Veronese webs for bihamiltonian structures of higher corank

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To the memory of Stanislaw Zakrzewski,
with the respect and gratitude

0 Introduction.

A C^∞- manifold M is endowed by a Poisson pair if two linearly independent smooth bivectors c_1, c_2 are defined on M and $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. A bihamiltonian structure $J = \{c_\lambda\}$ is the whole 2-dimensional family of bivectors. The structure J is degenerate if $\operatorname{rank} c_\lambda < \dim\ M, \lambda \in \mathbb{R}^2$.

An intensive study of such objects was done by I.M.Gelfand and I.S.Zakharevich ([8], [9], [10]) in a particular case of bihamiltonian structures in general position on an odd-dimensional M (the corresponding Poisson pairs are necessarily degenerate: rank $c_\lambda = 2n, \lambda \in \mathbb{R}^2 \setminus \{0\}$, if $\dim M = 2n+1$). In [9] there was introduced a notion of a Veronese web, i.e. a 1-parameter family of 1-codimensional foliations such that the corresponding family of annihilators is represented by the Veronese curve in the cotangent space at each point. It turns out that Veronese webs form a complete system of local invariants for bihamiltonian structures of general position. More precisely, it was proved in [9] that any such structure $J = \{c_\lambda\}$ in $\mathbb{R}^{2n+1}$ admits a local reduction to a Veronese web $\mathcal{W}_J$ and that for any Veronese web $\mathcal{W}$ one can locally construct a bihamiltonian structure $J(\mathcal{W})$ of general position in $\mathbb{R}^{2n+1}$ with the reduction equal to $\mathcal{W}$. In the real analytic case J and $J(\mathcal{W}_J)$ are isomorphic.

The aim of this paper is to introduce a wider class of degenerate bihamiltonian structures that posess many features of the general position case and to generalize the notion of a Veronese web for this class. We call the bihamiltonian structures from this class complete since they are intemately connected with the completely integrable systems ([2]) on M. In particular, the Poisson pairs appearing in the well known method of argument translation (see [6], [7], and Example 1.11, below) generate complete bihamiltonian structures of higher (>1) corank.

The paper is organized as follows. In Section 1 we recall some definitions and facts about bihamiltonian structures and introduce the main definition of completeness. The last is based on one result of A.Brailov (Theorem 1.8). We show that complete bihamiltonian structures generalize the case of general position. Analyzing the corresponding Poisson pair $(c_1(x), c_2(x))$ at a point $x \in M$ we deduce that it consists of finite number of the so called Kronecker blocks (Corollary 1.14); the general position is characterized by the case of the sole block. Section 2 is devoted to distinguishing the invariants for the sum of k Kronecker blocks. In the next section we define local Veronese webs for complete bihamiltonian structures under some assumption of regularity. This last means that the number of Kronecker blocks does not change from point to point and the corresponding subspaces vary smoothly "sweeping" k subbundles in the tangent bundle. In general, these distributions are nonintegrable (Example 3.5); consequently, the bihamiltonian structure does not split to direct product of the bihamiltonian structures of corank 1, i.e. of general position. We conclude the paper calculating the Veronese web for the method of argument translation (Section 5). In the case of normal noncompact real form of complex semisimple Lie algebra this web is a product of flat Veronese webs of codimension 1.

Off course, the most interesting (open) question is the folowing. Does the Veronese web of complete bihamiltonian structure determine it up to isomorphism?

1 Bihamiltonian structures and completeness.

Let M be a C^∞- manifold. In the sequel, all considered Poisson bivectors will have maximal rank on an open dense subset in M. Given a Poisson bivector c, define $\operatorname{rank} c$ as $\max_{x\in M} \operatorname{rank} c(x)$.

1.1. Definition Two linearly independent Poisson bivectors c_1, c_2 on M form a *Poisson pair* if $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

1.2. Proposition *A pair of linearly independent Poisson bivectors (c_1, c_2) is Poisson if and only if $[c_1, c_2] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket.*

1.3. Definition A *bihamiltonian structure* on M is defined as a two-dimensional linear subspace $J = \{c_\lambda\}_{\lambda\in\mathcal{S}}$ of Poisson bivectors on M parametrized by a two-dimensional vector space $\mathcal{S}$ over $\mathbb{R}$. We say that J is *degenerate* if $\operatorname{rank} c_\lambda < \dim M$ for any $c_\lambda \in J$.

It is clear that every Poisson pair generates a bihamiltonian structure and the transition from the latter one to a Poisson pair corresponds to a choice of basis in $\mathcal{S}$. We shall write (J, c_1, c_2) for a bihamiltonian structure J with a chosen Poisson pair (c_1, c_2) generating J.

1.4. Definition Let J be a bihamiltonian structure. Introduce a *subfamily* $J_0 \subset J$ of Poisson bivectors of maximal rank R_0 (the set $J \setminus J_0$ is at most a finite sum of 1-dimensional subspaces), and a *set* of functions $\mathcal{F}_0 = \mathrm{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_c(M))$, where $Z_c(M)$ stands for the space of the Casimir functions of c on M. We take Span in order to obtain a vector space: a sum of two Casimir functions for different $c_1, c_2 \in J_0$ need not be a Casimir function.

The following proposition shows how the degenerate bihamiltonian structures can be applied for constructing the completely integrable systems.

1.5. Proposition *Let J be a degenerate bihamiltonian structure on M. A family $\mathcal{F}_0$ is involutive with respect to any $c_\lambda \in J$.*

Proof Let $c_1, c_2 \in J_0$ be linearly independent, $f_i \in Z_{c_i}, i = 1, 2$. Then

$$\{f_1, f_2\}_{c_\lambda} = (\lambda_1 c_1(f_1) + \lambda_2 c_2(f_1))f_2 = -\lambda_2 c_2(f_2) f_1 = 0. \tag{1.5.1}$$

Now it remains to prove that for any $c \in J_0, f_i \in Z_c, i = 1, 2$, one has $\{f_1, f_2\}_{c_\lambda} = 0$. For that purpose we first rewrite (1.5.1) as

$$c_\lambda(x)(\phi_1, \phi_2) = 0, \tag{1.5.2}$$

where $\phi_i \in \ker c_i(x),\ i = 1, 2,\ x \in M$, and the lefthandside denotes the contraction of the bivector with two covectors. Second, we fix x such that $\mathrm{rank}\, c(x) = R_0$ and approximate $df_2|_x$ by a sequence of elements $\{\phi^i\}_{i=1}^\infty,\ \phi^i \in \ker c^i(x)$, where $c^i \in J_0,\ i = 1, 2, \ldots$, is linearly independent with c. Finally, by (1.5.2) we get $c_\lambda(x)(df_1|_x, \phi^i) = 0$ and by the continuity $\{f_1, f_2\}_{c_\lambda}(x) = 0$. Since the set of such points x is dense in M, the proof is finished. q.e.d.

In fact this proposition is true for the local Casimir functions (for the germs of Casimir functions). The corresponding family of functions (germs) $\mathrm{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_c(U))$ ($\mathrm{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_{c,x})$ is denoted by $\mathcal{F}_0(U)$ ($\mathcal{F}_{0,x}$).

In order to obtain a completely integrable system from Casimir functions one should require additional assumptions on the bihamiltonian structure J. Off course, the condition of completeness given below concerns the local Casimir functions (in fact their germs) and may be insufficient for obtaining the completely integrable system. However, it is of use if the local Casimir functions are restrictions of the global ones (see Example 1.11, below).

Given a Poisson bivector $c_\lambda \in J$, let $S_\lambda(x)$ denote the symplectic leaf of c_λ through a point $x \in M$.

1.6. Definition ([3]) Let J be a bihamiltonian structure; fix some $c_\lambda \in J$.

J is called *complete at a point $x \in M$ with respect to c_λ* if the linear subspace of $T_x^* M$ generated by the differentials of the germs $f \in \mathcal{F}_{0,x}$ restricted to $S_\lambda(x)$ has dimension $\frac{1}{2} \dim S_\lambda(x)$.

1.7. Proposition *A bihamiltonian structure J is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $S_\lambda(x)$ is of maximal dimension if and only if* $\dim(\bigcap_{c_\lambda \in J_0} T_x S_\lambda(x))$ $= \frac{1}{2}\dim S_\lambda(x)$.

The following theorem is due to A.Brailov (see [3], Theorem 1.1 and Remark after it).

1.8. Theorem *A bihamiltonian structure (J, c_1, c_2) is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $S_\lambda(x)$ is of maximal dimension if and only if the following condition holds*

$(*)$ $\operatorname{rank}(\lambda_1 c_1 + \lambda_2 c_2)(x) = R_0$ *for any* $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{0\}$.

Here the bivector $c_\lambda = (\lambda_1 c_1 + \lambda_2 c_2)(x)$ is regarded as an element of $\bigwedge^2 T_x^{\mathbb{C}} M$, where $T^{\mathbb{C}}M$ is the complexified tangent bundle, and its rank is defined as that of the associated sharp map $c_\lambda^\sharp(x) : (T_x^{\mathbb{C}}M)^* \longrightarrow T_x^{\mathbb{C}}M$.

The theorem shows that J is complete with respect to a fixed $c_\lambda \in J_0$ at a point x such that the dimension $S_\lambda(x)$ is maximal if and only if $J = J_0 \bigcup \{0\}$ and J is complete at x with respect to any nontrivial $c_\lambda \in J$. This motivates the next definition.

1.9. Definition Let (J, c_1, c_2) be a bihamiltonian structure. The structure J (the pair (c_1, c_2)) is *complete at a point* $x \in M$ if condition $(*)$ of Theorem 1.8 holds at x. J ((c_1, c_2)) is called *complete* if it is so at any point from some open and dense subset in M.

1.10. Proposition *Let J be complete on M and let $x \in M$ be a point of completeness. Then there exists a neighbourhood $U \ni x$ such that the foliation $\mathcal{L}$ defined on U by $\mathcal{F}_0(U)$ is lagrangian in any $S_\lambda(y), \lambda \neq 0, y \in U$ (by Proposition 1.7 this foliation can be defined as the intersection of the foliations of symplectic leaves for $c_\lambda \in J_0$).*

1.11. Example (Method of argument translation, see [6], [3].) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual space. Fix a basis $\{e_1, \ldots, e_n\}$ in $\mathfrak{g}$ with the structure constants $\{c_{ij}^k\}$; write $\{e^1, \ldots, e^n\}$ for the dual basis in $\mathfrak{g}^*$. The standard linear Poisson bivector on $\mathfrak{g}^*$ is defined as

$$c_1(x) = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\{x_k\}$ are linear coordinates in $\mathfrak{g}^*$ corresponding to $\{e^1, \ldots, e^n\}$. In more invariant terms c_1 is described as an operator dual to the Lie-multiplication map $[\,,] : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$. It is well-known that the symplectic leaves of c_1 are the coadjoint orbits in $\mathfrak{g}^*$. Now define c_2 as a bivector with constant coefficients $c_2 = c(a)$, where a is a fixed point on any leaf of maximal dimension. It turns out that c_1, c_2 form a Poisson pair and it is easy to describe the set I of points x for which condition $(*)$ fails. Consider the complexification $(\mathfrak{g}^*)^{\mathbb{C}} \cong (\mathfrak{g}^{\mathbf{C}})^*$ and the sum $\operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^*$ of symplectic leaves of nonmaximal dimension for the complex linear bivector $c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, where $z_j = x_j + \mathrm{i} y_j$, $j = 1, \ldots, n$, are the corresponding complex coordinates in $(\mathfrak{g}^*)^{\mathbb{C}}$. Then I is equal to the intersection of the sets $\mathfrak{g}^* \subset (\mathfrak{g}^*)^{\mathbb{C}}$ and $\overline{a, \mathrm{Sing}(\mathfrak{g}^{\mathbb{C}})^*}$, where $\overline{a, \mathrm{Sing}(\mathfrak{g}^{\mathbb{C}})^*}$ denotes a cone of complex 2-dimensional subspaces passing through a and $\mathrm{Sing}(\mathfrak{g}^{\mathbb{C}})^*$.

In particular, (c_1, c_2) is complete for a semisimple $\mathfrak{g}$ ($\mathrm{codim}\, \mathrm{Sing}(\mathfrak{g}^{\mathbb{C}})^* \geq 3$, see [1], Corollary 4.42, and codimension of I in $\mathfrak{g}^*$ is not less than 2). Note, that this gives rise to completely integrable systems since the local Casimir functions on $\mathfrak{g}^*$ are restrictions of the global ones, i.e. the invariants of the coadjoint action.

1.12. Example (Bihamiltonian structure of general position on an odd-dimensional manifold, see [9].) Consider a pair of bivectors (a_1, a_2), $a_i \in \bigwedge^2 V, i = 1, 2$, where V is a $(2m+1)$-dimensional vector space; (a_1, a_2) is in general position if and only if is represented by the Kronecker block of dimension $2m+1$, i.e.

$$\begin{aligned} a_1 &= p_1 \wedge q_1 + p_2 \wedge q_2 + \cdots + p_m \wedge q_m \\ a_2 &= p_1 \wedge q_2 + p_2 \wedge q_3 + \cdots + p_m \wedge q_{m+1} \end{aligned} \tag{1.12.0}$$

in an appropriate basis $p_1, \ldots p_m, q_1, \ldots q_{m+1}$ of V. A bihamiltonian structure J on a $(2m+1)$-dimensional M is in general position if and only if the pair $(c_1(x), c_2(x))$ is so for any $x \in M$. Such J is complete. In general, a complete Poisson pair at a point is a direct sum of the Kronecker blocks and the zero pair as the corollary of the next theorem shows. This theorem is a reformulation of the classification result for pairs of 2-forms in a vector space ([8], [10]).

1.13. Theorem *Given a finite-dimensional vector space V over $\mathbb{C}$ and a pair of bivectros (c_1, c_2), $c_i \in \bigwedge^2 V$, there exists a direct decomposition $V = \oplus V_j$, $c_i = \sum c_i^{(j)}$, $c_i^{(j)} \in \bigwedge^2 V_j$, $i = 1, 2$, such that each triple $(V_j, c_1^{(j)}, c_2^{(j)})$ is from the following list:*

(a) *the zero block: $c_1^{(j)} = c_2^{(j)} = 0$;*

(b) *the Jordan block: $\dim V_j = 2n_j$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to*

$$\begin{pmatrix} 0 & A_i \\ -A_i^T & 0 \end{pmatrix}, i = 1, 2,$$

where $A_1 = I_{n_j}$ (the unity $n_j \times n_j$-matrix) and $A_2 = J_{n_j}^{\lambda}$ (the Jordan block with the eigenvalue λ);

(c) *the Kronecker block: $\dim V_j = 2n_j + 1$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to*

$$\begin{pmatrix} 0 & B_i \\ -B_i^T & 0 \end{pmatrix}, i = 1, 2,$$

where $B_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$ *($(n_j + 1) \times n_j$-matrices).*

1.14. Corollary *Let (J, c_1, c_2) be a bihamiltonian structure. It is complete at a point $x \in M$ if and only if the pair $(c_1(x), c_2(x))$, $c_i(x) \in \bigwedge^2(T_x^{\mathbb{C}} M), i = 1, 2$, does not contain the Jordan blocks in its decomposition.*

PROOF The statement follows from the definition of completeness. q.e.d.

2 Complete bihamiltonian structure at a point.

Now, we shall examine a linear bihamiltonian structure $(J, c_1, c_2), c_i \in \bigwedge^2 V$ such that the decomposition $V = \oplus_{j=0}^k V_j,\ c_i = \sum_{j=0}^k c_i^{(j)}$ from Theorem 1.13 consists of the zero block $V_0, \dim V_0 = n_0$ and k Kronecker blocks $V_1, \ldots, V_k, \dim V_j = 2n_j + 1$. The aim is to introduce the infinithesimal approximation to Verones webs (these last will be defined in the next section).

2.1. Definition ([9]) Let $\mathcal{S}, V$ be vector spaces of dimensions 2 and $n+1$ respectively. A *Veronese inclusion* of $\mathbb{P}(\mathcal{S})$ in $\mathbb{P}(V)$ is a map $i : \mathbb{P}(\mathcal{S}) \longrightarrow \mathbb{P}(V)$ such that there exists a linear isomorphism $\phi : \mathbb{P}(V) \longrightarrow \mathbb{P}(S^n\mathcal{S})$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{S}) & \stackrel{i}{\longrightarrow} & \mathbb{P}(V) \\ \| & & \uparrow \phi \\ \mathbb{P}(\mathcal{S}) & \stackrel{\mathbb{P}(S^n(\cdot))}{\longrightarrow} & \mathbb{P}(S^n\mathcal{S}). \end{array}$$

Here S^n denotes the n-th symmetric power; the standard model of the mapping $S^n(\cdot)$ is described as follows. Let $\mathcal{S}$ be a space of linear functions f in two variables t_1, t_2. Then $S^n\mathcal{S}$ is a space of homogeneous polynomials in t_1, t_2 and $S^n(f) = f^n$.

2.2. Theorem *([8],[9]) Let $\mathcal{V}_{\lambda,j} \subset V$ be the characteristic subspace (i.e. the symplectic leaf passing through 0) of the bivector $c_\lambda^{(j)} = \lambda_1 c_1^{(j)} + \lambda_2 c_2^{(j)}$.*

a) The folloving objects are defined invariantly (j=1,...,k):

(i) *the subspace $V_j \subset V$;*

(ii) *the intersection of the characteristic subspaces $L_j = \bigcap_{\lambda \neq 0} \mathcal{V}_{\lambda,j} \subset V_j$ and its annihilator $W^j = L_j^{\perp_j} = \mathrm{Span}_{\mathbb{R}}(\bigcup_{\lambda \neq 0} \ker c_\lambda^{(j)}) \subset V_j^*$;*

(iii) *the spaces $W_j = V_j/L_j$ and $\mathcal{W}_{\lambda,j} = \mathcal{V}_{\lambda,j}/L_j \subset W_j$ $(\dim W_j/\mathcal{W}_{\lambda,j} = 1)$;*

(iv) *the map $\mathbb{R}^2 \setminus \{0\} \ni \lambda \stackrel{\phi_j}{\mapsto} \ker c_\lambda^{(j)} \subset W^j$ (under the canonical identification $W_j^* \cong W^j$ the map ϕ_j can be written also as $\lambda \stackrel{\phi_j}{\mapsto} \mathcal{W}_{\lambda,j}^{\perp_j} \subset W_j^*$).*

b) The projectivization $\mathbb{P}(\phi_j) : \mathbb{P}(\mathbb{R}^2) \cong \mathbb{P}(\mathcal{S}) \longrightarrow \mathbb{P}(W^j)$ is a Veronese inclusion, $j = 1, \ldots, k$.

2.3. Remark In fact all objects mentioned in Theorem 2.2 are invariants of the bihamiltonian structure J itself, i.e. they do not depend on the choice of the generating Poisson pair (c_1, c_2).

2.4. Definition An *infinithesimal Veronese web* of type $(n_0, n_1, \ldots, n_k)$ on a vector space $W, \dim W = n_0 + n_1 + \cdots + n_k + k$, is a 1-parameter family $\{\mathcal{W}_\lambda\}_{\lambda \in \mathbb{P}(\mathcal{S})}$ of linear subspaces $\mathcal{W}_\lambda \subset W, \operatorname{codim} \mathcal{W}_\lambda = n_0 + k$, satisfying the following conditions:

(i) there is a distinguished subspace $W^0 \subset W^*, \dim W^0 = n_0$, contained in any annihilator $\mathcal{W}_\lambda^\perp \subset W^*, \lambda \in \mathbb{P}(\mathcal{S})$;

(ii) there is a direct decomposition $W_K = \oplus_{j=1}^k W_j, \dim W_j = n_j + 1$ of the subspace $W_K = (W^0)^\perp \subset W$; the corresponding decomposition $\mathcal{W}_\lambda = \oplus_{j=1}^k \mathcal{W}_{\lambda,j}, \lambda \in \mathbb{P}(\mathcal{S})$, of the subspace $\mathcal{W}_\lambda \subset W_K$ is such that codimension of $\mathcal{W}_{\lambda,j}$ in W_j is 1 for $j = 1, \ldots, k$;

(iii) if $W^*/W^0 = \oplus_{j=1}^k W^j, W^j \cong W_j^*$, is the corresponding decomposition of the space $(W_K)^* \cong W^*/W^0$, then the map $\mathbb{P}(\mathcal{S}) \ni \lambda \overset{\psi_j}{\mapsto} \mathcal{W}_{\lambda,j}^{\perp_j} \in \mathbb{P}(W^j)$, where $\mathcal{W}_{\lambda,j}^{\perp_j}$ denotes the (1-dimensional) annihilator of $\mathcal{W}_{\lambda,j} \subset W_j$ in W^j, is a Veronese inclusion for $j = 1, \ldots, k$.

2.5. Proposition *Let (J, c_1, c_2) be as above. Then the vector space $W = V/L$, where $L = \oplus_{j=1}^k L_j$ (see Theorem 2.2), has a structure of an infinithesimal Veronese web of type $(n_0, n_1, \ldots, n_k)$.*

(Note that $L = \bigcap_{\lambda \neq 0} \mathcal{V}_\lambda \subset V$, where $\mathcal{V}_\lambda = \oplus_{j=1}^k \mathcal{V}_{\lambda,j}$ is the characteristic subspace of the bivector c_λ.)

Proof Set $V^0 = (\oplus_{j=1}^k V_j)^\perp \subset V^*$, and $W_K = \oplus_{j=1}^k W_j$ (W_j taken from condition (iii) of Theorem 2.2). We first make the following two remarks: 1) W_K can be regarded as a subspace of $W = V/L$; 2) if $W^0 = (W_K)^\perp \subset W^*$, then $V^0 = W^0 \subset W^*$ under the canonical identification $(V/L)^* \cong L^\perp \subset V^*$. Now the proof follows from Theorem 2.2.q.e.d.

2.6. Remark Let $\mathcal{W} = (W, \{\mathcal{W}_\lambda\}_{\lambda \in \mathbb{P}(\mathbb{R}^2)})$ be an infinithesimal Veronese web of type $(n_0, n_1, \ldots, n_k)$. Using the construction of Gelfand and Zakharevich ([9],p.165-166) to each component $(W_j, \{(\mathcal{W}_\lambda)_j\}), j = 1, \ldots, k$, one can obtain a linear bihamiltonian structure $J(\mathcal{W})$ on some vector space $V(\mathcal{W})$ of dimension $n_0 + 2n_1 + \cdots + 2n_k + k$. If $\mathcal{W} = \mathcal{W}_J$ is associated to a priori defined linear bihamiltonian structure J, then $J(\mathcal{W}_J)$ and J are linearly isomorphic.

3 Regular bihamiltonian structures and their Veronese webs.

In this section we shall define objects that generalize the Veronese webs introduced in [9] for the bihamiltonian structures of general posiiton. We shall show that any complete bihamiltonian structure satisfying some additional (cf. Example 3.7) conditions of regularity admits the local reduction to such an object.

3.1. Definition Let J be a complete bihamiltonian structure on M. A *type* of J at $x \in M$ is the vector $(n_0, n_1, \ldots, n_k)(x)$, where $n_0(x)$ is dimension of the zero block and $2n_1(x)+1, \ldots, 2n_k(x)+1$ are dimensions of the Kronecker blocks in the decomposition of $(c_1(x), c_2(x))$ for some generating J Poisson pair (c_1, c_2) (these dimensions do not depend on this pair, see 2.3). If this vector is independent of x we call it a type of J and say that J is *regular*.

3.2. Definition Consider a manifold U diffeomorphic to an open set in $\mathbb{R}^N$, where $N = n_0+(n_1+1)+\cdots+(n_k+1)$, and a family $\mathcal{W} = \{\mathcal{W}_\lambda\}_{\lambda\in\mathbb{P}(\mathcal{S})}$ of (n_0+k)-codimensional foliations on U parametrized by the projectivizaton of a two-dimensional vector space $\mathcal{S}$. We call $\mathcal{W}$ a *Veronese web* of type $(n_0, n_1, \ldots, n_k)$ if the following conditions are satisfied:

(i) there is a distinguished subbundle $W^0 \subset T^*U, \operatorname{rank} W^0 = n_0$, such that $W^0 \subset \mathcal{W}_\lambda^\perp$ for any $\lambda \in \mathbb{P}(\mathcal{S})$, where $\mathcal{W}_\lambda^\perp \subset T^*U$ is the subbundle annihilating $T\mathcal{W}_\lambda$;

(ii) there is a bundle decomposition $W_K = \oplus_{j=1}^k W_j, \operatorname{rank} W_j = n_j + 1$, of the subbundle $W_K = (W^0)^\perp \subset TU$; the corresponding bundle decomposition $T\mathcal{W}_\lambda = \oplus_{j=1}^k (T\mathcal{W}_\lambda)_j$ of the subbundle $T\mathcal{W}_\lambda \subset W_K$ is such that the fiber codimension of $(T\mathcal{W}_\lambda)_j$ in W_j is 1;

(iii) if $T^*U/W^0 = \oplus_{j=1}^k W^j, W^j \cong W_j^*$, is the corresponding decomposition of the bundle $(W_K)^* \cong T^*U/W^0$, then the map $\mathbb{P}(\mathcal{S}) \ni \lambda \overset{\psi_{j,x}}{\mapsto} (T_x\mathcal{W}_\lambda)_j^{\perp j} \in \mathbb{P}(W_x^j)$, where $(T_x\mathcal{W}_\lambda)_j^{\perp j}$ denotes (the 1-dimensional) annihilator of $(T_x\mathcal{W}_\lambda)_j \subset W_{j,x}$ in W_x^j, is a Veronese inclusion for any $j = 1, \ldots, k$ and $x \in U$.

In analogy with 1.13 we say that W^0 is the zero block and $W_j, j = 1, \ldots, k$, are the Kronecker blocks.

3.3. Theorem *Let J be a regular bihamiltonian structure of type $\mathbf{n} = (n_0, n_1, \ldots, n_k)$ and let $x \in M$ be a point of completeness for J. Write $\mathcal{V}_\lambda$ for the foliation of symplectic leaves of $c_\lambda \in J$. Then there exists a neighbourhood $\tilde{U} \ni x$ such that $U = \tilde{U}/\mathcal{L}$ (see 1.10) is diffeommorphic to an open set in $\mathbb{R}^N$ and $\{\mathcal{V}_\lambda|_{\tilde{U}}/\mathcal{L}\}_{\lambda\in\mathbb{P}(\mathcal{S})}$ is a Veronese web of type $\mathbf{n}$ on U.*

Proof The theorem follows from Proposition 2.5.q.e.d.

3.4. Example Let $U = \mathbb{R}^3$ with coordinates (x, y, z), $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, T_u\mathcal{W}_\lambda = \operatorname{Span}_{\mathbb{R}}\{(\lambda_1 v_1 + \lambda_2 v_2)(u)\}, u \in U$. Then $\Gamma(W^0) = \operatorname{Span}_{C^\infty(U)}\{xdy - dz\}$, where Γ stands for the space of sections, $W_K = W_1 \subset TU$ is the nonintegrable distribution generated by v_1, v_2. On $\tilde{U} = U \times \mathbb{R}$ one defines the corresponding bihamiltonian structure as $\{\frac{\partial}{\partial p} \wedge (\lambda_1 v_1 + \lambda_2 v_2)\}_{(\lambda_1,\lambda_2)\in\mathbb{R}^2}$, where p is a coordinate on $\mathbb{R}$.

3.5. Example Let $U=\mathbb{R}^4$ with coordinates (x,y,z,t), $v_1=\frac{\partial}{\partial x}, v_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}, v_3=\frac{\partial}{\partial z}, v_4=\frac{\partial}{\partial t}, T_u\mathcal{W}_\lambda=\mathrm{Span}_{\mathbb{R}}\{(\lambda_1v_1+\lambda_2v_2)(u),(\lambda_1v_3+\lambda_2v_4)(u)\}, u\in U$. Then $W^0=0, W_K=W_1\oplus W_2, \Gamma(W_1)=\mathrm{Span}_{C^\infty(U)}\{v_1,v_2\}, \Gamma(W_2)=\mathrm{Span}_{C^\infty(U)}\{v_3,v_4\}$. On $\tilde{U}=U\times\mathbb{R}^2$ the corresponding bihamiltonian structure is $\{\frac{\partial}{\partial p_1}\wedge(\lambda_1v_1+\lambda_2v_2)+\frac{\partial}{\partial p_2}\wedge(\lambda_1v_3+\lambda_2v_4)\}_{(\lambda_1,\lambda_2)\in\mathbb{R}^2}$, where p_1,p_2 are coordinates on $\mathbb{R}^2$.

3.6. Definition In the case when the zero block $W^0\subset T^*U$ is trivial a Veronese web admits the following local description. One has $TU=\oplus_{j=1}^k W_j, T^*U=\oplus_{j=1}^k W^j$, and one can choose linear coordinates (λ_1,λ_2) on $\mathcal{S}$ and a local coframe $\alpha_1^1,\ldots,\alpha_{n_1+1}^1$, $\ldots,\alpha_1^k,\ldots,\alpha_{n_k+1}^k,\alpha_1^j,\ldots,\alpha_{n_j+1}^j\in\Gamma(W^j)$ such that the annihilator $(T\mathcal{W}_\lambda)^\perp\subset T^*U$ is generated by $\alpha_\lambda^1,\ldots,\alpha_\lambda^k$, where $\alpha_\lambda^j=\lambda_1^{n_j}\alpha_1^j+\lambda_1^{n_j-1}\lambda_2\alpha_2^j+\cdots+\lambda_2^{n_j}\alpha_{n_k+1}^j$ (Veronese curve). If in a neighbourhood of any $x\in U$ there exists a holonomic coframe satisfying the above properties, the Veronese web is called *flat* (such a web splits to a direct product of flat Veronese webs of codimension 1).

In the general case a Veronese web $\{\mathcal{W}_\lambda\}$ of type $(n_0,n_1,\ldots,n_k)$ is called *flat* if there exists a foliation $\mathcal{C}$ of codimension n_0 on M such that $\{\mathcal{W}_\lambda\}$ can be restricted to some Veronese web on each leaf of $\mathcal{C}$ and the restriction is flat in the above sense.

The webs from Examples 3.4, 3.5 do not split to a direct product of 1-codimensional Veronese webs; in particular, they are not flat.

We conclude the section by an example of a complete bihamiltonian structure that is not regular.

3.7. Example Let $M=\mathbb{R}^6$ with coordinates $(p_1,p_2,q_1,\ldots,q_4)$, $c_1=\frac{\partial}{\partial p_1}\wedge\frac{\partial}{\partial q_1}+\frac{\partial}{\partial p_2}\wedge\frac{\partial}{\partial q_2}, c_2=\frac{\partial}{\partial p_1}\wedge(\frac{\partial}{\partial q_2}+q_1\frac{\partial}{\partial q_3})+\frac{\partial}{\partial p_2}\wedge\frac{\partial}{\partial q_4}$. Here we have: two 3-dimensional Kronecker blocks on $M\setminus H, H=\{q_1=0\}$; the 5-dimensional Kronecker block and the 1-dimensional zero block on the hyperplane H.

4 Veronese webs for the argument translation method.

The notations from Subsection 1.11 will be used below. We consider normal (déployable in terminology of Bourbaki, [5], IX,3) real form $\mathfrak{g}$ of complex simple Lie algebra. The generalization to the semisimple case is straightforward. Let $m_1,\ldots,m_r, r=\mathrm{rank}(\mathfrak{g})$ be the exponents of $\mathfrak{g}$.

4.1. Theorem *Let (c_1,c_2) be the Poisson pair from Example 1.11. Then the Veronese web $\{\mathcal{W}_\lambda\}_{\lambda\in\mathbb{R}^2}$ of the corresponding bihamiltonian structure J is of type $(0,m_1,\ldots,m_r)$ and is flat (see 3.6).*

Proof Let $g_1(x),\ldots,g_r(x), \deg g_j=m_j+1$, be a set of algebraically independent global homogeneous polynomial Casimir functions for c (see [4], VIII,8). Here we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form. Note that $g_1,\ldots,g_r$ are functionally independent on $\mathfrak{g}\setminus\mathrm{Sing}\,\mathfrak{g}$, where $\mathrm{Sing}\,\mathfrak{g}$ is the set of adjoint orbits of nonmaximal dimension. Indeed, their restrictions to a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ are algebraically independent and invariant with respect to the Weyl group W. Now, we can apply the result of R.Steinberg ([12]) to deduce the nondegeneracy for the Jacobi matrix of $g_1|_{\mathfrak{h}}, \dots, g_r|_{\mathfrak{h}}$ at a regular point.

Consider the subspace $d\mathcal{F}_0 \subset \Gamma(T^*\mathfrak{g}^*)$ generated by the differentials of functions from the involutive set $\mathcal{F}_0$ (see 1.4) corresponding to J. It turns out that $d\mathcal{F}_0$ is generated by $\{dg_j|_{\lambda_1 x+\lambda_2 a}, (\lambda_1, \lambda_2) \in \mathbb{R}^2, j = 1, \dots, r\}$. If $g_j^i(a, x), i = 0, \dots, m_j+1, j = 1, \dots, r$, are the coefficients of the Taylor expansions $g_j(x + \lambda a), j = 1, \dots, r$, with respect to $\lambda \in \mathbb{R}$, then one also has

$$d\mathcal{F}_0 = \mathrm{Span}\{dg_j^i(a, x), i = 0, \dots, m_j, j = 1, \dots, r\}. \tag{4.1.1}$$

Moreover, these differentials are linearly independent at any $x \in \mathfrak{g}^* \setminus I$. This follows from the fact that J is complete at $A \setminus I$, from (4.1.1), and from the formula $\sum_{j=1}^r m_j = \frac{1}{2}(\dim \mathfrak{g} - r)$ (cf. [11], formula (F1), p. 289).

Thus, we can regard $g_j^i(a, x), i = 0, \dots, m_j, j = 1, \dots, r$ as coordinates on the reduced space $(\mathfrak{g}^* \setminus I)/\mathcal{L}$ (see Theorem 3.3). Finally, $(T\mathcal{W}_\lambda)^\perp, \lambda = (\lambda_1, \lambda_2)$, is generated by

$$\lambda_1^{m_j} dg_j^0(a, x) + \lambda_1^{m_j - 1}\lambda_2 dg_j^1(a, x) + \dots + \lambda_2^{m_j} dg_j^{m_j}(a, x), j = 1, \dots, r.$$

q.e.d.

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