Varieties of representations of finitely generated groups

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Abstract

Varieties of low-dimensional representations of some finitely generated groups were considered and some theorems about the number of irreducible components were proved. As auxiliary results, dimensions of commutator varieties for matrices of dimensions 2, 3 and 4 were calculated.

1. Introduction

Let G be a finitely generated group. The set $\mathrm{Hom}\,(G, GL_n(\mathbb{C}))$ of all homomorphisms from G to $GL_n(\mathbb{C})$ (i. e. the space of all n-dimensional representations of G) is known to have a natural structure of an affine algebraic variety, and endowed with this structure it is called the *variety of representations* of G in $GL_n(\mathbb{C})$ and is denoted by $R_n(G)$ (see [6]).

In [1] the decompositon into irreducible components of the varieties of representations of the groups with presentations

$$\Gamma_g = \langle\, x_1, \dots, x_s, t_1, \dots, t_k, y_1, z_1, \dots, y_g, z_g \mid x_1^{m_1} = \dots = x_s^{m_s} = [y_1, z_1] \dots [y_g, z_g] W(x_1, \dots, x_s, t_1, \dots, t_k) = 1\,\rangle,$$

where $g \geq 2$, $m_i = 0$ or $m_i \geq 2$ for $i = 1, \dots, s$, and $W(x_1, \dots, x_s, t_1, \dots, t_k)$ is a cyclically reduced word in the free group F freely generated by $x_1, \dots, x_s, t_1, \dots, t_k$, were considered. It follows from [9] that the main results of [1] **cannot** be directly generalized to the case $g = 1$. Therefore we consider low-dimensional (for dimensions $n = 2, 3, 4$) representations of the groups

$$G = \langle\, x, y, z_1, \dots, z_m \mid [x, y] = W(z_1, \dots, z_m)\,\rangle,$$

where the word $W(z_1, \dots, z_m)$ does not belong to the commutator **subgroup** of the free group F freely generated by $z_1, \dots, z_m$.

The main results of this work are theorems 16 and 17 proved in section 6 and results about dimensions of commutator varieties

$$W(A) = \{\, (x, y) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \mid [x, y] = A\,\},$$

for all matrices $A \in SL_n(\mathbb{C})$ for $n = 2, 3, 4$.

2. Required facts and definitions

Definition 1. [3] Let K be a field, and let $f_1, \dots, f_s$ be polynomials in $K[x_1, \dots, x_n]$. Then we set

$$X = \{(a_1, \dots, a_n) \in K^n | f_i(a_1, \dots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

We call X the *affine variety* (defined by $f_1, \dots, f_s$).

Definition 2. [3] An affine variety X is called *irreducible* if whenever X is written in the form $X = X_1 \cup X_2$, where X_1 and X_2 are affine varieties, then either $X_1 = X$ or $X_2 = X$.

Lemma 1. [3] *In algebraically closed field an affine variety, defined by a single polynomial, is irreducible if and only if the polynomial is irreducible.*

Theorem 1. [5] *Let X be an affine variety. Then the following conditions are equivalent:*

1. *X is irreducible.*
2. *Any two nonempty open sets in X have nonempty intersection.*
3. *Any nonempty open set is dense in X.*

Definition 3. The affine n-space is topologizing by the next way: the closed sets are to be precisely the affine varieties. This is called the *Zariski topology*[5].

Let G be a finitely generated group.

Definition 4. [8] A *linear representation* of G in n-dimesional vector space V over the field K is a homomorphism ρ from the group G into the group $GL(V)$ (the group of authomorphisms of V).

If there is the basis in V, then $GL(V)$ identifies with $GL_n(K)$.

We say that n is the *degree of the representation* under consideration.

If $G = \langle x_1, \dots x_d | r_q, q \in Q \rangle$, then an n-dimensional representation $\rho \in \mathrm{Hom}\,(G)$ of G is determined by the d-tuple of matrices $(\rho(x_1), \dots, \rho(x_d))$. This defines an embedding $\mathrm{Hom}(G, GL_n(K)) \to GL_n(K)^d$. It is easy to see that the image is an affine algebraic variety $R_n(G)$, whose geometric structure is independent of the presentation chosen for G [6].

Definition 5. [5] With an irreducible variety X is associated its field $K(X)$ of rational functions. As a finitely generated field extension of K, $K(X)$ has finite transcendence degree over K. This number is called the *dimension of* X, written $\dim X$.

In case X has more than one irreducible component, $X = X_1 \cup \dots \cup X_r$, $\dim X = \max(\dim X_i)$.

Theorem 2. [5] *Let X, Y be irreducible varieties of respective dimension m, n. Then* $\dim(X \times Y) = m + n$.

Lemma 2. [5] *Let X be an irreducible variety, Y closed, irreducible subset. Then* $\dim Y \leq$ *dim$X$.*

Definition 6. [3] Let $X \in K^m$ and $Y \in K^n$ be varieties. A function $\phi : X \to Y$ is said to be a *regular mapping*, if there exist polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$ such that

$$\phi(a_1, \ldots, a_m) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in X$.

Theorem 3 (on the dimension of fibres)**.** [7] *If $f : X \to Y$ is a regular mapping of irreducible varieties, $f(X) = Y$, $\dim X = n$, $\dim Y = m$, then $m \leq n$ and*

1. $\dim f^{-1}(y) \geq n - m$ *for every point $y \in Y$.*
2. *in Y there exists a nonempty open set U such that* $\dim f^{-1}(y) = n - m$ *for $y \in U$.*

Theorem 4 (on the dimension of the intersection with a hypersurface)**.** [7] *If X is an irreducible algebraic variety of dimension n, f is a polynomial that does not vanish identically on X, and*

$$X_f = \{(a_1, \ldots, a_n) \in X | f(a_1, \ldots, a_n) = 0\}$$

is nonempty, then each of its components has dimension $(n-1)$.

Theorem 5. [7] *Let X and Y be irreducible affine varieties. If $f : X \to Y$ is a regular mapping and $f(X)$ is dense in Y, then $f(X)$ contains a set that is open in Y.*

Theorem 6. *Let X be an irreducible affine variety. If $f : X \to Y$ is a regular mapping and $f(X)$ is dense in Y, then Y is an irreducible variety.*

Proof. Proof by contradiction. Let Y be a reducible variety. By theorem 1 exists nonempty, open in Y subsets U_1 and U_2, such that $U_1 \cap U_2 = \emptyset$. Since $f(X)$ is dense in Y $f^{-1}(U_1) \neq \emptyset$ and $f^{-1}(U_2) \neq \emptyset$, but $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open in X, as prototypes of open subsets, and $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$. So X is reducible. There is a contradiction. □

Definition 7. [9] We will call the matrix $A \in GL_n(\mathbb{C})$ *regular*, if in its Jordan normal form every eigenvalue corresponds to a single Jordan block.

Definition 8. [5] We will call the matrix $A \in GL_n(\mathbb{C})$ *semisimple*, if it is diaganolizable.

Theorem 7. [9] *The set of all regular matrices is open in $GL_n(\mathbb{C})$.*

Theorem 8. *The set of all regular semisimple matrices is open in $GL_n(\mathbb{C})$.*

Proof. It is easy to see that regular semisimple matrices are matrices whose characteristic polynomial $f(\lambda)$ hasn't multiple roots. So rest of the matrices are the matrices for which $\mathrm{Res}(f, f'_\lambda) = 0$, where $\mathrm{Res}(f, f'_\lambda)$ is a resultant of polynomials f and f'_λ. □

Theorem 9. [9] *The commutator variety*

$$W(A) = \{(x, y) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) | [x, y] = A\}$$

is nonempty for every matrix $A \in SL_n(\mathbb{C})$.

Theorem 10. [9] *Let* $A \in SL_n(\mathbb{C})$,

$$W(A) = \{(x, y) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) | [x, y] = A\}$$

be the corresponding commutator variety. Then any $W_1 \subset W(A)$ *contains a point* (x, y) *such that both* x *and* y *are regular matrices.*

Theorem 11. [9] *For any* $A \in SL_n(\mathbb{C})$ *the dimension of any irreducible component* W_1 *of the commutator variety* $W(A)$ *is between* $(n^2 + 1)$ *and* $(n^2 + n)$.

Theorem 12. [9] *There exists an open set* $U \subset SL_n(\mathbb{C})$ *such that for any matrix* $A \in U$ *the commutator variety* $W(A)$ *is an irreducible variety of dimension* $n^2 + 1$.

Theorem 13. [9] *The commutator variety* $W(E_n)$, *where* E_n *is an identity* $n \times n$ *matrix, is an irreducible variety of dimension* $(n^2 + n)$.

Definition 9. [4] If Jordan normal form of matrix A is $diag(J_{k_1}(\alpha_1), \ldots, J_{k_l}(\alpha_l))$, $k_1 + \cdots + k_l = n$, then polynomials $(\lambda - \alpha_1)^{k_1}, \ldots, (\lambda - -\alpha_l)^{k_l}$ are called *elementary divisors* of the matrix A.

Theorem 14. [4] *Let* $(\lambda - \alpha_1)^{p_1}, \ldots, (\lambda - \alpha_u)^{p_u}$ *be elementary divisors of the matrix* A *and* $(\lambda - \mu_1)^{q_1}, \ldots, (\lambda - \mu_v)^{q_v}$ *be elementary divisors of the matrix* B.*Then the dimension of the space of solutions of the equation* $AX = XB$ *is determained by the formula*

$$\sum_{i=1}^{u} \sum_{j=1}^{v} \delta_{ij},$$

where δ_{ij} *denotes the degree of the greatest common divisor of* $(\lambda - \alpha_i)^{p_i}$ *and* $(\lambda - \mu_j)^{q_j}$.

For the matrix $A \in GL_n(\mathbb{C})$ we will write the characteristic polynomial in the form:

$$\det(A - \lambda E) = \lambda^n + \sigma_1(A)\lambda^{n-1} + \cdots + \sigma_n(A).$$

Lemma 3. [9] *The coefficient* $\sigma_t(A)$ *is equal (up to sign) to the sum of all principal minors of* A *of order* t.

Let us consider the following variety

$$T(A) = \{y \in GL_n(\mathbb{C}) | \sigma_i(Ay) - \sigma_i(y) = 0, i = 1, \ldots, n-1\}, A \in SL_n(\mathbb{C}),$$

and introduce the following projection:

$$\pi : W(A) \to T(A) : (x, y) \to y.$$

Lemma 4. *The projection* π *is a regular mapping.*

3. Calculation of dimensions of varieties $W(A)$, $A \in SL_2(\mathbb{C})$

In this section we calculate dimensions of commutator varieties

$$W(A) = \{(x, y) \in GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) | [x, y] = A\}, A \in SL_2(\mathbb{C}).$$

Let B is a Jordan normal form of A. Then $W(B)$ is isomorphic to $W(A)$. So we will assume that A is a Jordan matrix. There are two kind of the matrix A:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1\lambda_2 = 1 \text{ and } \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \lambda_1^2 = 1.$$

1. In the first case
$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1\lambda_2 = 1,$$
the variety $T(A)$ is defined by the polynomial $y_{11} + y_{22} - \lambda_1 y_{11} - \lambda_2 y_{22}$. This polynomial is identically zero if and only if $A = E_2$. Hence $\dim T(E_2) = 4$. In the other cases by the theorem on the dimension of the intersection with a hypersurface, $\dim T(A) = 3$.

2. In the second case
$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \lambda_1 = \pm 1,$$
the variety $T(A)$ is defined by the polynomial $y_{11} + y_{22} - \lambda_1 y_{11} - y_{21} - -\lambda_1 y_{22}$. This polynomial isn't identically zero, hence by the theorem on the dimension of the intersection with a hypersurface, $\dim T(A) = 3$.

Note that the variety $T(A)$ is irreducible for each A (since $A = E_2$, $T(A) = GL_2(\mathbb{C})$ and if $A \neq E_2$ then the variety $T(A)$ is defined by a single linear equation, and we can apply lemma 1).

Let $W(A) = W_1 \cup W_2 \cdots \cup W_k$ is the decomposition of the commutator variety into irreducible components. The $\pi(W_i) \subset T(A)$, so $\dim \overline{\pi(W_i)} \leq \dim T(A)$ by lemma 2. By theorem 14 $\dim \pi^{-1}(y) = 2$, if y is a regular matrix.

By theorem 10 the set $\pi(W_i)$ contains a regular matrix y_i for each i. Hence by the theorem on the dimension of fibres,

$$\dim W(A) = \max_i\{\dim W_i\} \leq \max_i\{\dim \overline{\pi(W_i)} + \dim \pi^{-1}(y_i)\} \leq \dim T(A) + 2.$$

Applying theorems 11 and 13 we obtain $\dim W(E_2) = 6$, $\dim W(A) = 5$ for each $A \neq E_2$, $A \in SL_2(\mathbb{C})$.

4. Calculation of dimensions of varieties $W(A)$, $A \in SL_3(\mathbb{C})$

In this section we calculate dimensions of commutator varieties

$$W(A) = \{(x, y) \in GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) | [x, y] = A\}, A \in SL_3(\mathbb{C}).$$

Without loss of generality, we may assume that A is a Jordan matrix. The variety $T(A)$ for matrix 3×3 is defined by two polynomials:

$$f_1 = \sigma_1(Ay) - \sigma_1(y) = \operatorname{tr} Ay - \operatorname{tr} y,$$

$$f_2 = \sigma_2(Ay) - \sigma_2(y).$$

For all matrices, except the identity matrix, we can express one of the variables from the polynomial f_1 (it is easy to see if we write polynomials). Hence by the theorem on the dimension of the intersection with a hypersurface, $\dim T(A) = 7$ if $A \neq E_3$. Since $T(E_3) = GL_3(\mathbb{C})$, $\dim T(E_3) = 9$.

Let $W(A) = W_1 \cup W_2 \cdots \cup W_k$ is the decomposition of the commutator variety into irreducible components. Then $\pi(W_i) \subset T(A)$, so by theorem 2, $\dim \overline{\pi(W_i)} \leq \dim T(A)$. By theorem 14 $\dim \pi^{-1}(y) = 3$, if y is a regular matrix.

By theorem 10 the set $\pi(W_i)$ contains a regular matrix y_i for each i. Hence by theorem on the dimension of fibres,

$$\dim W(A) = \max_i \{\dim W_i\} \leq \max_i \{\dim \overline{\pi(W_i)} + \dim \pi^{-1}(y_i)\} \leq \dim T(A) + 3.$$

Applying theorems 11 and 13 we obtain $\dim W(E_3) = 12$, $\dim W(A) = 10$ for each $A \neq E_3$, $A \in SL_3(\mathbb{C})$.

5. Calculation of dimensions of varieties $W(A)$, $A \in SL_4(\mathbb{C})$

In this section we calculate dimensions of commutator varieties

$$W(A) = \{(x,y) \in GL_3(\mathbb{C}) \times GL_4(\mathbb{C}) | [x,y] = A\}, A \in SL_4(\mathbb{C}).$$

Without loss of generality, we may assume that A is a Jordan matrix.

There are five kind of the matrix A:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}.$$

1. In the first case

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \lambda_1\lambda_2\lambda_3\lambda_4 = 1,$$

the variety is defined $T(A)$ by the polynomials

$$f_1 = \sigma_1(Ay) - \sigma_1(y) = -y_{11} - y_{22} - y_{33} - y_{44} + \lambda_1 y_{11} + \lambda_2 y_{22} + \lambda_3 y_{33} + \lambda_4 y_{44},$$

$$f_2 = \sigma_2(Ay) - \sigma_2(y) = y_{12}y_{21} - y_{11}y_{22} + y_{13}y_{31} - y_{11}y_{33} + y_{23}y_{32} - y_{22}y_{33} +$$

$$+ y_{14}y_{41} + y_{24}y_{42} + y_{34}y_{43} - y_{11}y_{44} - y_{22}y_{44} - y_{33}y_{44} - \lambda_1\lambda_2 y_{12}y_{21} + \lambda_1\lambda_2 y_{11}y_{22} -$$

$$- \lambda_1\lambda_3 y_{13}y_{31} + \lambda_1\lambda_3 y_{11}y_{33} - \lambda_2\lambda_3 y_{23}y_{32} + \lambda_2\lambda_3 y_{22}y_{33} - \lambda_1\lambda_4 y_{14}y_{41} -$$

$$- \lambda_2\lambda_4 y_{24}y_{42} - \lambda_3\lambda_4 y_{34}y_{43} + \lambda_1\lambda_4 y_{11}y_{44} + \lambda_2\lambda_4 y_{22}y_{44} + \lambda_3\lambda_4 y_{33}y_{44},$$

$$f_3 = \sigma_3(Ay) - \sigma_3(y) = y_{13}y_{22}y_{31} - y_{12}y_{23}y_{31} - y_{13}y_{21}y_{32} + y_{11}y_{23}y_{32} +$$

$$+ y_{12}y_{21}y_{33} - y_{11}y_{22}y_{33} + y_{14}y_{22}y_{41} - y_{12}y_{24}y_{41} + y_{14}y_{33}y_{41} - y_{13}y_{34}y_{41} -$$

$$- y_{14}y_{21}y_{42} + y_{11}y_{24}y_{42} + y_{24}y_{33}y_{42} - y_{23}y_{34}y_{42} - y_{14}y_{31}y_{43} - y_{24}y_{32}y_{43} +$$

$$+ y_{11}y_{34}y_{43} + y_{22}y_{34}y_{43} + y_{12}y_{21}y_{44} - y_{11}y_{22}y_{44} + y_{13}y_{31}y_{44} + y_{23}y_{32}y_{44} -$$

$$- y_{11}y_{33}y_{44} - y_{22}y_{33}y_{44} - \lambda_1\lambda_2\lambda_3 y_{13}y_{22}y_{31} + \lambda_1\lambda_2\lambda_3 y_{12}y_{23}y_{31} +$$

$$+ \lambda_1\lambda_2\lambda_3 y_{13}y_{21}y_{32} - \lambda_1\lambda_2\lambda_3 y_{11}y_{23}y_{32} - \lambda_1\lambda_2\lambda_3 y_{12}y_{21}y_{33} + \lambda_1\lambda_2\lambda_3 y_{11}y_{22}y_{33} -$$

$$- \lambda_1\lambda_2\lambda_4 y_{14}y_{22}y_{41} + \lambda_1\lambda_2\lambda_4 y_{12}y_{24}y_{41} + \lambda_1\lambda_2\lambda_4 y_{14}y_{21}y_{42} - \lambda_1\lambda_2\lambda_4 y_{11}y_{24}y_{42} -$$

$$- \lambda_1\lambda_2\lambda_4 y_{12}y_{21}y_{44} + \lambda_1\lambda_2\lambda_4 y_{11}y_{22}y_{44} - \lambda_1\lambda_3\lambda_4 y_{14}y_{33}y_{41} + \lambda_1\lambda_3\lambda_4 y_{13}y_{34}y_{41} +$$

$$+ \lambda_1\lambda_3\lambda_4 y_{14}y_{31}y_{43} - \lambda_1\lambda_3\lambda_4 y_{11}y_{34}y_{43} - \lambda_1\lambda_3\lambda_4 y_{13}y_{31}y_{44} + \lambda_1\lambda_3\lambda_4 y_{11}y_{33}y_{44} -$$

$$-\lambda_2\lambda_3\lambda_4 y_{24}y_{33}y_{42} + \lambda_2\lambda_3\lambda_4 y_{23}y_{34}y_{42} + \lambda_2\lambda_3\lambda_4 y_{24}y_{32}y_{43} - \lambda_2\lambda_3\lambda_4 y_{22}y_{34}y_{43} -$$
$$-\lambda_2\lambda_3\lambda_4 y_{23}y_{32}y_{44} + \lambda_2\lambda_3\lambda_4 y_{22}y_{33}y_{44}.$$

The case $A = E_4$ is obvious, so we will assume that $A \neq E_4$. Then $\lambda_i \neq 1$ exists. Without loss of generality, lets $\lambda_1 \neq 1$. Then from $f_1 = 0$ one can obtain

$$y_{11} = \frac{(1-\lambda_2)y_{22} + (1-\lambda_3)y_{33} + (1-\lambda_4)y_{44}}{\lambda_1 - 1}.$$

Suppose that $A \neq -E_4$ (this case will be described later). Then λ_j, $j \neq 1$ such that $\lambda_1\lambda_j \neq 1$ exists. Lets $\lambda_1\lambda_2 \neq 1$. Then, substitute expression for y_{11} in f_2, we obtain

$$f_2 = (y_{21} - \lambda_1\lambda_2 y_{21})y_{12} + f_0,$$

where

$$f_0 = y_{13}y_{31} + y_{23}y_{32} - y_{22}y_{33} + y_{14}y_{41} + y_{24}y_{42} + y_{34}y_{43} - y_{22}y_{44} - y_{33}y_{44} +$$
$$+\frac{y_{22}^2}{1-\lambda_1} + \frac{2y_{22}y_{33}}{1-\lambda_1} + \frac{y_{33}^2}{1-\lambda_1} + \frac{2y_{22}y_{44}}{1-\lambda_1} + \frac{2y_{33}y_{44}}{1-\lambda_1} + \frac{y_{44}^2}{1-\lambda_1} - \frac{\lambda_2 y_{22}^2}{1-\lambda_1} -$$
$$-\frac{\lambda_2 y_{22}y_{33}}{1-\lambda_1} - \frac{\lambda_2 y_{22}y_{44}}{1-\lambda_1} - \frac{\lambda_1\lambda_2 y_{22}^2}{1-\lambda_1} - \frac{\lambda_1\lambda_2 y_{22}y_{33}}{1-\lambda_1} - \frac{\lambda_1\lambda_2 y_{22}y_{44}}{1-\lambda_1} + \frac{\lambda_1\lambda_2^2 y_{22}^2}{1-\lambda_1} -$$
$$-\frac{\lambda_3 y_{22}y_{33}}{1-\lambda_1} - \frac{\lambda_3 y_{33}^2}{1-\lambda_1} - \frac{\lambda_3 y_{33}y_{44}}{1-\lambda_1} - \lambda_1\lambda_3 y_{13}y_{31} - \frac{\lambda_1\lambda_3 y_{22}y_{33}}{1-\lambda_1} - \frac{\lambda_1\lambda_3 y_{33}^2}{1-\lambda_1} -$$
$$-\frac{\lambda_1\lambda_3 y_{33}y_{44}}{1-\lambda_1} - \lambda_2\lambda_3 y_{23}y_{32} + \lambda_2\lambda_3 y_{22}y_{33} + \frac{2\lambda_1\lambda_2\lambda_3 y_{22}y_{33}}{1-\lambda_1} + \frac{\lambda_1\lambda_3^2 y_{33}^2}{1-\lambda_1} -$$
$$-\frac{\lambda_4 y_{22}y_{44}}{1-\lambda_1} - \frac{\lambda_4 y_{33}y_{44}}{1-\lambda_1} - \frac{\lambda_4 y_{44}^2}{1-\lambda_1} - \lambda_1\lambda_4 y_{14}y_{41} - \frac{\lambda_1\lambda_4 y_{22}y_{44}}{1-\lambda_1} - \frac{\lambda_1\lambda_4 y_{33}y_{44}}{1-\lambda_1} -$$
$$-\frac{\lambda_1\lambda_4 y_{44}^2}{1-\lambda_1} - \lambda_2\lambda_4 y_{24}y_{42} + \lambda_2\lambda_4 y_{22}y_{44} + \frac{2\lambda_1\lambda_2\lambda_4 y_{22}y_{44}}{1-\lambda_1} - \lambda_3\lambda_4 y_{34}y_{43} +$$
$$+\lambda_3\lambda_4 y_{33}y_{44} + \frac{2\lambda_1\lambda_3\lambda_4 y_{33}y_{44}}{1-\lambda_1} + \frac{\lambda_1\lambda_4^2 y_{44}^2}{1-\lambda_1}.$$

Therefore f_0 doesn't contain y_{12}, and term $(y_{21} - \lambda_1\lambda_2 y_{21})y_{12}$ in f_2 doesn't reduce for considered matrices. Thus either f_0 devides into y_{21}, or f_2 is irreducible. One can see that if $A \neq E_4$, $A \neq -E_4$ then f_0 devides into y_{21}. Hence f_2 is irreducible. Lets proof that f_2 doesn't divide f_3. Really for all considered λ_i, roots of f_2, which aren't roots of

f_3 exist:

y_{12}	0	0	0	0
y_{13}	0	0	1	1
y_{14}	1	1	0	0
y_{21}	0	1	0	1
y_{22}	0	0	0	0
y_{23}	1	0	0	0
y_{24}	0	0	0	0
y_{31}	1	1	0	0
y_{32}	0	1	1	1
y_{33}	0	0	0	0
y_{34}	1	1	1	0
y_{41}	0	0	1	1
y_{42}	1	1	1	1
y_{43}	0	0	0	0
y_{44}	0	0	0	0
f_2	0	0	0	0
f_3	$\lambda_2\lambda_3\lambda_4 - 1$	$\lambda_1\lambda_2\lambda_4 - 1$	$\lambda_1\lambda_3\lambda_4 - 1$	$\lambda_1\lambda_2\lambda_3 - 1$

Hence, applying the theorem on the dimension of the intrsection with hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13$$

for all $A \neq E_4, A \neq -E_4$, $A \in SL_4(\mathbb{C})$.

For the matrix $(-E_4)$ the polynomial f_2 is identically zero, therefore

$$\dim T(-E_4) = 16 - 1 - 1 = 14$$

(the polynomial f_1 is linear, the polynomial f_2 defined a hypersurface). Note that $T(-E_4)$ is irreducible by lemma 1 (we can express one coordinate from the $f_1 = 0$ and substitute it in f_2, the irreducible polynomial will be obtained).

For the matrix E_4 we have $T(E_4) = GL_4(\mathbb{C})$, hence

$$\dim T(E_4) = 4^2 = 16.$$

2. In the second case

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \lambda_1\lambda_2\lambda_3^2 = 1,$$

the variety $T(A)$ is defined by the polynomials

$$f_1 = \sigma_1(Ay) - \sigma_1(y) = \operatorname{tr} Ay - \operatorname{tr} y,$$

$$f_2 = \sigma_2(Ay) - \sigma_2(y),$$

$$f_3 = \sigma_3(Ay) - \sigma_3(y).$$

From $f_1 = 0$ we obtain

$$y_{43} = (1-\lambda_1)y_{11} + (1-\lambda_2)y_{22} + (1-\lambda_3)y_{33} + (1-\lambda_3)y_{44}.$$

Substitute expression for y_{43} in f_2, we obtain

$$f_2 = (y_{21} - \lambda_1\lambda_2 y_{21})y_{12} + f_0,$$

where f_0 doesn't contain y_{34}, and the first term doesn't reduce in the case $\lambda_1\lambda_2 \neq 1$. In the cases $\lambda_1\lambda_2 = 1$ we can assume that $\lambda_1\lambda_3 \neq 1$ or $\lambda_2\lambda_3 \neq 1$. If $\lambda_1\lambda_2 = \lambda_1\lambda_3 = \lambda_2\lambda_3 = 1$, then we obtain matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which will be considered later. Now we will assume without loss of generality that $\lambda_1\lambda_2 \neq 1$.

One can see (if write f_0), that f_0 doesn't devide into y_{21}. Hence f_2 is irreducible. Now we will prove that f_2 doesn't divide f_3. Really, for all considered λ_i, the roots of f_2, which aren't roots of f_3 exist:

y_{11}	0
y_{12}	1
y_{13}	0
y_{14}	0
y_{21}	0
y_{22}	0
y_{23}	1
y_{24}	0
y_{31}	0
y_{32}	0
y_{33}	0
y_{34}	1
y_{41}	1
y_{42}	0
y_{44}	0
f_2	0
f_3	$\lambda_1\lambda_2$

Hence, applying the theorem on the dimension of the intrsection with hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13.$$

Lets consider the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can write polynomials f_1, f_2, f_3 for these matrices, and express one of the coordinates from f_1. Then we substitute it in f_2 and f_3. We obtain the polynomials f_2' and f_3' which containe less variables. Then, applying a division algorithm [3] (it wiil be easy to apply it, because polynomials don't contain parameters), we obtain that f_2' doesn't divide f_3'.

For example, lets consider the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

For it we have

$$f_1 = -2y_{11} - 2y_{22} - 2y_{33} + y_{43} - 2y_{44},$$

$$f_2 = y_{13}y_{41} + y_{23}y_{42} - y_{11}y_{43} - y_{22}y_{43},$$

$$f_3 = 2y_{13}y_{22}y_{31} - 2y_{12}y_{23}y_{31} - 2y_{13}y_{21}y_{32} + 2y_{11}y_{23}y_{32} + 2y_{12}y_{21}y_{33}-$$

$$-2y_{11}y_{22}y_{33} - y_{13}y_{22}y_{41} + 2y_{14}y_{22}y_{41} + +y_{12}y_{23}y_{41} - 2y_{12}y_{24}y_{41} + 2y_{14}y_{33}y_{41}-$$

$$-2y_{13}y_{34}y_{41} + y_{13}y_{21}y_{42} - 2y_{14}y_{21}y_{42} - y_{11}y_{23}y_{42} + 2y_{11}y_{24}y_{42} + 2y_{24}y_{33}y_{42}-$$

$$-2y_{23}y_{34}y_{42} - y_{12}y_{21}y_{43} + y_{11}y_{22}y_{43} - 2y_{14}y_{31}y_{43} - 2y_{24}y_{32}y_{43} + 2y_{11}y_{34}y_{43}+$$

$$+2y_{22}y_{34}y_{43} + 2y_{12}y_{21}y_{44} - 2y_{11}y_{22}y_{44} + 2y_{13}y_{31}y_{44} + 2y_{23}y_{32}y_{44} - 2y_{11}y_{33}y_{44}$$

$-2y_{22}y_{33}y_{44}$.

Hence, from $f_1 = 0$, we obtain

$$y_{43} = 2(y_{11} + y_{22} + y_{33} + y_{44}).$$

Then

$$f_2' = -2y_{11}^2 - 4y_{11}y_{22} - 2y_{22}^2 - 2y_{11}y_{33} - 2y_{22}y_{33} + y_{13}y_{41} + y_{23}y_{42} - 2y_{11}y_{44}-$$

$-2y_{22}y_{44}$,

$$f_3' = -2y_{11}y_{12}y_{21} + 2y_{11}^2y_{22} - 2y_{12}y_{21}y_{22} + 2y_{11}y_{22}^2 - 4y_{11}y_{14}y_{31} + 2y_{13}y_{22}y_{31}-$$

$$-4y_{14}y_{22}y_{31} - 2y_{12}y_{23}y_{31} - 2y_{13}y_{21}y_{32} + 2y_{11}y_{23}y_{32} - 4y_{11}y_{24}y_{32} - 4y_{22}y_{24}y_{32}-$$

$$-4y_{14}y_{31}y_{33} - 4y_{24}y_{32}y_{33} + 4y_{11}^2y_{34} + 8y_{11}y_{22}y_{34} + 4y_{22}^2y_{34} + 4y_{11}y_{33}y_{34}+$$

$$+4y_{22}y_{33}y_{34} - y_{13}y_{22}y_{41} + 2y_{14}y_{22}y_{41} + y_{12}y_{23}y_{41} - 2y_{12}y_{24}y_{41} + 2y_{14}y_{33}y_{41}-$$

$$-2y_{13}y_{34}y_{41} + y_{13}y_{21}y_{42} - 2y_{14}y_{21}y_{42} - y_{11}y_{23}y_{42} + 2y_{11}y_{24}y_{42} + 2y_{24}y_{33}y_{42}-$$

$$-2y_{23}y_{34}y_{42} + 2y_{13}y_{31}y_{44} - 4y_{14}y_{31}y_{44} + 2y_{23}y_{32}y_{44} - 4y_{24}y_{32}y_{44} - 2y_{11}y_{33}y_{44}-$$

$-2y_{22}y_{33}y_{44} + 4y_{11}y_{34}y_{44} + 4y_{22}y_{34}y_{44}$.

Applying a division algorithm [3] for divide f'_3 by f'_2 with lexicographic monomial order $\{y_{11}, y_{12}, y_{13}, y_{14}, y_{21}, y_{22}, y_{23}, y_{24}, y_{31}, y_{32}, y_{33}, y_{34}, y_{41}, y_{42}, y_{44}\}$, we obtain

$$f'_3 = f'_2(-y_{22} - 2y_{34}) - 2y_{11}y_{12}y_{21} - 2y_{12}y_{21}y_{22} - 2y_{11}y_{22}^2 - 2y_{22}^3 - 4y_{11}y_{14}y_{31}+$$

$$+2y_{13}y_{22}y_{31} - 4y_{14}y_{22}y_{31} - 2y_{12}y_{23}y_{31} - 2y_{13}y_{21}y_{32} + 2y_{11}y_{23}y_{32} - 4y_{11}y_{24}y_{32}-$$

$$-4y_{22}y_{24}y_{32} - 2y_{11}y_{22}y_{33} - 2y_{22}^2y_{33} - 4y_{14}y_{31}y_{33} - 4y_{24}y_{32}y_{33} + 2y_{14}y_{22}y_{41}+$$

$$+y_{12}y_{23}y_{41} - 2y_{12}y_{24}y_{41} + 2y_{14}y_{33}y_{41} + y_{13}y_{21}y_{42} - 2y_{14}y_{21}y_{42} - y_{11}y_{23}y_{42}+$$

$$+y_{22}y_{23}y_{42} + 2y_{11}y_{24}y_{42} + 2y_{24}y_{33}y_{42} - 2y_{11}y_{22}y_{44} - 2y_{22}^2y_{44} + 2y_{13}y_{31}y_{44}-$$

$$-4y_{14}y_{31}y_{44} + 2y_{23}y_{32}y_{44} - 4y_{24}y_{32}y_{44} - 2y_{11}y_{33}y_{44} - 2y_{22}y_{33}y_{44}.$$

The same calculations may be considered for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence for two considered matrix, applying theorem on the dimension of the intersection with a hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13.$$

3. In the third case

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \lambda_1^2\lambda_2^2 = 1,$$

the varity $T(A)$ is defined by the polynomials

$$f_1 = \sigma_1(Ay) - \sigma_1(y) = \operatorname{tr} Ay - \operatorname{tr} y,$$

$$f_2 = \sigma_2(Ay) - \sigma_2(y),$$

$$f_3 = \sigma_3(Ay) - \sigma_3(y).$$

From $f_1 = 0$ we obtain

$$y_{21} = (1 - \lambda_1)y_{11} + (1 - \lambda_1)y_{22} + (1 - \lambda_2)y_{33} + (1 - \lambda_2)y_{44} - y_{43}.$$

Lets λ_i such that $\lambda_i^2 \neq 1$ exists (other matrix will be considered later). Without loss of generality lets $\lambda_2^2 \neq 1$. Substituting expression for y_{21} in f_2, we obtain

$$f_2 = (y_{43} - \lambda_2^2y_{43})y_{34} + f_0,$$

where f_0 doesn't contain y_{34} and the first term isn't identically zero. One can see (if write f_0), that f_0 doesn't devide into y_{43}. Hence f_2 is irreducible. Now we will prove that f_2

doesn't divide f_3. Really, for all considered λ_i, the roots of f_2, which aren't roots of f_3 exist:

y_{11}	0
y_{12}	0
y_{13}	0
y_{14}	1
y_{22}	1
y_{23}	0
y_{24}	0
y_{31}	1
y_{32}	0
y_{33}	0
y_{34}	1
y_{41}	0
y_{42}	1
y_{43}	0
y_{44}	0
f_2	0
f_3	$-(\lambda_1 - 1)(\lambda_1^2\lambda_2 - 1)$

Hence, applying the theorem on the dimension of the intrsection with hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13.$$

Lets consider the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Calculations, similar to the calculations in the second case, show that $\dim T(A) = 13$ for these matrices.

4. In the fourth case
$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \lambda_1\lambda_2^3 = 1,$$
the variety $T(A)$ is defined by the polynomials
$$f_1 = \sigma_1(Ay) - \sigma_1(y) = \operatorname{tr} Ay - \operatorname{tr} y,$$
$$f_2 = \sigma_2(Ay) - \sigma_2(y),$$
$$f_3 = \sigma_3(Ay) - \sigma_3(y).$$
From $f_1 = 0$ we obtain
$$y_{32} = (1 - \lambda_1)y_{11} + (1 - \lambda_2)y_{22} + (1 - \lambda_2)y_{33} + (1 - \lambda_2)y_{44} - y_{43}.$$

Substitute expression for y_{32} in f_2, we obtain

$$f_2 = (y_{42} - \lambda_2^2 y_{42}) y_{24} + f_0,$$

where f_0 doesn't contain y_{34}. Lets the first term isn't identically zero (another matrices will be considered later). One can see (if write f_0), that f_0 doesn't devide into y_{42} for considered matrices. Therefore f_2 is irreducible. Now we will prove that f_2 doesn't divide f_3. Really, for all considered λ_i, the roots of f_2, which aren't roots of f_3 exist:

y_{11}	0
y_{12}	1
y_{13}	0
y_{14}	0
y_{21}	0
y_{22}	0
y_{23}	0
y_{24}	1
y_{31}	0
y_{33}	0
y_{34}	0
y_{41}	1
y_{42}	0
y_{43}	0
y_{44}	0
f_2	0
f_3	$(\lambda_1 \lambda_2^2 - 1)$

Hence, applying the theorem on the dimension of the intrsection with hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13.$$

Lets consider the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Calculations, similar to the calculations in the second case, show that $\dim T(A) = 13$ for these matrices.

5. The fifth case

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \lambda_1^4 = 1,$$

the variety $T(A)$ is defined by the polynomials

$$f_1 = \sigma_1(Ay) - \sigma_1(y) = \operatorname{tr} Ay - \operatorname{tr} y,$$

$$f_2 = \sigma_2(Ay) - \sigma_2(y),$$

$$f_3 = \sigma_3(Ay) - \sigma_3(y).$$

From $f_1 = 0$ we obtain

$$y_{32} = (1-\lambda_1)y_{11} + (1-\lambda_1)y_{22} + (1-\lambda_1)y_{33} + (1-\lambda_1)y_{44} - y_{21} - y_{43}.$$

Substitute expression for y_{32} in f_2, we obtain

$$f_2 = (y_{41} - \lambda_1^2 y_{41})y_{14} + f_0,$$

where f_0 doesn't contain y_{14}. Lets the first term isn't identically zero (another matrices will be considered later). One can see (if write f_0), that f_0 doesn't devide into y_{41}. Hence f_2 is irreducible. Now we will prove that f_2 doesn't divide f_3. Really, for all considered λ_i, the roots of f_2, which aren't roots of f_3 exist:

y_{11}	0
y_{12}	1
y_{13}	0
y_{14}	0
y_{21}	0
y_{22}	0
y_{23}	0
y_{24}	0
y_{31}	0
y_{33}	1
y_{34}	1
y_{41}	1
y_{42}	0
y_{43}	0
y_{44}	0
f_2	0
f_3	$(\lambda_1 + \lambda_1^2)$

Hence, applying the theorem on the dimension of the intrsection with hypersurface, we obtain

$$\dim T(A) = 16 - 1 - 1 - 1 = 13.$$

Lets consider the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Calculations, similar to the calculations in the second case, show that $\dim T(A) = 13$ for these matrices.

Hence we have

$$\dim T(E_4) = 16,$$

$$\dim T(-E_4) = 14.$$

For other matrices $A \in SL_4(\mathbb{C})$,

$$\dim T(A) = 13.$$

Let $W(A) = W_1 \cup W_2 \cdots \cup W_k$ is the decomposition of the commutator variety into irreducible components. Then $\pi(W_i) \subset T(A)$, so by theorem 2, $\dim \overline{\pi(W_i)} \leq \dim T(A)$. By theorem 14 $\dim \pi^{-1}(y) = 4$, if y is a regular matrix.

By theorem 10 the set $\pi(W_i)$ contains a regular matrix y_i for each i. Hence by theorem on the dimension of fibres,

$$\dim W(A) = \max_i\{\dim W_i\} \leq \max_i\{\dim \overline{\pi(W_i)} + \dim \pi^{-1}(y_i)\} \leq \dim T(A) + 3.$$

Applying theorems 11 and 13 we obtain $\dim W(E_4) = 20$ and $\dim W(A) = 17$ if $A \neq E_4, -E_4$.

For $A = -E_4$ we obtain the estimation $\dim W(-E_4) \leq 18$.

Theorem 15. $\dim W(-E_4) = 18$.

Lemma 5. *An irreducible component W_i of the variety $W(-E_4)$, such that $\overline{\pi(W_i)} = T(-E_4)$, exists.*

Proof. Lets T_0 is the set of regular semisimple elements in T. $T_0 \neq \emptyset$, for example, T_0 contains a matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

(multiply this matrix by $-E_4$ we obtain itself accurate within conjugation, therefore, this matrix lies in $\pi(W)$, all the more so in $T(-E_4)$).

By theorem 8, T_0 is an open subset in $T(-E_4)$and $T_0 \subset \cup_i \pi(W_i)$. Hence $\overline{T_0} \subset \overline{\cup_i \pi(W_i)} = \cup_i \overline{\pi(W_i)}$ and $\overline{T}_0 = T(-E_4)$ by theorem 1. Therefore $T(-E_4) = \cup_i \overline{\pi(W_i)}$. Thus if W_i such that $\overline{\pi(W_i)} = T(-E_4)$ doesn't exist, then we obtain a contradiction with the fact that $T(-E_4)$ is irreducible. □

Proof of the theorem. Lets consider the mapping

$$\pi : W_i \to \overline{\pi(W_i)},$$

$$(x, y) \to y.$$

The set V of all regular elements is open in $GL_4(\mathbb{C})$ by theorem 7. Hence it is open in$\overline{\pi(W_i)}$. The set V isn't empty by theorem 10. By theorem on the dimension of fibres, in $\overline{\pi(W_i)}$ nonempty open set U, such that for all $y \in U$ $\dim \pi^{-1}(y) = \dim W_i - \overline{\pi(W_i)}$, exists. The variety $\overline{\pi(W_i)}$

is irreducible as closure of the image of the irreducible variety, hence $U \cap V \neq \emptyset$. For regular element y we have $\pi^{-1}(y) = 4$ by theorem 14. Therefore $\dim W_i = 14 + 4 = 18$. Using theorem 11, the estimation $\dim W(-E_4) \leq 18$ and definition of the dimension of the variety, we obtain, by on the dimension of fibres, that $\dim W(-E_4) = 18$. $\square$

6. About the decomposition into irreducible components of the low-dimensional representation varieties of some group

Let us conider the group $G = \langle x, y, z_1, \ldots, z_m | [x, y] = W(z_1, \ldots, z_m)\rangle$, where the word $W(z_1, \ldots, z_m)$ does not belong to the commutator subgroup of the free group $F(z_1, \ldots, z_m)$ freely generated by $z_1, \ldots, z_m$. In other words it means that in the factor group F/F' the word W can be wrote as

$$W = z_1^{k_1}, \ldots, z_m^{k_m},$$

where $k_i \neq 0$ exists.

In this chapter we will prowe some theorem about the low-dimensional (n=2,3,4) representation varieties $R_n(G)$.

Let us consider the projection

$$\rho : R_n(G) \to GL_n(\mathbb{C})^m,$$

$$(x, y, z_1, \ldots, z_m) \to (z_1, \ldots, z_m).$$

Then $\operatorname{Im} \rho \subset X = \{(z_1, \ldots, z_m) | \det W(z_1, \ldots, z_m) = 1\}$. Let us describe X.

Let $d = \mathrm{GCD}(k_1, \ldots, k_m), \widetilde{k_i} = k_i/d, \varepsilon$ — primitive root of 1 of degree d.

Let us denote

$$X_i = \{(z_1, \ldots, z_m) | (\det z_1)^{\widetilde{k_1}}, \ldots, (\det z_m)^{\widetilde{k_m}} = \varepsilon^i\}.$$

Then $X = \cup_{i=1}^d X_i$.

Lemma 6. *X_i is an irreducible variety.*

]

Proof. Note that

$$X_i = \{(z_1, \ldots, z_m) | \det(z_1^{\widetilde{k_1}}, \ldots, z_m^{\widetilde{k_m}}) = \varepsilon^i\}.$$

Without loss of generality let us consider that $\widetilde{k_j} \geq 0$ for all j. The proof will be by induction by the number $\sum_{j=1}^m \widetilde{k_j}$. Let $\sum_{j=1}^m \widetilde{k_j} = 1$. The variety $H = \{x \in GL_n(\mathbb{C}) | \det H = \varepsilon^i\}$ is irreducible by lemma 1. Then

$$X_i = GL_n(\mathbb{C})^{m-1} \times H$$

is irreducible.

Let $\sum_{j=1}^m \widetilde{k_j} \geq 1$. Let us denote $\widetilde{k_{i_0}} = \min\{\widetilde{k_1}, \ldots, \widetilde{k_m}\}$ (where minimum is taken over $k_i \neq 0$) and $l \neq i_0$. Let $z'_j = z_j$ if $j \neq i_0$, and $z'_{i_0} = z_l z_{i_0}$ is a biregular substitute. Then we obtain

$$X_i = \{(z'_1, \ldots, z'_m) | \det({z'_1}^{\widetilde{k_1}'}, \ldots, {z'_m}^{\widetilde{k_m}'}) = \varepsilon^i\},$$

where $\sum_{j=1}^m \widetilde{k_j}' < \sum_{j=1}^m \widetilde{k_j}$ and we can apply the assumption of the induction. The lemma is proved. □

Lemma 7. *For all X_i single component W_i of the variety $R_n(G)$ such that $\overline{\rho(W_i)} = X_i$ and $\dim W_i = n^2(m+1)$ exists.*

Proof. Settle X_i and part components of $R_n(G)$ into two groups:

$R_1, \dots, R_s$ such that $\overline{\rho(R_j)} = X_i$, $j = \overline{1, s}$;

$R_{s+1}, \dots, R_k$ such that $\overline{\rho(R_j)} \neq X_i$, $j = \overline{s+1, k}$.

Let $U_i = R_i \backslash \{\cup_{j \neq i} R_j\}, i = \overline{1, s}$.

Then $\overline{\rho(U_j)} = X_i, j = \overline{1, s}$ and by the theorem 5 $\rho(U_j)$ contains open subset $V_j \subset X_i$.

Let $M_1 = X_i \backslash \{\cup_{j=s+1}^{k} \overline{\rho(R_j)}\}$ is open in X_i subset.

Let us consider a mapping

$$\alpha : X_i \to SL_n(\mathbb{C}),$$

$$(z_1, \dots, z_m) \to W(z_1, \dots, z_m).$$

Under our propositions about the word W the mapping α is surjective. Really for anyl $A \in SL_n(\mathbb{C})$ we put $z_2 = \dots = z_n = E$ and obtain an equation $z_1^{k_1} = A$, which have a solution[4]. Let U is an open subset in $SL_n(\mathbb{C})$ such that $\forall z \in U$ the variety $W(z)$ is an irreducible variety of the dimension $n^2 + 1$ (see the theorem 12).

Let $M_2 = \alpha^{-1}(U) \subset X_i$ is an open in X_i subset.

Let us cosider the intersection $V_1 \cap \dots \cap V_s \cap M_1 \cap M_2 = V$ which is nonempty open subset in X_i (as X_i is an irreducible variety).

For the point $(a_1, \dots, a_m) \in V$ the fibre

$$\rho^{-1}(a_1, \dots, a_m) =$$

$$= \{(x, y, a_1, \dots, a_m) | [x, y] = A, \text{where } A = W(a_1, \dots, a_m)\} = W(A)$$

is an irreducible variety since $\alpha(a_1, \dots, a_m) = A \in U$. Therefore $\rho^{-1}(a_1, \dots, a_m)$ is contained in one of the component $R_1, \dots, R_s$, for example, $\rho^{-1}(a_1, \dots, a_m) \subset R_1$. Then

$$\rho^{-1}(a_1, \dots, a_n) \cap (R_2 \backslash R_1) = \emptyset.$$

We obtain a contradiction, since by construction $(a_1, \dots, a_m) \in \rho(R_2 \backslash R_1)$. Thus, we have proved that no more then one component of $W_i \subset R_n(G)$ with the property $\overline{\rho(W_i)} = X_i$ exists.

Obviously, at least one component of W_i with this property exists, since $\rho(R_n(G)) = \cup_i X_i$.

$\dim W_i = \dim X_i + \dim W(A) = (n^2 m - 1) + (n^2 + 1) = n^2(m+1)$ by the theorem on the dimension of fibres.

□

Lemma 8. *Each component of $R_n(G)$ has dimension no less than $n^2(m+1)$.*

Proof. The variety $R_n(G)$ is nonempty and defined in $GL_n(\mathbb{C})^{m+2}$ by n^2 equations. Therefore, by the theorem on the dimension of the intersection with a hypersurface, the dimension of each component of $R_n(G)$ is no less than $n^2(m+2) - n^2 = n^2(m+1)$. □

Hereinafter we set $n = 2, 3$ and we consider this case in detail.

Lemma 9. *If W is an irreducible component of $R_n(G)$ and $\overline{\rho(W)}$ does not coincide with any of the X_i then $\alpha(\rho(W)) = E_n$. It means that for any point $(x, y, z_1, \ldots, z_m) \in W$ we have $[x, y] = W(z_1, \ldots z_m) = E_n$.*

Proof. Assume the contrary (i. e. $\overline{\alpha(\rho(W))} \neq E_n$). Let $T = \overline{\alpha(\rho(W))} \backslash \{E_n\}$ is an open in $\overline{\alpha(\rho(W))}$ subset. Then $\alpha^{-1}(T)$ is open in $\overline{\rho(W)}$ and for any point $z = (z_1, \ldots, z_m) \in \alpha^{-1}(T)$ we have $\dim \rho^{-1}(z) = n^2 + 1$.

By the theorem on the dimension of fibres we have

$$\dim W = \dim \overline{\rho(W)} + (n^2 + 1) < \dim X_i + (n^2 + 1) =$$

$$= (n^2 m - 1) + (n^2 + 1) = n^2(m + 1).$$

It its a contradiction with the lemma 8. □

Let $C_2 = \{(x, y) | [x, y] = E_n\}$ is an irreducible commutator variety of dimension $(n^2 + n)$ (in the previous chapters this variety has been denoted $W(E_n)$), and let

$$R(W) = \{(z_1, \ldots, z_m) | W(z_1, \ldots, z_m) = E_n\}.$$

Note that really $R(W)$ is the variety of n-dimensional representations $R_n(G_1)$ of the group

$$G_1 = \langle z_1, \ldots, z_m | W(z_1, \ldots, z_m) = 1 \rangle.$$

Lemma 10. *Let H is an irreducible component of $R(W)$ and $\dim H \geq n^2 m - n$. Then $C_2 \times H$ is an irreducible component of $R_n(G)$. If $\dim H < n^2 m - n$ then $C_2 \times H \subset R_n(G)$ but it isn't an irrducible component of $R_n(G)$.*

Proof. It is clear that $C_2 \times H \subset R_n(G)$. If $\dim H \geq n^2 m - n$, then $\dim C_2 \times H \geq (n^2 + n) + (n^2 m - n) = n^2(m + 1)$.

It is clear that if $H_1 \neq H_2$ are the different irreducible components of $R(W)$, then $C_2 \times H_1 \not\subset C_2 \times H_2$ and $C_2 \times H_2 \not\subset C_2 \times H_1$ (since $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$).

Owing to dimension reasons, $C_2 \times H \not\subset W_i$ for every W_i which is an irreducible component of $R_n(G)$ such that $\overline{\rho(W_i)} = X_i$. Also $W_i \not\subset C_2 \times H$ since $\alpha(\rho(W_i)) = SL_n(\mathbb{C})$, but $\alpha(\rho(C_2 \times H)) = \{E_n\}$. Since the sets $C_2 \times H$, where H runs all irreducible components of $R(W)$, and sets W_i cover $R_n(G)$, so in the case $\dim H \geq n^2 m - n$ the sets $C_2 \times H$ are irreducible components of $R_n(G)$.

If $\dim H < n^2 m - n$, then $\dim C_2 \times H < n^2(m + 1)$ and the set $C_2 \times H$ isn't an irreducible component of $R_n(G)$ by lemma 2. In particular, in this case $C_2 \times H$ contains in one of the W_i. □

Thus, we have proved the theorem

Theorem 16. *The number of irreducible components of representation variety $R_n(G)$ is equal to $d+s$, where s is the number of irreducible components of representation variety $R_n(G_1)$, whose dimension is no less than n^2m-n.*

Corollary 1. *Let $m=1$ and $W(z)=z^t$, where $t\geq 2$, i. e.*

$$G=\langle x,y,z|[x,y]=z^t\rangle.$$

Then the number of irreducible components of the variety $R_2(G)$ is equal to $t+C_t^2=t+\frac{t(t-1)}{2}=\frac{t(t+1)}{2}$. The dimension of each component is 8.

In contrast with the corollary if we onsider the group

$$G_2=\langle x_1,x_2,y_1,y_2|[x_1,y_1][x_2,y_2]=z^t\rangle,$$

thenn $R_2(G)$ has t irreducible components [1].

Proof of the corollary. Let ε is a primitive root of 1 of degree t. Then in $R_2(G)$ there are t components $W_1,\dots,W_t$ such that $\overline{\rho(W_i)}=X_i$, where $X_i=\{z\in GL_2(\mathbb{C})|\det z=\varepsilon^i\}$.

The components of the variety $R(W)=\{z\in GL_2(\mathbb{C}),z^t=E_2\}$ are orbits of the matrices:

$$H_{ij}=\left\{X\begin{pmatrix}\varepsilon^i & 0\\ 0 & \varepsilon^j\end{pmatrix}X^{-1}|X\in GL_2(\mathbb{C})\right\}$$

The varieties H_{ij} are closed and irreducible in $GL_2(\mathbb{C})$. If $i=j$, then

$$H_{ii}=\left\{\begin{pmatrix}\varepsilon^i & 0\\ 0 & \varepsilon^i\end{pmatrix}\right\}$$

and $\dim H_{ii}=0$. If $i\neq j$, $\dim H_{ij}=2\geq 4-2$. Thus, if $i\neq j$ then the set $C_2\times H_{ij}$ is an irreducible $R_2(G)$. □

Similarly,

Corollary 2. *Let $m=1$ and $W(z)=z^t$, where $t\geq 2$, i. e.*

$$G=\langle x,y,z|[x,y]=z^t\rangle.$$

Then the number of irreducible components of the variety $R_3(G)$ is equal to $t+C_t^3=t+\frac{t(t-1)(t-2)}{6}$. The dimension of each component is 18.

Corollary 3. *Let $r,t\in\mathbb{N},(r,t)=1$ and $G=\langle x,y,z_1,z_2|[x,y]=z_1^rz_2^t\rangle$. Then $R_2(G)$ is an irreducible variety.*

Proof. If $G_1=\langle z_1,z_2|z_1^rz_2^t=1\rangle$, then the variety $R_2(G_1)$ has components of dimension 4 and 5 [2]. Therefore in the theorem we have $s=0$ and $d=(r,t)=1$, i. e. $R_2(G)$ is an irreducible variety. □

Now we consider the case $n=4$ in detail.

Lemma 11. *If W is an irreducible component of $R_4(G)$ and $\overline{\rho(W)}$ does not coincide with any of the X_i, then $\alpha(\rho(W)) \in \{E_4, -E_4\}$. it means that for any point $(x, y, z_1, \ldots, z_m) \in W$ we have $[x, y] = W(z_1, \ldots z_m) = E_4$ or $[x, y] = W(z_1, \ldots z_m) = -E_4$.*

Proof. Assume the contrary (i. e. $\overline{\alpha(\rho(W))} \notin \{E_4, -E_4\}$). Let $T = \overline{\alpha(\rho(W))} \setminus \{E_4, -E_4\}$ is an open in $\overline{\alpha(\rho(W))}$ subset. Then $\alpha^{-1}(T)$ is open in $\overline{\rho(W)}$ and for any point $z = (z_1, \ldots, z_m) \in \alpha^{-1}(T)$ we have $\dim \rho^{-1}(z) = n^2 + 1 = 17$.

By the theorem on the dimension of fibres we have

$$\dim W = \dim \overline{\rho(W)} + 17 < \dim X_i + 17 =$$

$$= (16m - 1) + 17 = 16(m + 1).$$

It is a contradiction with the lemma 8. ☐

Let us introduce the following varieties:

$$C_2 = \{(x, y) | [x, y] = E_4\}$$

is an irreducible commutator variety of the dimension 20 (in the previous chapters this variety has been denoted $W(E_4)$); and

$$R(W) = \{(z_1, \ldots, z_m) | W(z_1, \ldots, z_m) = E_4\}.$$

Let $R(W) = \cup_{i=1}^{s} H_i$ is a decomposition of the variety $R(W)$ into irreducible components.

$$C_2' = \{(x, y) | [x, y] = -E_4\}$$

is an irreducible commutator variety of the dimension 18 (in the previous chapters this variety has been denoted $W(-E_4)$) and $C_2' = \cup_{i=1}^{k_1} R_i \cup_{i=1}^{k_2} S_i$ is its decomposition into irreducible components,where $\dim R_i = 17, \dim S_i = 18$. Note that in decomposition of C_2' may not exist components of dimension 17 (may be $k_1 = 0$); and

$$R'(W) = \{(z_1, \ldots, z_m) | W(z_1, \ldots, z_m) = -E_4\}$$

Let $R'(W) = \cup_{i=1}^{s'} H_i'$ is a decomposition of the variety $R'(W)$ into irreducible components.

Lemma 12. *Let H is an irreducible component of $R(W)$ and $\dim H \geq 16m - 4$. Then $C_2 \times H$ is an irreducible component of $R_4(G)$. If $\dim H < 16m - 4$, then $C_2 \times H \subset R_4(G)$, but it isn't an irreducible component of $R_4(G)$.*

Let H' is an irreducible component of $R(W)$ and $\dim H' \geq 16m - 1$. Then $R_i \times H'$ is an irreducible component of $R_4(G)$ for every i. If $\dim H < 16m - 1$, then $R_i \times H' \subset R_4(G)$, but it isn't an irreducible component of $R_4(G)$.

Let H' is an irreducible component of $R(W)$ and $\dim H' \geq 16m - 2$. Then $S_i \times H'$ is an irreducible component of $R_4(G)$ for every i. If $\dim H' < 16m - 2$, then $S_i \times H' \subset R_4(G)$, but it isn't an irreducible component of $R_4(G)$.

Proof. It is clear that $C_2 \times H, R_i \times H', S_i \times H' \subset R_4(G)$.

It is clear that if $H_1 \neq H_2$ are different irreducible components of $R(W)$, then $C_2 \times H_1 \not\subset C_2 \times H_2$ and $C_2 \times H_2 \not\subset C_2 \times H_1$ (since $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$). Similarly, $V_1 \times H_1' \not\subset V_2 \times H_2'$ and $V_2 \times H_2' \not\subset V_1 \times H_1'$ for all V_1 and V_2 — irreducible components of C_2', if $V_1 \neq V_2$ or $H_1' \neq H_2'$. $C_2 \times H_1 \not\subset V_1 \times H_1'$ and $V_1 \times H_1' \not\subset C_2 \times H_1$ since $\alpha(\rho(C_2 \times H_1)) = \{E_4\}$, but $\alpha(\rho(V_1 \times H_1')) = \{-E_4\}$.

If $\dim H \geq 16m - 4$, then $\dim C_2 \times H \geq 20 + (16m - 4) = 16m + 16$. Owing to dimension reasons, $C_2 \times H \not\subset W_i$ for every W_i which is an irreducible component of $R_4(G)$, such that $\overline{\rho(W_i)} = X_i$. Also $W_i \not\subset C_2 \times H$ since $\alpha(\rho(W_i)) = SL_4(\mathbb{C})$, but $\alpha(\rho(C_2 \times H)) = \{E_4\}$.

If $\dim H' \geq 16m - 1$, then $\dim R_i \times H' \geq 16 + 16m$. Owing to dimension reasons, $R_i \times H' \not\subset W_i$ for every W_i which is an irreducible component of $R_4(G)$, such that $\overline{\rho(W_i)} = X_i$. Also $W_i \not\subset R_i \times H'$ since $\alpha(\rho(W_i)) = SL_4(\mathbb{C})$, but $\alpha(\rho(R_i \times H')) = \{-E_4\}$.

If $\dim H' \geq 16m - 2$, then $\dim S_i \times H' \geq 16 + 16m$. Owing to dimension reasons, $S_i \times H' \not\subset W_i$ for every W_i which is an irreducible component of $R_4(G)$, such that $\overline{\rho(W_i)} = X_i$. Also $W_i \not\subset S_i \times H'$ since $\alpha(\rho(W_i)) = SL_4(\mathbb{C})$, but $\alpha(\rho(S_i \times H')) = \{-E_4\}$.

Since the sets $C_2 \times H$, where H runs all irreducible components of $R(W)$, $R_i \times H', S_i \times H'$, where H' runs all irreducible components of $R'(W)$, and sets W_i cover $R_4(G)$, then the first part of the statement is proved.

If $\dim H < 16m - 4$, then $\dim C_2 \times H < 16m + 16$ and the set $C_2 \times H$ isn't an irreducible component of $R_4(G)$ by lemma 8. In particular, in this case $C_2 \times H$ contains in one of the W_i.

If $\dim H' < 16m - 1$, then $\dim R_i \times H' < 16m + 16$ and the set $R_i \times H'$ isn't an irreducible component of $R_4(G)$ by lemma 8. In particular, in this case $R_i \times H'$ contains in one of the W_i.

If $\dim H' < 16m - 2$, then $\dim S_i \times H' < 16m + 16$ and the set $S_i \times H'$ isn't an irreducible component of $R_4(G)$ by lemma 8. In particular, in this case $S_i \times H'$ contains in one of the W_i. □

Thus, we have proved the theorem

Theorem 17. *The number of irreducible components of representation variety $R_4(G)$ is equal to $d + s + k_1 t_1 + k_2 t_2$, where*

- *s is the number of irreducible components of the variety $R(W)$, whose dimension is no less than $16m - 4$,*
- *t_1 is the number of irreducible components of the variety $R'(W)$, whose dimension is no less than $16m - 1$,*
- *t_2 is the number of irreducible components of the variety $R'(W)$, whose dimension is no less than $16m - 2$.*

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