tJ[v_i, v_j] + t(-J[Jv_i, Jv_j] + t[Jv_i, Jv_j]) \\
&\qquad = (\mathrm{Id} + tJ)[v_i, v_j] + t(\mathrm{Id} + tJ)(-J[Jv_i, Jv_j]) \\
&\qquad = \sum_l \alpha_{ij}^l (v_l + tJv_l) + t\sum_l \beta_{ij}^l (v_l + tJv_l) \\
&\qquad = \sum_l (\alpha_{ij}^l + t\beta_{ij}^l)(v_l + tJv_l).
\end{aligned}$$

□

NOTATION. Given a real-analytic manifold M, let $M^{\mathbb{C}}$ denote the germ along M of a complexification of M, i.e. a complex-analytic manifold Z such that M is embedded in Z as a completely real submanifold. The germ $M^{\mathbb{C}}$ is defined uniquely up to (a germ of) a biholomorphic map (see [1]). If φ is a real-analytic function on M, φ^c will stand for the unique germ along M of a complex-analytic function on $M^{\mathbb{C}}$ such that $\varphi^c|_M = \varphi$.

THEOREM 1. *Let $M^{k(n+1)}$ be a manifold of dimension $m = k(n+1)$ and let $\{w(t)\}_{t\in\mathbb{RP}^1}$ be an integrable generalized Veronese curve of distributions of codimension k on M. Then for any point $x \in M$ there exist a coordinate map (U, φ), $M \supset U \ni x$, $\varphi = (\varphi_1, \dots, \varphi_m)$, and a germ along U of an integrable distribution $F \subset TU^{\mathbb{C}}$ (T stands for the real tangent bundle) such that for any $t \in \mathbb{RP}^1$:*

(1) *The distribution $(\mathrm{Id} + tJ)F \subset TU^{\mathbb{C}}$ is integrable (we assume that the value $t = \infty$ corresponds to JF).*
(2) $\mathrm{rank}((\mathrm{Id} + tJ)F) = kn = \mathrm{rank}\, w(t)$.
(3) *The distribution $(\mathrm{Id} + tJ)F$ is projectable on U along the germ of the foliation $\mathcal{Y} = \{\mathrm{Re}\,\varphi_1^c = const, \dots \mathrm{Re}\,\varphi_m^c = const\}$.*
(4) *The projection of $(\mathrm{Id} - tJ)F$ coincides with $w(t)|_U$.*

Proof: Let $a_1, \dots, a_{n+1} \in \mathbb{R}$ be different nonzero numbers. Then the foliations $\mathcal{W}(a_1), \dots, \mathcal{W}(a_{n+1})$ are in general position and for any point one can find a neighbourhood U and functions

$$\begin{gathered} \varphi_1 = \psi_1^1, \dots, \ \varphi_{n+1} = \psi_{n+1}^1, \\ \varphi_{(n+1)+1} = \psi_1^2, \dots, \varphi_{2n+2} = \psi_{n+1}^2, \\ \dots \\ \varphi_{(k-1)(n+1)+1} = \psi_1^k, \dots, \qquad \varphi_m = \psi_{n+1}^k \end{gathered}$$

on U such that $\mathcal{W}(a_j) = \{\psi_j^1 = \mathrm{const}, \dots, \psi_j^k = \mathrm{const}\}$, $j \in \overline{1, n+1}$.

We define a new family of distributions $F(t) \subset TU^{\mathbb{C}}$, $t \in \mathbb{RP}^1$, of codimension k by

$$\Gamma(F(t))^\perp = \langle(\mathrm{Id} - tJ^*)\pi^*\gamma^1(t), \dots, (\mathrm{Id} - tJ^*)\pi^*\gamma^k(t)\rangle.$$

Here $\langle\cdot\rangle$ stands for the linear span, $J^* : T^*U^{\mathbb{C}} \to T^*U^{\mathbb{C}}$ is the adjoint operator to the complex structure $J : TU^{\mathbb{C}} \to TU^{\mathbb{C}}$, $\pi : U^{\mathbb{C}} \to U$ is the projection along the foliation $\mathcal{Y}$ defined in (3), and $\gamma^1(t), \dots, \gamma^k(t)$ are the annihilating $w(t)$ 1-forms (see Definition 1).

Now, let us define the distribution F as

$$F = \bigcap_{t\in\mathbb{RP}^1} F(t)$$

or

$$\Gamma F^\perp = \langle(\mathrm{Id} - tJ^*)\pi^*\gamma^1(t), \dots, (\mathrm{Id} - tJ^*)\pi^*\gamma^k(t) | t \in \mathbb{RP}^1\rangle.$$

Notice that the 1-forms $\pi^*\gamma_0^i, \pi^*\gamma_1^i - J^*\pi^*\gamma_0^i, \dots, \pi^*\gamma_n^i - J^*\pi^*\gamma_{n-1}^i, -J^*\pi^*\gamma_n^i$ corresponding to different powers of t in $(\mathrm{Id} - tJ^*)\pi^*\gamma^i(t)$, $i \in \overline{1,k}$, are linearly independent. Therefore the standard properties of the Veronese curve (of degree $n+2$) imply that

$$\Gamma F^\perp = \langle (\mathrm{Id} - tJ^*)\pi^*\gamma^1(t), \dots, (\mathrm{Id} - tJ^*)\pi^*\gamma^k(t) | t \in \{0, a_1, \dots, a_{n+1}\}\rangle$$

or

$$F = \bigcap_{j \in \overline{0,n+1}} F(a_j)$$

(here we put $a_0 = 0$).

This allows us to prove the integrability of F by showing the integrability of $F(a_0), \dots, F(a_{n+1})$.

Evidently, $\gamma^i(a_j) = \sum_{s=1}^k \beta_{js}^i d\psi_j^s, i \in \overline{1,k}, j \in \overline{1,n+1}$ for some functions β_{js}^i. Similarly, $\gamma^i(a_0) = \sum_{s=1}^k \beta_{0s}^i d\psi_0^s$, $i \in \overline{1,k}$ for some functions β_{0s}^i, ψ_0^s. Thus

$$\begin{aligned}(\mathrm{Id} - a_jJ^*)\pi^*\gamma^i(a_j) &= (\mathrm{Id} - a_jJ^*)\sum_{s=1}^k \pi^*\beta_{js}^i d\pi^*\psi_j^s \\ &= \sum_{s=1}^k \pi^*\beta_{js}^i (d(\mathrm{Re}(\psi_j^s)^c) + a_j d(\mathrm{Im}\,\psi_j^s)^c)) \\ &= \sum_{s=1}^k \pi^*\beta_{js}^i d(\mathrm{Re}(\psi_j^s)^c + a_j(\mathrm{Im}\,\psi_j^s)^c), \quad i \in \overline{1,k}, \quad j \in \overline{1,n+1}\end{aligned}$$

and

$$(\mathrm{Id} - a_0J^*)\pi^*\gamma^i(a_0) = \pi^*\gamma^i(a_0) = \sum_{s=1}^k \pi^*\beta_{0s}^i d\pi^*\psi_0^s, \qquad i \in \overline{1,k}$$

(here we used the obvious facts that $\pi^*\varphi_j = \mathrm{Re}\,\varphi_j^c$, $j \in \overline{1,m}$, and that $J^*d(\mathrm{Re}\,\varphi_j^c) = -d(\mathrm{Im}\,\varphi_j^c)$). Now it is easy to check the Frobenius integrability conditions using the nondegeneracy of the matrix $\|\beta_{js}^i\|_{i,s}$ for any $j \in \overline{0,n+1}$. So the distributions $F(a_j)$ are indeed integrable.

To prove (1) we choose another set of generators for $\Gamma F^\perp$ as follows

$$\Gamma F^\perp = \langle (\mathrm{Id} - tJ^*)\pi^*\gamma^1(t), \dots, (\mathrm{Id} - tJ^*)\pi^*\gamma^k(t) | t \in \{a_1, \dots, a_{n+2}\}\rangle,$$

where $a_{n+2} := \infty$, and notice that

$$\Gamma J^*F^\perp = \langle (J^* + t\,\mathrm{Id})\pi^*\gamma^1(t), \dots, (J^* + t\,\mathrm{Id})\pi^*\gamma^k(t) | t \in \{a_1, \dots, a_{n+2}\}\rangle.$$

Now the integrability for $JF = ((J^*)^{-1}F^\perp)^\perp = (J^*F^\perp)^\perp$ can be proved by the same considerations as for F and the integrability of $(\mathrm{Id} + tJ)F$ follows from Lemma.

In order to prove (2) we mention that $\operatorname{rank} F = kn$ by the construction and that $\mathrm{Id} + tJ : TU^{\mathbb{C}} \to TU^{\mathbb{C}}$ is the isomorphism for any $t \in \mathbb{RP}^1$: the inverse operator is given by the formula

$$(\mathrm{Id} + tJ)^{-1} = \frac{1}{1+t^2}(\mathrm{Id} - tJ), \qquad t \neq \infty,$$

$$J^{-1} = -J.$$

Now we are able to prove (3). We need to show that the distribution $(\mathrm{Id} - tJ)F + T\mathcal{Y} \subset TU^{\mathbb{C}}$ is integrable for any $t \in \mathbb{RP}^1$. We fix $t = t_0 \in \mathbb{R}$, choose $b_0 = t_0, b_1, \ldots, b_{n+1}$ to be different real numbers and calculate the annihilators:

$$((\mathrm{Id} - t_0 J)F + T\mathcal{Y})^{\perp} = ((\mathrm{Id} - t_0 J)F)^{\perp} \cap (T\mathcal{Y})^{\perp},$$

$$\Gamma(T\mathcal{Y})^{\perp} = \langle d \operatorname{Re} \varphi_1^c, \ldots, d \operatorname{Re} \varphi_m^c \rangle,$$

$$\begin{aligned}
\Gamma((\mathrm{Id} - t_0 J)F)^{\perp} &= \Gamma((\mathrm{Id} - t_0 J^*)^{-1} F^{\perp}) \\
&= (\mathrm{Id} + t_0 J^*)\langle (\mathrm{Id} - tJ^*)\pi^*\gamma^1(t), \ldots, (\mathrm{Id} - tJ^*)\pi^*\gamma^k(t) | t \in \{b_0, \ldots, b_{n+1}\}\rangle \\
&= \langle (1+t_0^2)\pi^*\gamma^1(t_0), \ldots, (1+t_0^2)\pi^*\gamma^k(t_0), \\
&\quad (1+t_0 b_j)\pi^*\gamma^1(b_j) + (t_0 - b_j)J^*\pi^*\gamma^1(b_j), \ldots, \\
&\quad (1+t_0 b_j)\pi^*\gamma^k(b_j) + (t_0 - b_j)J^*\pi^*\gamma^k(b_j) | j \in \overline{1, n+1}\rangle.
\end{aligned}$$

It is easy to see that the collection of 1-forms

$$\{\pi^*\gamma^i(b_j), J^*\pi^*\gamma^i(b_j)\}_{1 \le i \le k, 1 \le j \le n+1}$$

is a coframe on $U^{\mathbb{C}}$, hence the 1-forms $(t_0 - b_j)J^*\pi^*\gamma^i(b_j)$, $1 \le i \le k$, $1 \le j \le n+1$, cannot be linearly expressed by the 1-forms $d \operatorname{Re} \varphi_1^c, \ldots, d \operatorname{Re} \varphi_m^c$ which are combinations of $\pi^*\gamma^i(b_j)$. So, finally,

$$((\mathrm{Id} - t_0 J)F + T\mathcal{Y})^{\perp} = \langle \pi^*\gamma^1(t_0), \ldots, \pi^*\gamma^k(t_0)\rangle$$

and the distribution

$$(\mathrm{Id} - t_0 J)F + T\mathcal{Y} = \pi^* w(t_0)$$

is integrable. In the same manner one can show that

$$JF + T\mathcal{Y} = \pi^* w(\infty).$$

Simultaneously, the last two equations prove (4). □

3. Application to integrability

THEOREM 2. *Let $M^{k(n+1)}$ be a manifold and let $\{w(t)\}_{t\in\mathbb{RP}^1}$ be a generalized Veronese curve of distributions of codimension k on M. Then in order that $\{w(t)\}_{t\in\mathbb{RP}^1}$ is a generalized Veronese web it is sufficient that it is integrable at $n+3$ different points $a_0,\dots,a_{n+2}\in\mathbb{RP}^1$.*

Proof: A careful analysis of the proof of Theorem 1 shows that this proof uses only integrability of $\{w(t)\}$ at $0,\infty$ and arbitrary different nonzero finite points $a_1,\dots,a_{n+1}$ for the construction of the distribution $F\subset TU^{\mathbb{C}}$ such that $(\mathrm{Id}+tJ)F$ is integrable for any $t\in\mathbb{RP}^1$ and is projectable onto $w(t)$. Thus it remains to map 0 and ∞ to a_0 and a_{n+2} respectively by an appropriate automorphism of $\mathbb{RP}^1$. $\square$

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