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BI-POISSON STRUCTURES AND INTEGRABILITY OF GEODESIC FLOW ON HOMOGENEOUS SPACES

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Abstract. Let G/K be a semisimple orbit of the adjoint representation of a real connected reductive Lie group G. Let K_1 be any closed subgroup of K containing the commutant of the identity component of K. We prove that the geodesic flow on the symplectic manifold $T^*(G/K_1)$, corresponding to a G-invariant pseudo-Riemannian metric on G/K_1 which is induced by a bi-invariant pseudo-Riemannian metric on G, is completely integrable in the class of real analytic functions, polynomial in momenta. To this end we study the Poisson geometry of the space of G-invariant functions on $T^*(G/K)$ using a one-parameter family of moment maps.

Introduction

Let M be a homogeneous space of a real connected reductive Lie group G, i.e., $M = G/K$. Suppose K is a (closed) reductive subgroup of G. Consider the space A^G of all G-invariant real analytic functions on the cotangent bundle T^*M. This space is an algebra with respect to the canonical Poisson bracket on the symplectic manifold T^*M. Let C^G be a center of the algebra A^G. Denote by $\operatorname{ddim} A^G$ (resp. $\operatorname{ddim} C^G$) the maximal number of functionally independent functions from the set A^G (resp. C^G). Put $\operatorname{ddim}(A^G/C^G) = \operatorname{ddim} A^G - \operatorname{ddim} C^G$.

One calls a Hamiltonian system on T^*M (completely) integrable if it admits a maximal number of independent integrals in involution, i.e., $\dim M$ functions commuting with respect to the Poisson bracket on T^*M whose differentials are independent in an open dense subset of T^*M. By Liouville's theorem the integral curves of an integrable Hamiltonian system under a certain additional compactness hypothesis are quasiperiodic (are the orbits of a constant vector field on an invariant torus).

The natural extension of the action of G on M to an action on the symplectic manifold T^*M is Hamiltonian with the moment mapping $\mu^{\rm can} : T^*M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. The functions of type $h \circ \mu^{\rm can}$, $h : \mathfrak{g}^* \to \mathbb{R}$, are integrals for any G-invariant Hamiltonian flow on T^*M, in particular, for the geodesic flow corresponding to any G-

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invariant pseudo-Riemannian metric on M. The maximal number of independent real analytic functions in involution on T^*M of type $h\circ\mu^{\mathrm{can}}$ is equal to $\dim(G/K)-\boldsymbol{\varepsilon}$, where the nonnegative integer $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(G,K)$ has the following two equivalent definitions [Myk3, Vin]:

(1) $\boldsymbol{\varepsilon}$ is the complexity of the complex affine variety $G^{\mathbb{C}}/K^{\mathbb{C}}$, i.e., equals to codimension of the maximal dimension orbits of the Borel subgroup $B\subset G^{\mathbb{C}}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, if G and K are algebraic Lie groups;

(2) $2\boldsymbol{\varepsilon}=\operatorname{ddim}(A^G/C^G)$.

From this fact it follows immediately that if $\boldsymbol{\varepsilon}(G,K)=0$, then any G-invariant Hamiltonian flow on T^*M is integrable in the class of integrals generated only by the symmetries of the considered dynamical system. In this case the subgroup $K^{\mathbb{C}}$ is called a spherical subgroup of $G^{\mathbb{C}}$. All spherical subgroups of semisimple complex Lie groups are described in [Kra, Myk2, Bri]. Remark also that if $\boldsymbol{\varepsilon}(G,K)=0$ and the group K is compact, then the homogeneous space G/K is a weakly symmetric space (see [Vin]). The complete integrability of the geodesic flows on the cotangent bundles to symmetric spaces was considered in papers [Thi, Mish, GS, Myk2].

In the paper [MS] it was observed that if $\boldsymbol{\varepsilon}(G,K)=1$, then again any Hamiltonian flow on T^*M with a G-invariant Hamiltonian H is integrable: for the integrability we can use either H- or another G-invariant function as one additional integral (to the integrals of the form $h\circ\mu^{\mathrm{can}}$). All pairs $(G^{\mathbb{C}},K^{\mathbb{C}})$ with $\boldsymbol{\varepsilon}(G,K)=1$ are enumerated in [Pan2, MS] (the case of simple G) and in [AC] (the semisimple case). Remark also that some spaces G/K from these lists were found by Thimm [Thi] (the space $\mathrm{SO}(n)/\mathrm{SO}(n-2)$) and by Paternain and Spatzier [PS] (the space $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$).

So the problem of constructing a maximal commutative set of real analytic functions on $T^*(G/K)$ is reduced to the problem of finding a maximal commutative set $\mathcal{I}^G$ of real analytic functions in the set A^G (containing $\boldsymbol{\varepsilon}(G,K)$ functions additional to the functions of type $h\circ\mu^{\mathrm{can}}$).

For the homogeneous space G/K with compact G (of an arbitrary complexity) Bolsinov and Jovanovic [BJ2] showed that the geodesic flow on $T^*(G/K)$ of the bi-invariant metric on G/K is integrable in the class of smooth integrals. The proof of this fact is based on their paper [BJ1], where they proved the so-called noncommutative integrability of this geodesic flow (in the class of real analytic integrals).

Let G/K be a semisimple orbit of the adjoint representation of the Lie group G, i.e., $G/K=\mathrm{Ad}(G)\cdot a$, where a is a semisimple element of the Lie algebra $\mathfrak{g}$. Denote by K_1 any closed subgroup of G such that $K'\subset K_1\subset K$, where K' is the commutant of the identity component of K. In this paper we prove (Theorem 3.10) the complete integrability in the class of real analytic integrals of the geodesic flow on the symplectic manifold $T^*(G/K_1)$ corresponding to the following two classes of metrics: (a) G-invariant pseudo-Riemannian metrics on G/K_1 which are induced by bi-invariant pseudo-Riemannian metrics on G; (b) G-invariant pseudo-Riemannian metrics on G/K_1 which arise from the so-called Mishchenko–Fomenko sectional operators $\varphi_{a,b,D}$ (see Subsection 3.1). The analogous result for the unitary group $G=\mathrm{U}(n)$ was obtained by Bolsinov and Jovanovic in their paper [BJ3, Theorem 3.4]. The proof of their theorem is based on a verification of some sufficient conditions using canonical matrix representations of semisimple elements of the Lie algebra $u(n)$ and, as remarked in [BJ3], may be generalized for the case of compact classical Lie groups $G=\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$.

Our method of proof is different and works also in the case of an exceptional Lie group G although the set $\mathcal{I}^G$ of integrals, which we obtain, is the same as in [BJ3]. These G-invariant integrals $\{H^\lambda(x) = h(x+\lambda a)\}$ viewed as $\mathrm{Ad}^*(K)$-invariant functions on the cotangent space $T^*_{\{K\}}(G/K)$ are obtained by the argument translation method from the invariants $\{h\}$ of the Lie algebra $\mathfrak{g}$. Note that the integrability of the geodesic flows on G/T, where the Lie group G is compact and T is its maximal torus, was first proved, using this method, by Bordemann [Bor] and later independently in [BJ1].

If G is a compact Lie group, all the metrics from the class (a) and a dense subset of the metrics from the class (b) are Riemannian. For such metrics the theorem of Liouville applies (all common level surfaces of the integrals are compact). This is of special interest since it allows for the possibility of a qualitative study of the flow.

Crucial ingredients in our proof are 1) a method of investigation of the Poisson algebra A^G using a one-parameter family of moment maps on $T^*(G/K)$ with the same locally free group action of G (the method is based on the Gelfand-Zakharevich theory of bihamiltonian structures [GZ, Zakh] and on the technique of their reductions [Pana1, Pana2]); 2) the reduction of the Poisson algebra A^G on $T^*(G/K)$ to a Poisson algebra $A^{\hat{G}}$ on $T^*(\hat{G}/\hat{K})$, such that the "effective part" of the action of $\hat{G}$ on $T^*(\hat{G}/\hat{K})$ is locally free (this part of the proof originates from [Myk3]).

1. G-invariant bi-Poisson structures and moment maps

1.1. Some definitions, conventions, and notation

All objects in this paper are real analytic or complex analytic, X stands for a connected manifold, $\mathcal{E}(X)$ for the space of respectively real analytic or holomorphic functions on X. We shall write $\mathbb{F}$ for $\mathbb{R}$ or $\mathbb{C}$ depending on the category.

We will say that some functions from the set $\mathcal{E}(X)$ are independent if their differentials are independent at each point of some open dense subset in X. For any subset $\mathcal{F} \subset \mathcal{E}(X)$ denote by $\operatorname{ddim}_x \mathcal{F}$ the maximal number of independent functions from the set $\mathcal{F}$ at a point $x \in X$. Put $\operatorname{ddim} \mathcal{F} \stackrel{\text{def}}{=} \max_{x\in X} \operatorname{ddim}_x \mathcal{F}$.

Definition 1.1. A pair (η_1, η_2) of linearly independent bi-vector fields (bi-vectors for short) on a manifold X is called *Poisson* if $\eta^t \stackrel{\text{def}}{=} t_1\eta_1 + t_2\eta_2$ is a Poisson bi-vector for any $t = (t_1, t_2) \in \mathbb{F}^2$, i.e., each bi-vector η^t determines on X a Poisson structure with the Poisson bracket $\{\ ,\ \}^t : (f_1, f_2) \mapsto \eta^t(df_1, df_2)$; the whole family of Poisson bi-vectors $\{\eta^t\}_{t\in\mathbb{F}^2}$ is called a *bi-Poisson structure*.

A bi-Poisson structure $\{\eta^t\}$ (we shall often skip the parameter space) can be viewed as a two-dimensional vector space of Poisson bi-vectors, the Poisson pair (η_1, η_2) as a basis in this space.

The following two definitions are motivated by Proposition 1.4, which is due to Bolsinov (see below), and by the Gelfand–Zakharevich theory of bi-Poisson (bi-Hamiltonian) structures [GZ, Zakh].

Definition 1.2. A bi-Poisson structure $\{\eta^t\}$ on X is *Kronecker* at a point $x \in X$ if $\operatorname{rank}_{\mathbb{C}}(t_1\eta_1 + t_2\eta_2)|_x$ is constant with respect to $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$ (in the real analytic category we consider $(\eta_j)_x$ as a skew-symmetric bilinear form on the complexified cotangent space $(T_x^*X)^{\mathbb{C}}$). We say that $\{\eta^t\}$ is *micro-Kronecker* if it is Kronecker at any point of some open dense subset in X.

Let G be a Lie group acting on a manifold X. Denote by A_X^G the space of all G-invariant functions from the set $\mathcal{E}(X)$. We say that the bi-Poisson structure $\{\eta^t\}$ is G-invariant if each bi-vector η^t, $t \in \mathbb{F}^2$ is. Put $D_x^G \stackrel{\mathrm{def}}{=} \{df_x \mid f \in A_X^G\} \subset T_x^*X$ for any $x \in X$. Let B_x^t denote the restriction of η_x^t to this subspace D_x^G. If $\mathbb{F} = \mathbb{R}$, we mean $B_x^t, t \in \mathbb{C}^2$, as the complex bilinear form $t_1 B_x^{(1,0)} + t_2 B_x^{(0,1)}$.

Definition 1.3. Let $\{\eta^t\}$ be a G-invariant bi-Poisson structure. We say that the pair $(A_X^G, \{\eta^t\})$ is *Kronecker* at a point $x \in X$, where $\operatorname{ddim}_x A_X^G = \operatorname{ddim} A_X^G$, if the linear space $\{B_x^t, t \in \mathbb{F}^2\}$ is two-dimensional and $\operatorname{rank}_{\mathbb{C}} B_x^t$ is constant with respect to $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$. We say that $(A_X^G, \{\eta^t\})$ is *micro-Kronecker* if it is Kronecker at any point of some open dense subset in X.

Proposition 1.4. [Bol] *Let B_1 and B_2 be two linearly independent skew-symmetric bilinear forms on a vector space V. Suppose that the kernel of each form $B^t = t_1 B_1 + t_2 B_2$, $t \in \mathbb{F}^2$, is nontrivial, i.e., $0 < r_0 \stackrel{\mathrm{def}}{=} \min_{t \in \mathbb{F}^2} \dim \ker B^t$. Put $T_0 = \{t \in \mathbb{F}^2 \mid \dim \ker B^t = r_0\}$. Then*

1. *the subspace $L_0 \stackrel{\mathrm{def}}{=} \sum_{t \in T_0} \ker B^t$ is isotropic with respect to any form B^t, $t \in \mathbb{F}^2$, i.e., $B^t(L_0, L_0) = 0$;*
2. *the space L_0 is maximal isotropic with respect to any form B^{t_0}, $t_0 \in T_0$, i.e., $\dim L_0 = \frac{1}{2}(r_0 + \dim V)$ iff $\dim_{\mathbb{C}} \ker B^t = r_0$ for all $t \in \mathbb{C}^2 \setminus \{0\}$.*

By Proposition 1.4 if the set $\{B^t\} = \{B_x^t\}$ is associated with some bi-Poisson structure as in Definition 1.3, then the subspace $L_0 \subset V = D_x^G$ is spanned by differentials of functions at x and these functions are involutive at x with respect to any Poisson bracket $\{\ ,\ \}^t$. Note also that the space $\ker B_x^t$ contains the differentials of the functions from the center $C_X^{G,t}$ of the algebra (A_X^G, η^t). But L_0 is generated by a finite set of spaces $\{\ker B_x^{t_j}\}_{j=1}^N$. So if the pair $(A_X^G, \{\eta^t\})$ is micro-Kronecker in some neighborhood of x and each $\ker B_x^{t_j}$ is generated by the differentials of the functions from C_X^{G,t_j}, we have $\frac{1}{2}(\dim \ker B_x^t + \dim D_x^G)$ involutive independent functions on X, i.e., these functions form a maximal involutive subset of functions in A_X^G.

If a Poisson bi-vector η on X is nondegenerate, then there exists a unique symplectic form ω such that $\eta(\cdot,\cdot) = -\omega(\omega_\flat^{-1}(\cdot), \omega_\flat^{-1}(\cdot))$. Here $\omega_\flat : TX \to T^*X$ is the natural isomorphism given by the contraction with the 2-form ω on the first index. Such a Poisson bi-vector η will be denoted by ω^{-1}.

Definition 1.5. Let $\mathfrak{g}$ be the Lie algebra of the group G and $\eta = \omega^{-1}$ be a nondegenerate G-invariant Poisson bi-vector on X. For each vector $\xi \in \mathfrak{g}$ denote by ξ_X the fundamental vector field on X generated by the one-parameter diffeomorphism group $\exp(t\xi) \subset G$. The group G acts on the symplectic manifold (X, ω) in a Hamiltonian fashion if there is a G-equivariant map $\mu : X \to \mathfrak{g}^*$, such that for each $\xi \in \mathfrak{g}$, the field ξ_X is the Hamiltonian vector field with the Hamiltonian function $f_\xi : X \to \mathbb{F}$, $x \mapsto \mu(x)(\xi)$, i.e., $df_\xi = -\omega(\xi_X, \cdot)$.

The equivariance property $\mu(g^{-1}x)(\xi) = \mu(x)(\operatorname{Ad}(g)\xi)$, where $g \in G$, $x \in X$, of the moment map μ implies the identity $\{f_\xi, f_\zeta\} = f_{[\xi,\zeta]}$, where $\xi, \zeta \in \mathfrak{g}$ and $\{\ ,\ \}$ is the

Poisson bracket associated with η. In other words, the mapping μ is canonical with respect to the Poisson structure η on X and the standard linear Poisson structure on $\mathfrak{g}^*$. Moreover, by definition $\{f, h\circ\mu\} = 0$ for any G-invariant function $f \in A_X^G$ and $h \in \mathcal{E}(\mathfrak{g}^*)$. If $\eta \in \{\eta^t\}$, combining involutive functions of type $h\circ\mu$ with that from C_X^{G,t_j} we shall get complete involutive families on X.

1.2. Bi-Poisson structures $\{\eta^t(\alpha)\}$ on T^*M

Let M be a real (or complex) connected manifold. Denote by Ω the canonical symplectic form on the cotangent bundle T^*M. Let $\pi : T^*M \to M$ be the canonical projection.

Proposition 1.6. *Let α be a nontrivial closed 2-form on M. Put $\omega_1 = \Omega$ and $\omega_2 = \Omega + \pi^*\alpha$. Write $\eta_1 = \omega_1^{-1}$, $\eta_2 = \omega_2^{-1}$ for the inverse Poisson bi-vectors. Then the family $\{\eta^t(\alpha) = \eta^t = t_1\eta_1 + t_2\eta_2\}$, $t_1, t_2 \in \mathbb{F}$, is a bi-Poisson structure. The Poisson structure η^t is nondegenerate iff $t_1 + t_2 \neq 0$. If $t_1 + t_2 = 0$ and the 2-form α on M is nondegenerate, then the symplectic leaves of the degenerate structure η^t coincide with the fibers of π.*

Proof. Let us use the canonical local coordinates (p, q) on T^*M. In them the matrix of the 2-form $\omega_s \overset{\text{def}}{=} \Omega + (s-1)(\pi^*\alpha)$, $s \in \mathbb{F}$, is equal to $W_s = W + (s-1)B$ with the inverse matrix $W_s^{-1} = -W + (s-1)C$, where

$$W = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & A(q) \end{bmatrix}, \quad C = \begin{bmatrix} A(q) & 0 \\ 0 & 0 \end{bmatrix},$$

I_n is the identity $n \times n$-matrix ($n = \dim M$) and $A(q)$ is the matrix of the 2-form α. Therefore

$$t_1 W_1^{-1} + t_2 W_2^{-1} = (t_1 + t_2)\Big(W + \frac{t_2}{t_1 + t_2} B\Big)^{-1},$$

i.e., each bi-vector $t_1\eta_1 + t_2\eta_2$, $t_1 + t_2 \neq 0$, is Poisson (is determined by the nondegenerate closed (symplectic) 2-form proportional to some form ω_s, $s \in \mathbb{F}$). By continuity the bi-vector $\eta^{(t_1, -t_1)}$, $t_1 \in \mathbb{F}$, is also Poisson. Since it is defined by the matrix $-C$: $\eta^{(1,-1)} = -\sum_{jk} A_{jk}(q)\frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial p_k}$, we obtain the latest assertion of the proposition. □

Remark 1.7. Fix $x \in T^*M$ and put $V = \pi^{-1}(\pi(x))$. As we proved above, if the form α is symplectic, the degenerate Poisson bi-vector $\eta^{(1,-1)}(\alpha)$ determines on the linear space V a symplectic structure and this structure is independent of a point $v \in V$. In other words, at each $v \in V$ (under the natural identification of T_vV with V) this structure induces the same skew-symmetric nondegenerate bilinear form $\alpha' : V \times V \to \mathbb{F}$.

1.3. Hamiltonian actions and maximal involutive sets of functions

Let G be a reductive connected Lie group over a field $\mathbb{F}$ (real or complex numbers) with a closed reductive subgroup H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups G and H. Suppose that these Lie algebras are algebraic (see Subsection 2.2). Let η^{can} be the canonical (defined by the canonical symplectic form Ω) bi-vector on the cotangent bundle $X = T^*M$, where $M = G/H$. The natural action of G on X is Hamiltonian with the moment $\mu^{\text{can}} : X \to \mathfrak{g}^*$. For this moment map μ^{can} the corresponding Hamiltonian function f_ξ^{can}, $\xi \in \mathfrak{g}$, has the form $f_\xi^{\text{can}} = \theta(\xi_X)$, where θ is the canonical 1-form on $X = T^*(G/H)$.

Put $B_x^{\mathrm{can}} = \eta_x^{\mathrm{can}}|D_x^G$ for each $x \in X$. Remark that $\dim D_x^G = \operatorname{ddim}_x A^G$, where we set $A^G = A_X^G$. Since Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are algebraic, $\operatorname{ddim} A^G = \min_{x' \in X} \operatorname{codim} G\cdot x'$ and $\operatorname{ddim} C^G = \min_{x' \in X} \dim \ker B_{x'}^{\mathrm{can}}$, where C^G is a center of Poisson algebra $(A^G, \eta^{\mathrm{can}})$ (see also Lemma 2.4). The following proposition is known [Myk3, §2], [BJ1, Lemma 3], [Pana1].

Proposition 1.8. *Let $\mathbb{F} = \mathbb{R}$. Suppose that there exist $m = \frac{1}{2}(\operatorname{ddim} A^G + \operatorname{ddim} C^G)$ independent functions in involution $\{f_1, \dots, f_m\}$ (with respect to η^{can}) on $T^*(G/H)$ from the set A^G. Then there are $s = \dim(G/H) - m$ polynomials $\{h_1, \dots, h_s\}$ on $\mathfrak{g}^*$ such that the functions $\{f_1, \dots, f_m; h_1 \circ \mu^{\mathrm{can}}, \dots, h_s \circ \mu^{can}\}$ form a maximal involutive set of independent functions on $T^*(G/H)$.*

Remark 1.9. We will give some comments on this proposition. The set $\{f_1, \dots, f_m\}$ is a maximal involutive subset in A^G, i.e., any function $f \in A^G$ commuting with each f_j locally is a function of $\{f_1, \dots, f_m\}$. The number $\frac{1}{2}(\operatorname{ddim} A^G - \operatorname{ddim} C^G)$ equals the complexity $\varepsilon(\mathfrak{g}, \mathfrak{h})$ of complex algebraic variety $G^{\mathbb{C}}/H^{\mathbb{C}}$ (if the groups are not closed in the Zariski topology we complexify their closures). This number is calculated in [Myk3, §2] in terms of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$; similar expressions for $\varepsilon(\mathfrak{g}, \mathfrak{h})$ were obtained in [Pan1]. So by the condition of the proposition we have $m = \varepsilon(\mathfrak{g}, \mathfrak{h}) + \operatorname{ddim} C^G$ independent functions in involution from the set A^G. The maximal number of independent real analytic functions in involution of the form $h \circ \mu^{\mathrm{can}}$ on $T^*(G/H)$ is equal to $\dim(G/H) - \varepsilon(\mathfrak{g}, \mathfrak{h})$[Myk3]. Since at a generic point $x \in X$, the space $\{df_x \mid f \in C^G\} = \ker B_x^{\mathrm{can}}$ coincides with the intersection of two subspaces of T_x^*X spanned by the differentials of the functions from the sets A^G and $\{h \circ \mu^{\mathrm{can}} \mid h \in \mathcal{E}(\mathfrak{g}^*)\}$ respectively, [Myk3, Pana1], we can complete the involutive set of functions $\{f_1, \dots, f_m\}$ by integrals of the form $h \circ \mu^{\mathrm{can}}$ and get a maximal involutive set of independent functions on $T^*(G/H)$.

Let $\mathcal{O} \subset \mathfrak{g}^*$ be some $\mathrm{Ad}^*(G)$-orbit through a semisimple element $a \in \mathfrak{g} \simeq \mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Here we identified the reductive Lie algebra $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$ using some invariant nondegenerate form on $\mathfrak{g}$. Then $\mathcal{O} = G/K$, where K is a closed reductive subgroup of G (the isotropy group of a). Denote by $\mathfrak{k}$ the Lie algebra of K. The orbit $\mathcal{O} \subset \mathfrak{g}^*$ is a symplectic manifold with the Kirillov–Kostant–Souriau form $\omega_{\mathcal{O}}$. So we can consider the bi-Poisson structure $\{\eta^t(\omega_{\mathcal{O}})\}$ on the manifold $X = T^*\mathcal{O}$. As we noted above, the natural extension of Ad^*-action of G on $\mathcal{O}$ to the action on $(T^*\mathcal{O}, \Omega)$ is Hamiltonian with the moment map μ^{can}. Moreover, for the fundamental vector fields ξ_X and $\xi_{\mathcal{O}}$ on the manifolds $X = T^*\mathcal{O}$ and $\mathcal{O}$ respectively (associated with a vector $\xi \in \mathfrak{g}$) we have $\pi_*(\xi_X) = \xi_{\mathcal{O}}$, where, recall, $\pi : T^*\mathcal{O} \to \mathcal{O}$ is the natural projection.

We will formulate and prove the following theorem only in the complex case since for us its assertion is an auxiliary result for the proof of the main Theorem 3.9. The proof below is the verification of conditions of Theorem 4.2 in [Pana1].

Theorem 1.10. *Let $\mathbb{F} = \mathbb{C}$. Suppose that the action of the Lie group $\mathrm{Ad}(G)$ on the cotangent bundle $T^*\mathcal{O}$ is locally free (as in the case of a generic $\mathcal{O}$). Then the pair $(A^G, \{\eta^t(\omega_{\mathcal{O}})\})$ is micro-Kronecker.*

Proof. Since the Lie group G acts on $\mathfrak{g}^*$ by the coadjoint action, without loss of generality we may assume that the action of G itself on $T^*\mathcal{O}$ is locally free. Then, in particular, G is a semisimple Lie group.

It is well known that Ad^*-action of G on the symplectic manifold $(\mathcal{O}, \omega_{\mathcal{O}})$ is Hamiltonian. By the definition of the form $\omega_{\mathcal{O}}$ the natural embedding $\mathcal{O} \to \mathfrak{g}^*$ is (up to a sign) the corresponding moment map which we denote by $\mu^{\mathcal{O}}$. Then the Hamiltonian vector field of the function $f_\xi^{\mathcal{O}} = \mu^{\mathcal{O}}(\xi)$, $\xi \in \mathfrak{g}$, coincides with $\xi_{\mathcal{O}}$, i.e., $df_\xi^{\mathcal{O}} = -\omega_{\mathcal{O}}(\xi_{\mathcal{O}}, \cdot)$.

We claim that the group G acts on the symplectic manifold $(T^*\mathcal{O}, \Omega^s)$, where $\Omega^s = \Omega + s(\pi^*\omega_{\mathcal{O}})$, $s \in \mathbb{C}$, in a Hamiltonian fashion. Indeed, since $\pi_*(\xi_X) = \xi_{\mathcal{O}}$, it is easy to verify that the function $f_\xi^{\mathrm{can}} + s(\pi^* f_\xi^{\mathcal{O}})$ has ξ_X as its Hamiltonian vector field with respect to the form Ω^s. The mapping $\mu^s = \mu^{\mathrm{can}} + s(\pi^*\mu^{\mathcal{O}}) : T^*\mathcal{O} \to \mathfrak{g}^*$ is G-equivariant since so is the projection π. So μ^s is the corresponding moment map.

Let $x \in X = T^*\mathcal{O}$. The image $\mu_*^s(T_xX) \subset \mathfrak{g}^*$ coincides with the annihilator in $\mathfrak{g}^*$ of the Lie algebra of the isotropy group G^x of x, [GS]. Since this algebra vanishes at a generic point x, at such a point $\operatorname{rank} \mu^s(x) = \dim \mathfrak{g}^*$ and the image $\mu^s(X)$ contains an open subset of $\mathfrak{g}^*$.

Denote by $W_x \subset T_xX$ the tangent space to the G-orbit $G \cdot x$ in $X = T^*\mathcal{O}$. Let $W_x^{s\perp}$ be the (skew)orthogonal complement to W_x in T_xX with respect to the form Ω^s. Fix the nondegenerate Poisson structure $\eta^t = (\Omega^s)^{-1}$. It is easy to see that $\dim\ker B_x^t$ coincides with dimension of the intersection $W_x \cap W_x^{s\perp}$. But by the G-equivariance of the moment map μ^s, $\xi_X(x) \in W_x \cap W_x^{s\perp}$ iff $\mathrm{ad}^*\,\xi(\mu^s(x)) = 0$, i.e., $\dim\ker B_x^t$ is equal to codimension of the orbit $\mathrm{Ad}^*(G) \cdot \mu^s(x)$ in $\mathfrak{g}^*$. So $\dim\ker B_x^t = \operatorname{rank}\mathfrak{g}$ at a generic point $x \in T^*\mathcal{O}$.

Now consider the degenerate Poisson structure η^t, $t_1 + t_2 = 0$. Its symplectic leaves are the cotangent spaces $T_{o_1}^*\mathcal{O}$, $o_1 \in \mathcal{O}$. Since the action of G on the base $\mathcal{O}$ is transitive, it is sufficient to consider only one leaf $V = T_o^*\mathcal{O}$, where $o = \{K\} \in \mathcal{O} = G/K$. By Remark 1.7 the corresponding symplectic structure $\omega'_{\mathcal{O}}$ on V is independent of the point $v \in V$. Moreover, since the bi-vector η^t is G-invariant, the Poisson algebra (A^G, η^t) is isomorphic to the Poisson algebra $(A^G|V, (\omega'_{\mathcal{O}})^{-1})$. The action of G on $T^*\mathcal{O}$ induces a linear action of the subgroup K on $V = T_o^*\mathcal{O}$. It is clear that the space $A^G|V$ coincides with the space A_V^K of all analytic K-invariant functions on V. Therefore dimension $\dim\ker B_v^t$, $v \in V$ coincides with the dimension of kernel of the restriction of the (nondegenerate) bi-vector $(\omega'_{\mathcal{O}})_v^{-1}$ to the space $\{df_v \mid f \in A_V^K\}$. So we are in the similar to the above situation. Indeed, the action K on V is locally free because so is the action of G on $T^*\mathcal{O}$. Since the form $\omega'_{\mathcal{O}}$ on V is independent of $v \in V$ and K acts on V by linear (symplectic) transformations preserving $\omega'_{\mathcal{O}}$, this action of K on V is Hamiltonian. The corresponding K-equivariant moment map μ' has the form $\mu'(v)(\zeta) = \frac{1}{2}\omega'_{\mathcal{O}}(v, \zeta \cdot v)$, where $\zeta \in \mathfrak{k}$ and $\mathfrak{k}$ acts on V by some linear representation (see also Remark 3.2, where the exact expression for $\omega'_{\mathcal{O}}$ is calculated). Here we consider the vector $v \in V$ also as an element of $T_vV = V$. Therefore the number $\dim\ker B_v^t$ is equal to the codimension of orbit $\mathrm{Ad}^*(K) \cdot \mu'(v)$ in $\mathfrak{k}^*$, i.e., $\dim\ker B_v^t = \operatorname{rank}\mathfrak{k} = \operatorname{rank}\mathfrak{g}$ at a generic point $v \in V$. Since the degenerate structure η^t is G-invariant, $\dim\ker B_x^t = \operatorname{rank}\mathfrak{g}$ and $\dim G^x = 0$ for all x from some open dense G-invariant subset $U \subset T^*\mathcal{O}$.

Consider again the nondegenerate Poisson structure $\eta^t = (\Omega^s)^{-1}$. If for $x \in U$ the dimension $\dim\ker B_x^t$ is not minimal, then the coadjoint orbit through the element $\mu^s(x)$ in $\mathfrak{g}^*$ has nonmaximal dimension. Since the algebra Lie $\mathfrak{g}$ is reductive, the union $\mathfrak{g}_{\mathrm{sing}}^* \subset \mathfrak{g}^*$ of such (singular) orbits in $\mathfrak{g}^*$ has the codimension $\geqslant 3$. Thus the preimage $(\mu^s)^{-1}(\mathfrak{g}_{\mathrm{sing}}^*)$ has codimension $\geqslant 3$ in U because the mapping $\mu^s : U \to \mathfrak{g}^*$ is a submersion on U. The union of these subsets when the parameter s runs through all complex numbers gives us a set of codimension $\geqslant 2$ in U. Thus there exists an open dense subset in U where the pair $(A^G, \{\eta^t(\omega_{\mathcal{O}})\})$ is Kronecker. $\square$

2. Lie-algebraic auxiliary results

In this section we prove some Lie-algebraic assertions, allowing us to prove Theorem 1.10 in the case when the corresponding $\mathrm{Ad}(G)$-action is not locally free.

2.1. Pairs of reductive Lie algebras

Let $\mathfrak{g}$ be a reductive real (or complex) Lie algebra. There exists a faithful representation χ of $\mathfrak{g}$ such that its associated bilinear form Φ_χ is nondegenerate on $\mathfrak{g}$ (if $\mathfrak{g}$ is semisimple we can take the Killing form associated with the adjoint representation of $\mathfrak{g}$). Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra, i.e., the representation $x \mapsto \mathrm{ad}_{\mathfrak{g}}\, x$ of $\mathfrak{k}$ on $\mathfrak{g}$ is completely reducible. This subalgebra is necessarily reductive (in itself). Assume also that the form Φ_χ is nondegenerate on $\mathfrak{k}$. Denote by $\mathfrak{m}$ the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$ (in particular $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the direct sum decomposition of $\mathfrak{g}$). For each element $x \in \mathfrak{g}$ let $\mathfrak{g}^0(x)$ (respectively $\mathfrak{g}^x$) denote the set of all $z \in \mathfrak{g}$ which satisfy $(\mathrm{ad}\ x)^n(z) = 0$ for sufficiently large n (respectively $[x, z] = 0$). Let $\mathfrak{k}^x = \mathfrak{k} \cap \mathfrak{g}^x$. The set

$$R(\mathfrak{m}) = \{x \in \mathfrak{m} \mid \dim \mathfrak{g}^x = q(\mathfrak{m}), \dim \mathfrak{g}^0(x) = Q(\mathfrak{m}), \dim \mathfrak{k}^x = p(\mathfrak{m})\}, \tag{2.1}$$

where $q(\mathfrak{m})$ (respectively $Q(\mathfrak{m})$ and $p(\mathfrak{m})$) is the minimum of dimensions of the spaces $\mathfrak{g}^y$ (respectively $\mathfrak{g}^0(y)$ and $\mathfrak{k}^y$) over all $y \in \mathfrak{m}$, is a nonempty Zariski open subset of $\mathfrak{m}$. Since the number $p(\mathfrak{m})$ is determined only by ad representation of $\mathfrak{k}$ in $\mathfrak{m}$, we will denote it also by $p(\mathfrak{m}, \mathfrak{k})$. The set $R(\mathfrak{m})$ consists of semisimple elements of $\mathfrak{g}$, [Myk2, Prop. 1.2], i.e., $q(\mathfrak{m}) = Q(\mathfrak{m})$ and the centralizer $\mathfrak{g}^x, x \in R(\mathfrak{m})$ is a reductive (in $\mathfrak{g}$) subalgebra of $\mathfrak{g}$. Moreover, the maximal semisimple ideal $[\mathfrak{g}^x, \mathfrak{g}^x]$ of $\mathfrak{g}^x$ is contained in the algebra $\mathfrak{k}^x$, i.e.,

$$[\mathfrak{g}^x, \mathfrak{g}^x] = [\mathfrak{k}^x, \mathfrak{k}^x], \tag{2.2}$$

(see [Mish] or [Myk2, Prop. 1.1]). In particular, the subalgebra $\mathfrak{k}^x \subset \mathfrak{g}^x$ is reductive in $\mathfrak{g}$ and $\dim(\mathfrak{g}^x/\mathfrak{k}^x) = \mathrm{rank}\, \mathfrak{g} - \mathrm{rank}\, \mathfrak{k}^x$.

Now, let us consider an important subset of $\mathfrak{m}$. For any $x \in \mathfrak{m}$ define the subspace $\mathfrak{m}(x) \subset \mathfrak{m}$ putting

$$\mathfrak{m}(x) \stackrel{\text{def}}{=} \{z \in \mathfrak{m} \mid [x, z] \in \mathfrak{m}\}, \tag{2.3}$$

i.e., $\mathrm{ad}\, x(\mathfrak{m}(x)) \subset \mathfrak{m}$. By the invariance of Φ_χ

$$\mathfrak{m}(x) = \{z \in \mathfrak{m} \mid \Phi_\chi(z, \mathrm{ad}\, x(\mathfrak{k})) = 0\}. \tag{2.4}$$

Proposition 2.1. [Myk3] *For arbitrary element $x \in R(\mathfrak{m})$, we have $[\mathfrak{m}(x), \mathfrak{k}^x] = 0$.*

By the dimension arguments from this proposition and definition (2.1) we get

Corollary 2.2. *For $x \in R(\mathfrak{m})$ and each element $x' \in \mathfrak{m}(x) \cap R(\mathfrak{m})$, we have $\mathfrak{k}^{x'} = \mathfrak{k}^x$.*

The following proposition generalize some assertions of the proof of Theorem 11 in [Myk3].

Proposition 2.3. *Assume that $x_0 \in R(\mathfrak{m})$ and $\mathfrak{a}$ is a reductive (in $\mathfrak{g}$) subalgebra of $\mathfrak{k}^{x_0}$. Let $\hat{\mathfrak{g}} = \mathfrak{g}^{\mathfrak{a}}$ and $\hat{\mathfrak{k}} = \mathfrak{k}^{\mathfrak{a}}$ be the centralizers of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Let $\hat{\mathfrak{m}} \stackrel{\text{def}}{=} \{x \in \hat{\mathfrak{g}} \mid \Phi_\chi(x, \hat{\mathfrak{k}}) = 0\}$. Then*

1. *the subalgebras $\mathfrak{k}^{x_0}$, $\mathfrak{a}$, $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$ are reductive in $\mathfrak{g}$ subalgebras, the restrictions of the form Φ_χ to the subalgebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$ respectively are nondegenerate, in particular, $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{m}}$;*
2. *$\hat{\mathfrak{m}} = \hat{\mathfrak{g}} \cap \mathfrak{m}$ ($\hat{\mathfrak{k}} = \hat{\mathfrak{g}} \cap \mathfrak{k}$ by definition);*
3. *for any element x from Zariski open subset $\hat{\mathfrak{m}} \cap R(\mathfrak{m})$ (containing x_0) of $\hat{\mathfrak{m}}$ we have $\hat{\mathfrak{m}}(x) = \mathfrak{m}(x)$ and $\mathfrak{a} \subset \mathfrak{k}^x$.*

If in addition $\mathfrak{a} = \mathfrak{k}^{x_0}$, then for any element x of the (nonempty) set $\hat{\mathfrak{m}} \cap R(\mathfrak{m})$ we have $\mathfrak{k}^x = \mathfrak{k}^{x_0}$ and the centralizer $\hat{\mathfrak{k}}^x$ is contained in the center $\mathfrak{z}(\hat{\mathfrak{g}})$ of the reductive Lie algebra $\hat{\mathfrak{g}}$, i.e, $\hat{\mathfrak{k}}^x = \mathfrak{z}(\hat{\mathfrak{g}}) \cap \hat{\mathfrak{k}}$. Moreover, this element x is a regular element of the reductive Lie algebra $\hat{\mathfrak{g}}$ and $x \in R(\hat{\mathfrak{m}})$ (i.e., $(\hat{\mathfrak{m}} \cap R(\mathfrak{m})) \subset R(\hat{\mathfrak{m}})$).

Proof. As we noted above the element $x_0 \in R(\mathfrak{m})$ is semisimple and the algebras $\mathfrak{g}^{x_0}$, $\mathfrak{k}^{x_0}$ are reductive in $\mathfrak{g}$. Therefore the subalgebra $\mathfrak{a} \subset \mathfrak{k}^{x_0}$ is reductive in $\mathfrak{g}$ by transitivity of this property. To prove property (1) we will use the following well known method [Bou2, Ch. VII, §1]. Since the representation $z \mapsto \operatorname{ad}_{\mathfrak{g}} z$ of the Lie algebra $\mathfrak{a}$ is completely reducible and the algebra $\hat{\mathfrak{g}}$ is an intersection of the kernels of endomorphisms $\operatorname{ad}_{\mathfrak{g}}(z)$, $z \in \mathfrak{a}$, we have the following splitting

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus [\mathfrak{a}, \mathfrak{g}] \tag{2.5}$$

(see [Bou1, Ch. I, §3, Prop. 6]). Using the invariance of the form Φ_χ and the relation $[\mathfrak{a}, \hat{\mathfrak{g}}] = 0$, we obtain that $\Phi_\chi([\mathfrak{a}, \mathfrak{g}], \hat{\mathfrak{g}}) = \Phi_\chi(\mathfrak{g}, [\mathfrak{a}, \hat{\mathfrak{g}}]) = 0$, i.e., the subspaces $\hat{\mathfrak{g}}$ and $[\mathfrak{a}, \mathfrak{g}]$ are mutually orthogonal. Now it follows from the above mentioned splitting (2.5) that the form Φ_χ is nondegenerate on $\hat{\mathfrak{g}}$. Changing in the considerations above the algebra $\mathfrak{g}$ by $\mathfrak{k}$, we prove that the form Φ_χ is nondegenerate on $\hat{\mathfrak{k}}$.

The centralizer $Z(\mathfrak{b})$ of a semisimple subalgebra $\mathfrak{b} \subset \mathfrak{g}$ in $\mathfrak{g}$ is reductive in $\mathfrak{g}$ subalgebra [Bou2, Ch. VII, §1, Prop. 13]. So the algebra $\hat{\mathfrak{g}}$ is reductive (the center of $\mathfrak{a}$ consists of semisimple elements of the Lie algebra $\mathfrak{g}$). The center of $\hat{\mathfrak{g}}$ also consists of semisimple elements of the Lie algebra $\mathfrak{g}$ because this center is a Cartan subalgebra of the centralizer $Z([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] \oplus [\mathfrak{a}, \mathfrak{a}])$ (of a semisimple algebra). Similarly $\hat{\mathfrak{k}}$ is a reductive in $\mathfrak{k}$ and, consequently, in $\mathfrak{g}$ subalgebra.

Since $\mathfrak{k}$ and $\mathfrak{m}$ are stable under $\operatorname{ad}(\mathfrak{a})$, we obtain the splitting $\hat{\mathfrak{g}} = \hat{\mathfrak{g}} \cap \mathfrak{k} \oplus \hat{\mathfrak{g}} \cap \mathfrak{m}$ into a sum of two mutually orthogonal subspaces, so that $\hat{\mathfrak{m}} = \hat{\mathfrak{g}} \cap \mathfrak{m}$.

Let x be any element of the nonempty Zariski open set $\hat{\mathfrak{m}} \cap R(\mathfrak{m})$ containing x_0. By Proposition 2.1, $[\mathfrak{m}(x), \mathfrak{k}^x] = 0$. But $x \in \hat{\mathfrak{m}}$, hence $[x, \mathfrak{a}] = 0$. Therefore $\mathfrak{a} \subset \mathfrak{k}^x$ and $[\mathfrak{m}(x), \mathfrak{a}] = 0$, i.e., $\mathfrak{m}(x) \subset \hat{\mathfrak{m}}$. Let $x_1 \in \mathfrak{m}(x)$ and $z \in \mathfrak{a}$. Given three elements x, x_1, z, consider the Jacobi identity

$$[[x, x_1], z] + [[x_1, z], x] + [[z, x], x_1] = 0.$$

Since $[\mathfrak{m}(x), \mathfrak{a}] = 0$ and $[x, \mathfrak{a}] = 0$, the second and the third term in this identity vanish. Therefore $[[x, \mathfrak{m}(x)], \mathfrak{a}] = 0$, and, consequently, $[x, \mathfrak{m}(x)] \subset \hat{\mathfrak{g}}$. But $[x, \mathfrak{m}(x)] \subset \mathfrak{m}$ by definition. From these two inclusions it follows that $[x, \mathfrak{m}(x)] \subset \hat{\mathfrak{m}}$. Since $\mathfrak{m}(x) \subset \hat{\mathfrak{m}}$, we have $\mathfrak{m}(x) \subset \hat{\mathfrak{m}}(x)$ (see (2.3)). Using analogous arguments from the inclusion $\hat{\mathfrak{m}} \subset \mathfrak{m}$ and (2.3) we obtain that $\hat{\mathfrak{m}}(x) \subset \mathfrak{m}(x)$. Thus $\hat{\mathfrak{m}}(x) = \mathfrak{m}(x)$.

Suppose now that in addition $\mathfrak{a} = \mathfrak{k}^{x_0}$. By already proved condition (3) we have $\mathfrak{k}^{x_0} \subset \mathfrak{k}^x$. But $\dim \mathfrak{k}^{x_0} = \dim \mathfrak{k}^x$, so that $\mathfrak{k}^{x_0} = \mathfrak{k}^x$. Let $\mathfrak{z}(x_0)$ be the center of the Lie algebra $\mathfrak{a} = \mathfrak{k}^{x_0}$. By the definition of the algebra $\hat{\mathfrak{g}}$, we have $[\mathfrak{z}(x_0), \hat{\mathfrak{g}}] = 0$ and $\mathfrak{z}(x_0) \subset \hat{\mathfrak{g}}$. Since

$$\hat{\mathfrak{k}}^x \stackrel{\text{def}}{=} \hat{\mathfrak{g}}^x \cap \hat{\mathfrak{k}} = (\mathfrak{g}^x \cap \hat{\mathfrak{g}}) \cap (\hat{\mathfrak{g}} \cap \mathfrak{k}) = \hat{\mathfrak{g}} \cap \mathfrak{k}^x = \hat{\mathfrak{g}} \cap \mathfrak{k}^{x_0},$$

we have

$$[\hat{\mathfrak{k}}^x, \mathfrak{k}^{x_0}] = 0 \quad \text{and} \quad \hat{\mathfrak{k}}^x \subset \mathfrak{k}^{x_0}.$$

Thus $\mathfrak{z}(x_0)$ is a subspace of the center of $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}^x \subset \mathfrak{z}(x_0)$, i.e., $\hat{\mathfrak{k}}^x = \mathfrak{z}(\hat{\mathfrak{g}}) \cap \hat{\mathfrak{k}}$. By definition, $\hat{\mathfrak{g}}^x = \hat{\mathfrak{g}} \cap \mathfrak{g}^x$ and $[\hat{\mathfrak{g}}^x, \mathfrak{k}^x] = 0$ because $\mathfrak{k}^x = \mathfrak{k}^{x_0}$. Since the algebra $\mathfrak{k}^x$ contains the maximal semisimple ideal of the centralizer $\mathfrak{g}^x$, the algebra $\hat{\mathfrak{g}}^x$ is a subalgebra of the center of $\mathfrak{g}^x$. Thus x is a regular element of the reductive Lie algebra $\hat{\mathfrak{g}}$ because x is a semisimple element of $\mathfrak{g}$. Now taking into account definition (2.1) and the fact that dimension of the space $\hat{\mathfrak{k}}^x = \mathfrak{z}(\hat{\mathfrak{g}}) \cap \hat{\mathfrak{k}}$ is constant for all x from the open set $\hat{\mathfrak{m}} \cap R(\mathfrak{m})$, we obtain that $x \in R(\hat{\mathfrak{m}})$. $\square$

2.2. Pairs of reductive algebraic Lie algebras

Here we will use the notation of the previous subsection, but suppose in addition that the Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ are algebraic, i.e., there are algebraic connected in the Zariski topology (irreducible) Lie groups $K \subset G$ with these Lie algebras. Remark here that if $\mathbb{F} = \mathbb{C}$, these groups are connected in usual topology, and if $\mathbb{F} = \mathbb{R}$, they have a finite number of connected components. Denote by $P^K_{\mathfrak{m}}$ the space of $\mathrm{Ad}(K)$-invariant polynomial functions on the space $\mathfrak{m}$. For any smooth function f on $\mathfrak{m}$, write $\operatorname{grad} f$ (or $\operatorname{grad}_{\mathfrak{m}} f$ if f is the restriction of some function to $\mathfrak{m}$) for the vector field on $\mathfrak{m}$ such that

$$df_x(y) = \Phi_\chi(\operatorname{grad} f(x), y) \quad \text{for all} \quad y \in \mathfrak{m}. \tag{2.6}$$

Lemma 2.4. *Put* $P(\mathfrak{m}) = \{x \in R(\mathfrak{m}) \mid \operatorname{ddim}_x P^K_{\mathfrak{m}} = \operatorname{ddim} P^K_{\mathfrak{m}}\}$. *Then for each point x from the nonempty Zariski open subset $P(\mathfrak{m}) \subset R(\mathfrak{m})$, we have* $\operatorname{ddim}_x P^K_{\mathfrak{m}} = \dim \mathfrak{m}(x)$ *and the space $\mathfrak{m}(x)$ is generated by the vectors* $\{\operatorname{grad} f(x) \mid f \in P^K_{\mathfrak{m}}\}$.

Proof. The proof below is simple and standard, but we need it for further references on this method. Consider first the real case, i.e., $\mathbb{F} = \mathbb{R}$. Let $K^{\mathbb{C}}$ be the complexification of the Lie group K with a compact real form $K_0 \subset K^{\mathbb{C}}$. Denote by $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{k}_0$ the corresponding Lie algebras. The Lie groups $K^{\mathbb{C}}$ and K_0 are connected (in the usual topology) [VO, Ch.5, §2]. Since the algebra $\mathfrak{k}_0$ is a compactly embedded subalgebra of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}_0$ is contained in some compact real form $\mathfrak{g}_0$ of $\mathfrak{g}^{\mathbb{C}}$. The natural extension $\Phi^{\mathbb{C}}_\chi$ of the form Φ_χ is negative-definite on $\mathfrak{g}_0$. It is clear that the space $\mathfrak{m}^{\mathbb{C}} \stackrel{\text{def}}{=} \mathfrak{m} \oplus i\mathfrak{m}$ coincides with the space $\mathfrak{m}_0 \oplus i\mathfrak{m}_0$, where $\mathfrak{m}_0 = \mathfrak{k}_0^\perp$ in $\mathfrak{g}_0$ with respect to $\Phi^{\mathbb{C}}_\chi|\mathfrak{g}_0$.

Consider the space $P^{K_0}_{\mathfrak{m}_0}$ of $\mathrm{Ad}(K_0)$-invariant polynomial functions on $\mathfrak{m}_0$. Since the connected Lie group $\mathrm{Ad}(K_0)$ is compact, any two its orbits in $\mathfrak{m}_0$ are separated by some polynomial from $P^{K_0}_{\mathfrak{m}_0}$. So $\operatorname{ddim} P^{K_0}_{\mathfrak{m}_0}$ is equal to codimension in $\mathfrak{m}_0$ of an $\mathrm{Ad}(K_0)$-orbit of maximal dimension, i.e., to $\dim \mathfrak{m}_0 - (\dim \mathfrak{k}_0 - p(\mathfrak{m}_0, \mathfrak{k}_0))$. But each polynomial $f \in P^{K_0}_{\mathfrak{m}_0}$ determines the (complex) polynomial $\tilde{f}$ on $\mathfrak{m}^{\mathbb{C}}$. Since $\mathrm{Ad}(K_0)$ is a real form of $\mathrm{Ad}(K^{\mathbb{C}})$, this polynomial is $\mathrm{Ad}(K^{\mathbb{C}})$-invariant. So $\operatorname{ddim}_{\mathbb{R}} P_{\mathfrak{m}_0}^{K_0} \leqslant \operatorname{ddim}_{\mathbb{C}} P_{\mathfrak{m}^{\mathbb{C}}}^{K^{\mathbb{C}}}$. Taking into account that $\mathfrak{m}$ is a real form of $\mathfrak{m}^{\mathbb{C}}$ and considering the real and imaginary parts of the restriction $\tilde{f}|\mathfrak{m}$, we obtain that $\operatorname{ddim}_{\mathbb{C}} P_{\mathfrak{m}^{\mathbb{C}}}^{K^{\mathbb{C}}} \leqslant \operatorname{ddim}_{\mathbb{R}} P_{\mathfrak{m}}^{K}$. Since $p(\mathfrak{m}_0, \mathfrak{k}_0) = p(\mathfrak{m}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}) = p(\mathfrak{m}, \mathfrak{k})$, these two inequalities above are equalities. To prove the last assertion it is sufficient to note that for any $f \in P_{\mathfrak{m}}^{K}$ by definition $\operatorname{grad} f(x) \in \mathfrak{m}(x)$. Slightly modifying this considerations, we obtain the proof in the complex case. $\square$

Let x_0 be an element of the set $P(\mathfrak{m}) \subset R(\mathfrak{m})$. Let $\hat{\mathfrak{g}} = \mathfrak{g}^{\mathfrak{a}}$ and $\hat{\mathfrak{k}} = \mathfrak{k}^{\mathfrak{a}}$ be the centralizers determined by the algebra $\mathfrak{a} = \mathfrak{k}^{x_0}$. The Lie algebras $\hat{\mathfrak{k}}$ and $\hat{\mathfrak{g}}$ are Lie algebras of the connected in Zariski topology algebraic Lie subgroups $\hat{K}$ and $\hat{G}$ of the Lie group G, $\hat{K} \subset \hat{G}$. It is clear that $P_{\mathfrak{m}}^{K}|\hat{\mathfrak{m}} \subset P_{\hat{\mathfrak{m}}}^{\hat{K}}$. Since, by Proposition 2.3, $\mathfrak{m}(x_0) = \hat{\mathfrak{m}}(x_0)$ and $x_0 \in R(\hat{\mathfrak{m}})$, as an immediate consequence of Lemma 2.4 we obtain

Corollary 2.5. *For the point $x_0 \in P(\mathfrak{m}) \subset R(\mathfrak{m})$ we have $\hat{\mathfrak{m}} \cap P(\mathfrak{m}) \subset P(\hat{\mathfrak{m}})$. In particular, $x_0 \in \hat{\mathfrak{m}} \cap P(\mathfrak{m})$ and for any point $x \in \hat{\mathfrak{m}} \cap P(\mathfrak{m})$, the following equality holds:* $\operatorname{ddim}_x P_{\hat{\mathfrak{m}}}^{\hat{K}} = \operatorname{ddim}_x (P_{\mathfrak{m}}^{K}|\hat{\mathfrak{m}}) = \operatorname{ddim}_x P_{\mathfrak{m}}^{K}$.

Since the form $\Phi_\chi^{\mathbb{C}}$ is negative-definite on the compact form $\mathfrak{g}_0$, the form $\Phi_\chi^{\mathbb{C}}$ is nondegenerate on each complex subspace $\operatorname{ad} y(\mathfrak{k}^{\mathbb{C}}) \subset \mathfrak{m}^{\mathbb{C}}$ if $y \in \mathfrak{m}_0$, i.e., $\mathfrak{m}^{\mathbb{C}}(y) \oplus \operatorname{ad} y(\mathfrak{k}^{\mathbb{C}}) = \mathfrak{m}^{\mathbb{C}}$. Taking into account that $\mathfrak{m}$ is a real form of $\mathfrak{m}^{\mathbb{C}}$, we obtain

Corollary 2.6. *If the reductive Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ are algebraic, then a splitting $\mathfrak{m}(x) \oplus \operatorname{ad} x(\mathfrak{k}) = \mathfrak{m}$ holds for all elements x from some nonempty Zariski open subset of $\mathfrak{m}$.*

3. Reduction

In this section we will prove Theorem 1.10 in the general case, i.e., for an arbitrary orbit $\mathcal{O} \subset \mathfrak{g}$ of a semisimple element $a \in \mathfrak{g}$, thus obtaining the integrability of the geodesic flow for such spaces and for other homogeneous spaces.

3.1. The bi-Poisson structure $\{\eta^t(\omega_{\mathcal{O}})\}$: exact formulas and involutive sets of functions

Let G be a reductive connected Lie group over the field $\mathbb{F}$ (of real or complex numbers) with the Lie algebra $\mathfrak{g}$. Suppose that the Lie algebra $\mathfrak{g}$ is algebraic. Then, in particular, G can be chosen to be a connected component of some connected in the Zariski topology algebraic group. Consider the adjoint action of G on $\mathfrak{g}$ and some G-orbit $\mathcal{O} \subset \mathfrak{g}$ through a semisimple element $a \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. Then $\mathcal{O} = G/K$, where K is a closed reductive subgroup of G (the isotropy group of a). Denote by $\mathfrak{k}$ the (algebraic) Lie algebra of K.

Using the invariant form $\langle\ ,\ \rangle \stackrel{\text{def}}{=} \Phi_\chi$ on the Lie algebra $\mathfrak{g}$, we identify the dual space $\mathfrak{g}^*$ and $\mathfrak{g}$. So $\mathcal{O}$ is a symplectic (real or complex) manifold with the Kirillov–Kostant–Souriau symplectic structure $\omega_{\mathcal{O}}$. By definition the form $\omega_{\mathcal{O}}$ is G-invariant and at the point $a \in \mathcal{O}$ we have

$$\omega_{\mathcal{O}}(a)([a, \xi_1], [a, \xi_2]) = -\langle a, [\xi_1, \xi_2]\rangle, \quad \forall \xi_1, \xi_2 \in \mathfrak{g}, \tag{3.1}$$

where we consider the vectors $[a, \xi_1], [a, \xi_2] \in T_a\mathfrak{g} = \mathfrak{g}$ as tangent vectors to the orbit $\mathcal{O} \subset \mathfrak{g}$ at the point $a \in \mathcal{O}$.

Since $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}$ of maximal rank, the form $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where recall $\mathfrak{m} = \mathfrak{k}^\perp$ in $\mathfrak{g}$. The form $\langle\ ,\ \rangle$ defines a G-invariant $\mathbb{F}$-valued metric on G/K. This metric identifies the cotangent bundle $T^*\mathcal{O}$ and the tangent bundle $T\mathcal{O}$. Let $\varphi : T^*\mathcal{O} \to T\mathcal{O}$ be the corresponding morphism. Thus we can also talk about the canonical 2-form Ω on $T\mathcal{O}$. The symplectic form Ω is G-invariant with respect to the natural action of G on $T\mathcal{O}$ (extension of the action of G on $\mathcal{O}$).

Denote by $\tau : T\mathcal{O} \to \mathcal{O}$ the natural projection. It is clear that $\tau \circ \varphi = \pi$, where, recall, $\pi : T^*\mathcal{O} \to \mathcal{O}$ is also the natural projection. So $\pi^*\omega_{\mathcal{O}} = \varphi^*(\tau^*\omega_{\mathcal{O}})$ and by Proposition 1.6 the inverses η_1, η_2 to the closed 2-forms $\omega_1 = \Omega$ and $\omega_2 = \Omega + \tau^*\omega_{\mathcal{O}}$ ($\alpha = \omega_{\mathcal{O}}$) on $T\mathcal{O}$ define the bi-Poisson structure $\{\eta^t = \eta^t(\omega_{\mathcal{O}})\}_{t\in\mathbb{F}^2}$ on $T\mathcal{O}$.

Consider the trivial vector bundle $G \times \mathfrak{m}$ with the two commuting Lie group actions on it: the left G-action, $l_h : (g, w) \mapsto (hg, w)$, and the right K-action $r_k : (g, w) \mapsto (gk, \operatorname{Ad} k^{-1}(w))$. Let $p_1 : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$ be the natural projection. It is well known that $G \times_K \mathfrak{m}$ and $T(G/K)$ are isomorphic. Using the corresponding G-equivariant diffeomorphism $\varphi_1 : G \times_K \mathfrak{m} \to T(G/K)$, $[(g, w)] \mapsto \frac{d}{dt}\big|_0 g \exp(tw)K$ and the projection p_1 we define the G-equivariant submersion $\Pi : G \times \mathfrak{m} \to T(G/K)$, $\Pi = \varphi_1 \circ p_1$.

We can identify the tangent space $T_o(G/K)$ at the point $o = p(e)$ with the space $\mathfrak{m}$ by means of the canonical projection $p : G \to G/K$. Let A^G (resp. $A^K_{\mathfrak{m}}$) be the set of all $\mathbb{F}$-analytic G-invariant (resp. $\operatorname{Ad}(K)$-invariant) functions on $T(G/K)$ (resp. on $\mathfrak{m}$). There is a one-to-one correspondence between G-orbits in $T(G/K)$ and $\operatorname{Ad}(K)$-orbits in $\mathfrak{m}$. Thus using the submersion $\Pi : G \times \mathfrak{m} \to T(G/K)$ we can identify naturally the spaces of functions A^G and $A^K_{\mathfrak{m}}$.

Let ξ^l be the left-invariant vector field on the Lie group G defined by a vector $\xi \in \mathfrak{g}$. The form $\Pi^*\Omega$ on the manifold $G \times \mathfrak{m}$ has the following form, [Myk4]:

$$(\Pi^*\Omega)_{(g,x)}((\xi_1^l(g), y_1), (\xi_2^l(g), y_2)) = \langle \xi_2, y_1\rangle - \langle \xi_1, y_2\rangle - \langle x, [\xi_1, \xi_2]\rangle, \tag{3.2}$$

where $g \in G, x \in \mathfrak{m}$, $\xi_1, \xi_2 \in \mathfrak{g}$, $y_1, y_2 \in \mathfrak{m} = T_x\mathfrak{m}$.

The kernel $\mathcal{K} \subset T(G \times \mathfrak{m})$ of the 2-form $\Pi^*\Omega$ coincides with the kernel of Π_*, i.e.,

$$\mathcal{K}_{(g,x)} = \{(\zeta^l(g), [x, \zeta]) \in T_gG \times \mathfrak{m},\ \zeta \in \mathfrak{k}\}. \tag{3.3}$$

Now it is easy to verify using formulas (3.1), (3.2) and (left) G- and (right) K-invariance of the form $\Pi^*(s_1\Omega + s_2\tau^*\omega_{\mathcal{O}})$, $(s_1, s_2) \in \mathbb{F}^2$ that

$$\begin{aligned}\Pi^*(s_1\Omega + s_2\tau^*\omega_{\mathcal{O}})_{(g,x)}&((\xi_1^l(g), y_1), (\xi_2^l(g), y_2))\\ &= s_1\big(\langle \xi_2, y_1\rangle - \langle \xi_1, y_2\rangle - \langle x, [\xi_1, \xi_2]\rangle\big) - s_2\langle a, [\xi_1, \xi_2]\rangle. \end{aligned}\tag{3.4}$$

By Proposition 1.6 the form $\delta_{s_1,s_2} \stackrel{\text{def}}{=} s_1\Omega + s_2\tau^*\omega_{\mathcal{O}}$ is nondegenerate on $T(G/K)$ if $s_1 \neq 0$.

Using expressions (3.4) for the form $\Pi^*\delta_{s_1,s_2}$ and (3.3) for the kernel of Π_*, we obtain that the left G-invariant Hamiltonian vector field of the function $f \in A^K_{\mathfrak{m}} = A^G$ equals $\Pi_*(\check{X}_f)$, where

$$\check{X}_f(e, x) = \big(s_1^{-1} \operatorname{grad} f(x),\ s_1^{-1}\big[x + s_2 s_1^{-1} a,\ \operatorname{grad} f(x)\big]\big) \in \mathfrak{g} \times \mathfrak{m}$$

(the vector-field $\operatorname{grad} f$ on $\mathfrak{m}$ is determined by (2.6)). Therefore the Poisson bracket of two functions f_1, f_2 from the set $A^K_{\mathfrak{m}}$ with respect to the Poisson structure determined by the form δ_{s_1,s_2}, $s_1 \neq 0$, is equal to the function $-s_1^{-2}\langle s_1x + s_2a,\ [\operatorname{grad} f_1(x), \operatorname{grad} f_2(x)]\rangle$. So we have proved

Lemma 3.1. *Given an orbit $\mathcal{O} = G/K$, let $\{\ ,\ \}^t$ be the Poisson bracket on the tangent bundle $T\mathcal{O}$ corresponding to the Poisson structure $\eta^t = \eta^t(\omega_{\mathcal{O}})$, $t = (t_1, t_2) \in \mathbb{F}^2$. Then for arbitrary functions $f_1, f_2 \in A^K_{\mathfrak{m}} = A^G$ and a point $x \in \mathfrak{m} = T_o(G/K)$ we have*

$$\{f_1, f_2\}^t(x) = -\langle (t_1 + t_2)x + t_2 a,\ [\operatorname{grad} f_1(x), \operatorname{grad} f_2(x)]\rangle. \tag{3.5}$$

The G-invariant Hamiltonian vector field of the function f_j has the form $\Pi_*(\check{X}^t_{f_j})$, where

$$\check{X}^t_{f_j}(e, x) = \big((t_1 + t_2) \operatorname{grad} f_j(x),\ \big[(t_1 + t_2)x + t_2 a,\ \operatorname{grad} f_j(x)\big]\big) \in \mathfrak{g} \times \mathfrak{m}. \tag{3.6}$$

Remark 3.2. Consider on the linear space $\mathfrak{m}$ the nondegenerate bilinear form $\beta(y_1, y_2) = \langle y_1, \operatorname{ad}_a^{-1}(y_2)\rangle$, $y_1, y_2 \in \mathfrak{m}$, where $\operatorname{ad}_a^{-1} \stackrel{\text{def}}{=} (\operatorname{ad} a|\mathfrak{m})^{-1}$. Since the endomorphism $\operatorname{ad} a|\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric (with respect to the form $\langle\ ,\ \rangle$), the form β is also skew-symmetric. Identifying the tangent space $T_x\mathfrak{m}$ with $\mathfrak{m}$ for each $x \in \mathfrak{m}$, we can consider β as a symplectic form on $\mathfrak{m}$. It is easy to verify (using the invariance of the form $\langle\ ,\ \rangle$) that for arbitrary functions f_1, f_2 on $\mathfrak{m}$ the corresponding Poisson bracket has the form

$$\{f_1, f_2\}^\beta(x) = -\langle a,\ [\operatorname{grad} f_1(x), \operatorname{grad} f_2(x)]\rangle, \tag{3.7}$$

i.e., it coincides on the space of $\operatorname{Ad}(K)$-invariant functions with the Poisson bracket $\{\ ,\ \}^t$ (3.5) in the exceptional case $(t_1 + t_2) = 0$, $t_2 = 1$. By Proposition 1.6 this degenerate Poisson structure η^t has the space $\mathfrak{m} = T_o\mathcal{O}$ as a symplectic leaf and the reduced Poisson structure $\{\ ,\ \}'$ on it is nondegenerate. Using expression (3.4) and calculating the Hamiltonian vector fields of arbitrary functions on $T\mathcal{O}$ with respect to the symplectic form $s_1\Omega + s_2\tau^*\omega_{\mathcal{O}}$, we can show that this Poisson bracket $\{\ ,\ \}'$ coincides with $\{\ ,\ \}^\beta$ (3.7). Since the Ad action of K on $\mathfrak{m}$ preserves the form β, this action of K is Hamiltonian with the moment map $\mu^\beta : \mathfrak{m} \to \mathfrak{k}^*$, $\mu^\beta(x)(\zeta) = -\frac{1}{2}\langle \operatorname{ad}_a^{-1}(x), [\zeta, x]\rangle$, $\forall \zeta \in \mathfrak{k}$ (see the proof of Theorem 1.10).

The centralizer $\mathfrak{k} = \mathfrak{g}^a$ is a direct sum of Lie algebras $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}'$, where $\mathfrak{z}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}'$ its maximal semisimple ideal. It is clear that $a \in \mathfrak{z}$. Moreover, since the Lie group $\operatorname{Ad}(K)$ (resp. $\operatorname{Ad}(\overline{K^{\mathbb{C}}})$) is connected if $\mathbb{F} = \mathbb{C}$ (resp. if $\mathbb{F} = \mathbb{R}$), we have $\operatorname{Ad}(K)b = b$ for each $b \in \mathfrak{z}$. Let $\mathfrak{k}_1$ be some algebraic Lie subalgebra of $\mathfrak{k}$ containing the semisimple Lie algebra $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. There exist closed subgroups K' and K_1 of the group K with the Lie algebras $\mathfrak{k}'$ and $\mathfrak{k}_1$ respectively such that $K' \subset K_1$. The center $\mathfrak{z}_1$ of $\mathfrak{k}_1$ is a subalgebra of the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have the following orthogonal splittings with respect to the form $\langle\ ,\ \rangle$

$$\mathfrak{k}_1 = \mathfrak{z}_1 \oplus \mathfrak{k}', \quad \mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{k}_1, \quad \mathfrak{m}_1 = \mathfrak{z}_* \oplus \mathfrak{m}, \quad \mathfrak{z} = \mathfrak{z}_1 \oplus \mathfrak{z}_*, \tag{3.8}$$

which serve as definitions for $\mathfrak{z}_*$ and $\mathfrak{m}_1$.

Consider on the tangent bundle $T(G/K_1)$ the space A^G_1 of all G-invariant $\mathbb{F}$-analytic functions. As in the case of the subgroup K, we can identify naturally the spaces A^G_1 and $A^{K_1}_{\mathfrak{m}_1}$. Using the form $\langle\ ,\ \rangle$, identify $T^*(G/K_1)$ and $T(G/K_1)$. The canonical Poisson structure, determined by the symplectic form Ω_1 on $T(G/K_1)$, induces the bracket operation $\{\ ,\ \}^{\mathrm{can}}$ on the space $\mathcal{E}(T(G/K_1))$. Using an expression for the lift of Ω_1 to $G \times \mathfrak{m}_1$ similar to (3.2), we find that for arbitrary functions $f_1, f_2 \in A^{K_1}_{\mathfrak{m}_1}$ on $\mathfrak{m}_1$

$$\{f_1, f_2\}^{\mathrm{can}}(x) = -\langle x,\ [\operatorname{grad}_{\mathfrak{m}_1} f_1(x), \operatorname{grad}_{\mathfrak{m}_1} f_2(x)]\rangle. \tag{3.9}$$

Denote by $I(\mathfrak{g})$ the space of all $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$. If $h \in I(\mathfrak{g})$, then it is clear that the function h^λ, $h^\lambda(y) = h(y+\lambda a)$, $\lambda \in \mathbb{F}$, is $\mathrm{Ad}(K_1)$-invariant on $\mathfrak{g}$. Put $H^\lambda = h^\lambda|\mathfrak{m}_1$. For each element $b \in \mathfrak{z}$ and a symmetric (with respect to the form $\langle\ ,\ \rangle$ on $\mathfrak{g}$) endomorphism $D : \mathfrak{k} \to \mathfrak{k}$, $D(\mathfrak{z}) \subset \mathfrak{z}$, $D|\mathfrak{k}' = \mathrm{Id}_{\mathfrak{k}'}$, we can consider the endomorphism $\varphi_{a,b,D} : \mathfrak{g} \to \mathfrak{g}$ putting $\varphi_{a,b,D}(z) = Dz$ if $z \in \mathfrak{k}$ and $\varphi_{a,b,D}(x) = \mathrm{ad}_a^{-1}([b,x])$ if $x \in \mathfrak{m} \subset \mathfrak{g}$. It is clear that the endomorphism $\varphi_{a,b,D}$ is symmetric and the group $\mathrm{Ad}(K_1)$ commutes elementwise with $\varphi_{a,b,D}$ on $\mathfrak{g}$. So the function $h_{a,b,D}(y) = \frac{1}{2}\langle y, \varphi_{a,b,D}(y)\rangle$ on $\mathfrak{g}$ is $\mathrm{Ad}(K_1)$-invariant. Suppose in addition that the endomorphism D leave the subspace $\mathfrak{z}_* \subset \mathfrak{z} \subset \mathfrak{k}$ invariant, i.e., $D(\mathfrak{z}_*) \subset \mathfrak{z}_*$. Then $\varphi_{a,b,D}(\mathfrak{m}_1) \subset \mathfrak{m}_1$ and the function $H_{a,b,D}(x) = \frac{1}{2}\langle x, \varphi_{a,b,D}(x)\rangle$, $x \in \mathfrak{m}_1$, is a Hamiltonian function of the geodesic flow of some pseudo-Riemannian metric on G/K_1 if $\varphi_{a,b,D}|\mathfrak{m}_1$ is nondegenerate.

Lemma 3.3. [BJ1] *For any functions h_1, h_2, $h \in I(\mathfrak{g})$ and arbitrary parameters λ_1, λ_2, $\lambda \in \mathbb{F}$ we have $\{H_1^{\lambda_1}, H_2^{\lambda_2}\}^{\mathrm{can}} = 0$ and $\{H^\lambda, H_{a,b,D}\}^{\mathrm{can}} = 0$.*

Proof. Mainly to fix notation we shall prove this lemma here. The functions $h_1^{\lambda_1}, h_2^{\lambda_2}$ and $h_{a,b,D}$ commute on $\mathfrak{g} \simeq \mathfrak{g}^*$ with respect to the Lie–Poisson bracket [MF]. This means, for example for the functions $h_1^{\lambda_1}, h_2^{\lambda_2}$, that $\langle x, [\mathrm{grad}_{\mathfrak{g}}\, h_1^{\lambda_1}(x), \mathrm{grad}_{\mathfrak{g}}\, h_2^{\lambda_2}(x)]\rangle = 0$ for all $x \in \mathfrak{m}_1 \subset \mathfrak{g}$. But $\mathrm{grad}_{\mathfrak{m}_1} H_j^{\lambda_1}(x) = (\mathrm{grad}_{\mathfrak{g}}\, h_j^{\lambda_1}(x))_{\mathfrak{m}_1}$. Now taking into account that $\mathrm{grad}_{\mathfrak{m}_1} H_j^{\lambda_1}(x) \in \mathfrak{m}_1(x)$, $[x, \mathfrak{m}_1(x)] \subset \mathfrak{m}_1$ and $\mathfrak{m}_1 \perp \mathfrak{k}_1$, we obtain that

$$\langle x, [(\mathrm{grad}_{\mathfrak{g}}\, h_1^{\lambda_1}(x))_{\mathfrak{m}_1}, (\mathrm{grad}_{\mathfrak{g}}\, h_2^{\lambda_2}(x))_{\mathfrak{m}_1}]\rangle = 0,$$

i.e., $\{H_1^{\lambda_1}, H_2^{\lambda_2}\}^{\mathrm{can}}(x) = 0$. Similarly we can show that $\{H^\lambda, H_{a,b,D}\}^{\mathrm{can}} = 0$. □

Remark 3.4. Since the form $\langle\ ,\ \rangle$ is invariant, the quadratic form $x \mapsto \langle x, x\rangle$ is contained in the involutive function set $\{h^\lambda|\mathfrak{m}_1 \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{F}\} \subset A_{\mathfrak{m}_1}^{K_1}$. The functions h^λ and $h_{a,b,D}$ were considered in the papers [MF, BJ1]. Moreover, in [BJ3] the geodesic flow with the Hamiltonian $H_{a,b,D}$ on $T(G/K_1)$ was studied. If the symmetric operator $\varphi_{a,b,D}|\mathfrak{m}_1$ is positive-definite, the function $H_{a,b,D} \in A_{\mathfrak{m}_1}^{K_1} = A_1^G$ is a Hamiltonian function for some G-invariant geodesic flow on $T(G/K_1)$. Remark that such a Riemannian metric on G/K_1 exists also for noncompact G.

3.2. The bi-Poisson structure $\{\eta^t(\omega_{\mathcal{O}})\}$: maximal involutive subsets of functions

We continue with the notation of Subsection 3.1 but in this subsection it is assumed in addition that $\mathbb{F}$ is the field of complex numbers. Since $\mathbb{F} = \mathbb{C}$, the group G is algebraic and the isotropy group $\mathrm{Ad}(K)$ of $a \in \mathfrak{g}$ is connected [Kos, Lemma 5].

Let x be an element of $R(\mathfrak{m})$ which satisfies the following conditions:

$$\mathfrak{m}(x) \oplus \mathrm{ad}\, x(\mathfrak{k}) = \mathfrak{m}, \quad \mathrm{ddim}_x P_{\mathfrak{m}}^K = \mathrm{ddim}\, P_{\mathfrak{m}}^K \tag{3.10}$$

(see Lemma 2.4 and its corollaries). The bi-Poisson structure $\{\eta^t = \eta^t(\omega_{\mathcal{O}})\}$ determines at this point $x \in \mathfrak{m} = T_o(G/K)$ the bilinear forms $B_x^t : D_x^G \times D_x^G \to \mathbb{C}$, where recall the space D_x^G is spanned by differentials of functions from the set A^G and $B_x^t = \eta^t|D_x^G$ (see subsection 1.1). Since we identified the spaces A^G and $A_{\mathfrak{m}}^K$, B_x^t determines the following bilinear forms (which we denote also by B_x^t for short)

$$B_x^t : \mathfrak{m}(x) \times \mathfrak{m}(x) \to \mathbb{C}, \quad (y_1, y_2) \mapsto -\langle (t_1+t_2)x + t_2 a,\ [y_1, y_2]\rangle \tag{3.11}$$

on the space $\mathfrak{m}(x) = \{\operatorname{grad} f(x) \mid f \in A^K_{\mathfrak{m}}\}$. Here we used expression (3.5) and the form $\langle\ ,\ \rangle$ to identify the spaces $\mathfrak{m}(x)$ and $\mathfrak{m}(x)^*$ (this form is nondegenerate on $\mathfrak{m}(x)$). It is easy to see that the kernel of B^t_x is the subspace $V^t(x) \subset \mathfrak{m}(x)$ given by

$$V^t(x) = \{y \in \mathfrak{m}(x) \mid [(t_1+t_2)x + t_2 a,\ y] \in \operatorname{ad} x(\mathfrak{k})\}. \tag{3.12}$$

Remark here that by definition $[x, \mathfrak{m}(x)] \subset \mathfrak{m}$ and $[a, \mathfrak{m}] \subset \mathfrak{m}$. Moreover, since $[a, \mathfrak{k}] = 0$, we have

$$V^t(x) = (\mathfrak{g}^{(t_1+t_2)x+t_2 a})_{\mathfrak{m}}, \quad \text{where} \quad t_1 + t_2 \neq 0 \tag{3.13}$$

and $(\cdot)_{\mathfrak{m}}$ denotes the projection onto $\mathfrak{m}$ along $\mathfrak{k}$. In particular, for $t = (1, 0)$ (for the canonical Poisson structure on $T(G/K)$), $V^{(1,0)}(x) = (\mathfrak{g}^x)_{\mathfrak{m}}$. Since $x \in R(\mathfrak{m})$, dimension of the space $(\mathfrak{g}^x)_{\mathfrak{m}}$ is equal to the constant $r = q(\mathfrak{m}) - p(\mathfrak{m})$, where the numbers $q(\mathfrak{m})$ and $p(\mathfrak{m})$ are defined in Subsection 2.1.

Consider the set $R(\mathfrak{z} \oplus \mathfrak{m})$ determined by (2.1) for the pair $(\mathfrak{g}, \mathfrak{k}')$. By [Myk2, Proposition 1.2] the intersection $R(\mathfrak{z} \oplus \mathfrak{m}) \cap R(\mathfrak{m})$ is nonempty. If $x' \in \mathfrak{m}$ is an element of this intersection, then the whole line $ta + x'$, $t \in \mathbb{C}$, with the exception of a finite number of points belongs to the Zariski open subset $R(\mathfrak{z} \oplus \mathfrak{m})$ of $\mathfrak{z} \oplus \mathfrak{m}$. So we can choose an element $x \in R(\mathfrak{m})$ satisfying (3.10) and such that

$$a + x \in R(\mathfrak{z} \oplus \mathfrak{m}). \tag{3.14}$$

Lemma 3.5. *The number* $\min_{x' \in R(\mathfrak{m})} \dim V^t(x')$, $t \in \mathbb{C}^2$, *is equal to* r, *if* $t_1 + t_2 \neq 0$, *and* $\geqslant r$, *if* $t_1 + t_2 = 0$. *In particular, the pair* $(A^G, \{\eta^t\}_{t \in \mathbb{C}^2})$ *is Kronecker at the point* x *iff* $\dim V^t(x) = r$ *for all* $t \in \mathbb{C}^2 \setminus \{0\}$.

Proof. By [Myk2] (see the proof of Proposition 1.2) $q(\mathfrak{m}) = q(\mathfrak{z} \oplus \mathfrak{m})$, i.e., the centralizers $\mathfrak{g}^x$ and $\mathfrak{g}^{x+a}$ have the same dimension. Now the evident relation $\mathfrak{g}^{x+a} \cap \mathfrak{k} = \mathfrak{g}^x \cap \mathfrak{k}$ ($[a, \mathfrak{k}] = 0$) implies that $\dim(\mathfrak{g}^{x+a})_{\mathfrak{m}} = \dim(\mathfrak{g}^x)_{\mathfrak{m}}$. Since $\min_{x' \in R(\mathfrak{m})} \dim(\mathfrak{g}^{x'+\lambda a})_{\mathfrak{m}} = \min_{x' \in R(\mathfrak{m})} \dim(\mathfrak{g}^{x'+a})_{\mathfrak{m}}$, where $\lambda \in \mathbb{C}$, we obtain the assertion of the lemma. $\square$

Proposition 3.6. *Suppose that an element* $x \in R(\mathfrak{m})$ *satisfies conditions* (3.10) *and* (3.14). *If the pair* $(A^G, \{\eta^t\}_{t \in \mathbb{F}^2})$, $\mathbb{F} = \mathbb{C}$, *is Kronecker at the point* x, *then*

(1) *in the set* $\{h^\lambda|\mathfrak{m} \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{F}\} \subset A^K_{\mathfrak{m}} = A^G$ *there are* $\frac{1}{2}(r + \dim \mathfrak{m}(x))$ *functions functionally independent at this point;*

(2) *this pair* $(A^G, \{\eta^t\}_{t \in \mathbb{F}^2})$ *is Kronecker on some open subset of* $T(G/K)$ *for which the intersection with* $\mathfrak{m} = T_o(G/K)$ *is a Zariski open subset of* $\mathfrak{m}$.

Proof. Since $\dim V^t(x) = r$ for all $t \in \mathbb{C}^2 \setminus \{0\}$, it follows from Proposition 1.4 that the space $L_0(x) = \sum_{t \in \mathbb{C}^2 \setminus \{0\}} V^t(x)$ is a maximal isotropic subspace of $\mathfrak{m}(x)$ with respect to the form $B^{(1,0)}(x)$ (of maximal rank). In particular, $\dim L_0(x) = \frac{1}{2}(r + \dim \mathfrak{m}(x))$.

But the space $L_0(x)$ is generated by a finite subset of spaces from the set $\{V^t(x)\}$. Since the family $V^t(x)$ depends smoothly on the parameter $t \in \mathbb{C}^2 \setminus \{0\}$, we obtain that $L_0(x) = \sum_{j=1}^N V^j(x)$, where each space $V^j(x)$ is determined by (3.12) with $t_1 + t_2 = 1$ and $t_2 = \lambda_j \in \mathbb{C}$, $j = \overline{1, N}$. Moreover, by (3.14) we can choose these numbers $\{\lambda_j\}$ such that $x + \lambda_j a \in R(\mathfrak{z} \oplus \mathfrak{m})$.

Consider the space $I(\mathfrak{g})$ of all $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$. If $h \in I(\mathfrak{g})$, then the function $h^\lambda(x) = h(x+\lambda a)$ restricted to $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant. But $[x+\lambda a, \operatorname{grad}_\mathfrak{g} h(x+\lambda a)] = 0$ by invariance of the form $\langle\ ,\ \rangle$. So

$$\begin{aligned}[x+\lambda a, \operatorname{grad}_\mathfrak{m} h^\lambda(x)] &= -[x+\lambda a, (\operatorname{grad}_\mathfrak{g} h(x+\lambda a))_\mathfrak{k}] \\ &= -[x, (\operatorname{grad}_\mathfrak{g} h(x+\lambda a))_\mathfrak{k}] \in \operatorname{ad} x(\mathfrak{k}),\end{aligned}$$

i.e., $\operatorname{grad}_\mathfrak{m} h^{\lambda_j}(x) \in V^j(x)$. Denote by $V_I^j(x)$ the subspace of $V^j(x)$ spanned by vectors $\operatorname{grad}_\mathfrak{m} h^{\lambda_j}(x)$, $h \in I(\mathfrak{g})$.

We claim that $V_I^j(x) = V^j(x)$. Indeed, the elements $x+\lambda_j a \in R(\mathfrak{z}\oplus\mathfrak{m})$ are semisimple elements of $\mathfrak{g}$. So by [Myk1, Theorem 2.5], the vectors $\operatorname{grad}_\mathfrak{g} h(x+\lambda a)$, $h \in I(\mathfrak{g})$, span the center $\mathfrak{h}(x+\lambda_j a)$ of the centralizer $\mathfrak{g}^{x+\lambda_j a}$, i.e., $V_I^j(x) = (\mathfrak{h}(x+\lambda_j a))_\mathfrak{m}$. By (2.2), $(\mathfrak{h}(x+\lambda_j a))_\mathfrak{m} = (\mathfrak{g}^{x+\lambda_j a})_\mathfrak{m}$, but by (3.13), we have $V^j(x) = (\mathfrak{g}^{x+\lambda_j a})_\mathfrak{m}$, i.e., $V_I^j(x) = V^j(x)$. Thus $L_0(x) = \sum_j V_I^j(x)$, the assertion (1) is proved.

By Lemma 3.3 and by dimension arguments, the space $\sum_j V_I^j(x')$ is maximal isotropic with respect to the form $B^{(1,0)}(x')$, $\dim V^j(x') = r, j = \overline{1,N}$ for all x' from some Zariski open subset in $R(\mathfrak{m})$ containing x. Taking into account that $V_I^j(x') \subset V^j(x') \subset L_0(x')$, from Proposition 1.4 it follows that the pair $(A^G, \{\eta^t\}_{t\in\mathbb{C}^2})$ is Kronecker at x'. □

Consider again the pair $(\mathfrak{g}, \mathfrak{k}_1)$. For $x_1 \in R(\mathfrak{m}_1)$ we have $r_1 = q(\mathfrak{m}_1) - p(\mathfrak{m}_1) = \dim(\mathfrak{g}^{x_1}/\mathfrak{k}_1^{x_1})$. As we have shown above, r_1 is corank of the skew-symmetric bilinear form $B_{x_1}^{\mathrm{can}}$ associated with the bracket $\{\ ,\ \}^{\mathrm{can}}$ (3.9) on the set $A_{\mathfrak{m}_1}^{K_1} = A_1^G$, where recall A_1^G denotes the space of all G-invariant $\mathbb{C}$-analytic functions on $T(G/K_1)$.

Proposition 3.7. *Let $x \in R(\mathfrak{m})$ and $\mathbb{F} = \mathbb{C}$. Suppose that in the set $\{h^\lambda|\mathfrak{m} \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{F}\} \subset A_\mathfrak{m}^K$ there are $\frac{1}{2}(r + \dim \mathfrak{m}(x))$ functionally independent functions at x. Then there is a point $x_1 \in R(\mathfrak{m}) \cap R(\mathfrak{m}_1)$ such that in the set $\{h^\lambda|\mathfrak{m}_1 \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{F}\} \subset A_{\mathfrak{m}_1}^{K_1}$, there are $\frac{1}{2}(r_1 + \dim \mathfrak{m}_1(x_1))$ functionally independent functions at x_1. These functions form a maximal involutive subset of independent functions in the algebra $A_1^G = A_{\mathfrak{m}_1}^{K_1}$ with respect to the canonical Poisson structure on $T(G/K_1)$. Moreover, these functions are integrals of the geodesic flow on $T(G/K_1)$ determined by the form $\langle\ ,\ \rangle$ and of the Hamiltonian flow with the Hamiltonian $H_{a,b,D}$.*

Proof. By [Myk2, Proposition 1.2] the intersection $R(\mathfrak{m}_1) \cap R(\mathfrak{m})$ is nonempty. Since the functions considered are polynomials, we can suppose that the point $x = x_1$ is contained in the set $R(\mathfrak{m}_1) \cap R(\mathfrak{m})$. So by assumptions of the proposition the isotropic subspace $L_\mathfrak{m}(x) \stackrel{\mathrm{def}}{=} \{\operatorname{grad}_\mathfrak{m} h^\lambda(x) \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{C}\}$ of $\mathfrak{m}(x)$ has the maximal possible dimension $\frac{1}{2}(\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \dim \mathfrak{m}(x))$. From the inclusion $\mathfrak{z}_* \subset \mathfrak{k}$ it follows that $\dim \mathfrak{k}_1^x = \dim \mathfrak{k}^x - \dim(\mathfrak{k}^x)_{\mathfrak{z}_*}$. Taking into account that $x \in \mathfrak{m}$, we get

$$\begin{aligned}\dim \mathfrak{m}_1(x) &= \dim \mathfrak{m}_1 - \dim(\operatorname{ad} x(\mathfrak{k}_1)) \\ &= (\dim \mathfrak{m} + \dim \mathfrak{z}_*) - \big((\dim \mathfrak{k} - \dim \mathfrak{z}_*) - (\dim \mathfrak{k}^x - \dim(\mathfrak{k}^x)_{\mathfrak{z}_*})\big) \\ &= \dim \mathfrak{m}(x) + 2\dim \mathfrak{z}_* - \dim(\mathfrak{k}^x)_{\mathfrak{z}_*}.\end{aligned} \tag{3.15}$$

Thus

$$\begin{aligned}\tfrac{1}{2}(r_1 + \dim \mathfrak{m}_1(x_1)) &= \tfrac{1}{2}(\dim(\mathfrak{g}^x/\mathfrak{k}_1^x) + \dim \mathfrak{m}_1(x)) \\ &= \tfrac{1}{2}(\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \dim \mathfrak{m}(x) + 2\dim \mathfrak{z}_*) \\ &= \dim L_\mathfrak{m}(x) + \dim \mathfrak{z}_*.\end{aligned} \tag{3.16}$$

By Lemma 3.3 the space $L_{\mathfrak{m}_1}(x) \stackrel{\text{def}}{=} \{\operatorname{grad}_{\mathfrak{m}_1} h^\lambda(x) \mid h \in I(\mathfrak{g}), \lambda \in \mathbb{C}\}$ is an isotropic subspace for B_x^{can}. We claim that the space $L_{\mathfrak{m}_1}(x)$ contains the space $\mathfrak{z}_* = (\mathfrak{z})_{\mathfrak{m}_1}$. It is clear that it suffices to prove this fact for semisimple $\mathfrak{g}$. Let $h \in I(\mathfrak{g})$ be some homogeneous polynomial of degree n. Define the polynomials $v_j(x)$ by the identity $h(x+\lambda a) = \sum_{j=0}^n \lambda^j v_j(x)$. Since the space $L_{\mathfrak{m}_1}(x)$ contains all vectors $\operatorname{grad}_{\mathfrak{m}_1} h(x+\lambda a)$, $\lambda \in \mathbb{C}$, this space contains the vector $\operatorname{grad}_{\mathfrak{m}_1} v_{n-1}(x)$. But by [Bou2, Ch. VIII, §8, Theorem 1] each invariant homogeneous polynomial of degree n is a linear combination of the functions on $\mathfrak{g}$ of the form $y \mapsto \operatorname{Tr}(\rho(y))^n$, where ρ is a finite-dimensional representation of the semisimple Lie algebra $\mathfrak{g}$. Since for such h we have $v_{n-1}(x) = n \operatorname{Tr}(\rho(a)^{n-1}\rho(x))$, we obtain that $dh(a)(\xi), \xi \in \mathfrak{g}$, coincides with $v_{n-1}(\xi)$, i.e., $\operatorname{grad}_{\mathfrak{g}} h(a) = \operatorname{grad}_{\mathfrak{g}} v_{n-1}(x)$. Taking into account that the vectors $\operatorname{grad}_{\mathfrak{g}} h(a), h \in I(\mathfrak{g})$ span the center $\mathfrak{z}$ of $\mathfrak{k} = \mathfrak{g}^a$ (see [Myk1, Theorem 2.5]), we prove that $\mathfrak{z}_* = (\mathfrak{z})_{\mathfrak{m}_1} \subset L_{\mathfrak{m}_1}(x)$. But by definition, $L_{\mathfrak{m}}(x) = (L_{\mathfrak{m}_1}(x))_{\mathfrak{m}}$. Since $\mathfrak{m}_1 = \mathfrak{z}_* \oplus \mathfrak{m}$, we have $L_{\mathfrak{m}_1}(x) = L_{\mathfrak{m}}(x) + \mathfrak{z}_*$. So by (3.16) $\dim L_{\mathfrak{m}_1}(x) = \frac{1}{2}(\dim(\mathfrak{g}^x/\mathfrak{k}_1^x) + \dim \mathfrak{m}_1(x))$. $\square$

Remark 3.8. Let G_r be the identity component of a real form of the algebraic complex Lie group G. Put $K_r = K \cap G_r$. Here, for this real case we will use all notation of this subsection but with index $_r$. Suppose also in addition to the previous assumptions that $\mathcal{O} = \operatorname{Ad}(G) \cdot a$, where $a \in \mathfrak{g}_r \subset \mathfrak{g}$. Now using the method of the proof of Lemma 2.4 and the fact that a nonempty Zariski open subset of a complex linear space intersects its real form, we deduce that

(1) the pairs $(A^G, \{\eta^t\})_{t\in\mathbb{C}^2}$ and $(A^{G_r}, \{\eta_r^t\})_{t\in\mathbb{R}^2}$ are micro-Kronecker simultaneously;
(2) Propositions 3.6, 3.7 hold if we replace the complex pair (G, K) by the real one (G_r, K_r) and put $\mathbb{F} = \mathbb{R}$.

3.3. The bi-Poisson structure $\{\eta^t(\omega_{\mathcal{O}})\}$: reduction

We continue with the notation of Subsection 3.1 (in particular, $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$).

Theorem 3.9. *The pair $(A^G, \{\eta^t(\omega_{\mathcal{O}})\})$ on the cotangent bundle $T^*\mathcal{O}$ is micro-Kronecker.*

Proof. Let x_0 be some element from the set $R(\mathfrak{m})$ satisfying (3.10) and let $\mathfrak{k}^{x_0} = \{z \in \mathfrak{k} \mid [x_0, z] = 0\}$ be the isotropy subalgebra of the point x_0 for $\operatorname{Ad}(K)$-action. As in Subsection 2.1 we consider two reductive Lie algebras

$$\hat{\mathfrak{g}} = \{y \in \mathfrak{g} \mid [y, z] = 0,\ \forall z \in \mathfrak{k}^{x_0}\} \quad \text{and} \quad \hat{\mathfrak{k}} = \hat{\mathfrak{g}} \cap \mathfrak{k}.$$

Denote by $\hat{G}$ the connected closed Lie subgroup of G with the Lie algebra $\hat{\mathfrak{g}}$. It is clear that for each $g \in \hat{G}$ we have $\operatorname{Ad} g(z) = z$, $\forall z \in \mathfrak{k}^{x_0}$. Put $\hat{K} = \hat{G} \cap K$. Since $[a, \mathfrak{k}^{x_0}] = 0$, the element a belongs to the algebra $\hat{\mathfrak{k}} \subset \hat{\mathfrak{g}}$ and $x_0 \in \hat{\mathfrak{m}} \subset \hat{\mathfrak{g}}$ by definition.

Now consider the $\hat{G}$-orbit $\hat{\mathcal{O}} = \operatorname{Ad}(\hat{G}) \cdot a$ in $\hat{\mathfrak{g}} \subset \mathfrak{g}$. Then $\hat{\mathcal{O}} = \hat{G}/\hat{K}$. Using the form Φ_χ (restricted to the Lie subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$) for identifications, we obtain the following two closed 2-forms on $T\hat{\mathcal{O}}$: canonical $\hat{\Omega}$ and the pull-back $\hat{\tau}^*\hat{\omega}_{\mathcal{O}}$ of the Kirillov–Kostant–Souriau symplectic form on $\hat{\mathcal{O}}$. These forms determine the bi-Poisson structure $\hat{\eta}^t(\hat{\omega}_{\mathcal{O}})$ on $T\hat{\mathcal{O}}$.

Let $f_1, f_2 \in A_{\mathfrak{m}}^K$. Then $\operatorname{grad} f_j(x') \in \mathfrak{m}(x')$ for $x' \in R(\mathfrak{m})$. By Proposition 2.3, $\mathfrak{m}(x) = \hat{\mathfrak{m}}(x) \subset \hat{\mathfrak{m}}$ for all points x from the nonempty open set $\hat{\mathfrak{m}} \cap R(\mathfrak{m})$ (containing the point x_0). But $\operatorname{grad} f_j(x) \in \mathfrak{m}(x) \subset \hat{\mathfrak{m}}$. So $\operatorname{grad} f_j(x) = \operatorname{grad}_{\hat{\mathfrak{m}}} f_j(x)$ and by (3.5), for the brackets $\{\ ,\ \}^t_\eta$ and $\{\ ,\ \}^t_{\hat\eta}$ on the spaces $A^K_{\mathfrak{m}}$ and $A^{\hat K}_{\hat{\mathfrak{m}}}$ respectively, the following relation holds:

$$\{f_1, f_2\}^t_\eta(x) = \{f_1|\hat{\mathfrak{m}}, f_2|\hat{\mathfrak{m}}\}^t_{\hat\eta}(x) \quad \text{for all} \quad x \in \hat{\mathfrak{m}} \subset \mathfrak{m}. \tag{3.17}$$

Since $\operatorname{ddim} A^K_{\mathfrak{m}} = \operatorname{ddim}(A^K_{\mathfrak{m}}|\hat{\mathfrak{m}}) = \operatorname{ddim} A^{\hat K}_{\hat{\mathfrak{m}}} = \dim \mathfrak{m}(x)$ for $x \in \hat{\mathfrak{m}} \cap R(\mathfrak{m})$, these two brackets induce on $\mathfrak{m}(x) = \hat{\mathfrak{m}}(x)$ the same two-dimensional linear space of skew-symmetric bilinear forms $B^t_x : \mathfrak{m}(x) \times \mathfrak{m}(x) \to \mathbb{F}$ determined by (3.11). Since by (3.10) the set $\operatorname{Ad}(K)(\hat{\mathfrak{m}}) \supset \operatorname{Ad}(K)(\mathfrak{m}(x))$ contains some open subset of $\mathfrak{m}$, we obtain that

$$\text{the pair } (A^K_{\mathfrak{m}}, \{\eta^t\}) \text{ is micro-Kronecker iff so is the pair } (A^{\hat K}_{\hat{\mathfrak{m}}}, \{\hat\eta^t\}). \tag{3.18}$$

But by the latest assertion of Proposition 2.3 the centralizer $\hat{\mathfrak{k}}^x$ of $x \in \hat{\mathfrak{m}} \cap R(\mathfrak{m})$ is contained in the center $\mathfrak{z}(\hat{\mathfrak{g}})$ of the algebra $\hat{\mathfrak{g}}$. Moreover, $\hat{\mathfrak{k}}^x$ coincides with this center because the Lie algebra $\hat{\mathfrak{k}} = \hat{\mathfrak{g}}^a$ contains $\mathfrak{z}(\hat{\mathfrak{g}})$. Since $\operatorname{ad}_{\hat{\mathfrak{g}}}(\hat{\mathfrak{k}}^x) = 0$ is the Lie algebra of the isotropy group of the element $x \in \hat{\mathfrak{m}} = T_o(\hat G/\hat K)$ for the action of the semisimple Lie group $\operatorname{Ad}(\hat G)$ on $T\hat{\mathcal{O}} = T^*\hat{\mathcal{O}}$, from Theorem 1.10 it follows that the pair $(A^{\hat G}, \{\hat\eta^t\}) = (A^{\hat K}_{\hat{\mathfrak{m}}}, \{\hat\eta^t\})$ is micro-Kronecker if $\mathbb{F} = \mathbb{C}$. In the real case this pair is micro-Kronecker by Remark 3.8. □

3.4. Integrable geodesic flows

Here we will use notation of Subsection 3.1 but suppose that $\mathbb{F} = \mathbb{R}$. Consider the orbit $\mathcal{O} = \operatorname{Ad}(G) \cdot a \simeq G/K$ and a closed subgroup $K_1 \subset K$. The Lie algebra of K_1 contains maximal semisimple ideal of the Lie algebra $\mathfrak{k}$.

Theorem 3.10. *There exists a maximal involutive set of independent real analytic functions on $(T(G/K_1), \Omega_1)$. These functions are integrals for* 1) *the geodesic flow determined by the pseudo-Riemannian metric $\langle\ ,\ \rangle$ on G/K_1;* 2) *the Hamiltonian flow with the Hamiltonian function $H_{a,b,D}$ on $T(G/K_1)$.*

Proof. By Theorem 3.9 the pair $(A^G, \{\eta^t(\omega_{\mathcal{O}})\})$ is micro-Kronecker. By Propositions 3.6 and 3.7, there exists $m = \frac{1}{2}(r_1 + \operatorname{ddim} A^G_1)$ independent involutive functions from the set A^G_1. These functions form a maximal involutive subset of independent functions in the algebra $A^G_1 = A^{K_1}_{\mathfrak{m}_1}$ with respect to the canonical Poisson structure on $T(G/K_1)$. Moreover, these functions are integrals of 1) the geodesic flow on $T(G/K_1)$ determined by the form $\langle\ ,\ \rangle$; 2) the Hamiltonian flow with the Hamiltonian function $H_{a,b,D}$ on $T(G/K_1)$. Now the assertion of the theorem follows immediately from Proposition 1.8. □

Remark 3.11. Suppose that the group G is compact. Then the form $\langle\,,\rangle$ defines on G/K_1 Riemannian metric; in the set $\{\varphi_{a,b,D}, b \in \mathfrak{z} \subset \mathfrak{k}\}$, where the endomorphism $D : \mathfrak{z}_* \to \mathfrak{z}_*$ is positive-definite if $\mathfrak{z}_* \neq 0$, there is a dense subset of positive definite operators, i.e., the corresponding bilinear form $\langle\cdot, \varphi_{a,b,D}\cdot\rangle$ defines G-invariant Riemannian metric on G/K_1.

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