Problem set in geometry. No. 3

1. Determine which transformations are described by the given formulas. For a rotation determine the center and angle, for a glide reflection the axis and translation vector, etc.:

   (a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; (b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$; (c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$;

   (d) $\varphi(z) = -\overline{z} + 2$, (e) $\varphi(z) = -\overline{z} + 2i$; (f) $\varphi(z) = i \cdot \overline{z}$; (g) $\varphi(z) = i \cdot z + 2i$; (h) $\varphi(z) = (\sqrt{2} + \sqrt{2}i) \cdot \overline{z}$.

2. Give the analytic form of the following plane transformations:

   (a) $\mathbf{R}_A^\alpha$ where $A = (1, -1)$, $\alpha = \frac{\pi}{4}$. (b) A dilative symmetry with axis $x + y = 1$, center $(0, 1)$ and scale 2. (c) A spiral similarity with rotation angle $\frac{\pi}{4}$, scale $\sqrt{2}$ and center $(2, 1)$.

3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \frac{1}{9} \begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix}$. Show that A is a rotation matrix, B a rotatory reflection matrix, and C a reflection in a plane. Determine the axes and angles of rotation (for A and B), and the planes of symmetry (for B and C).

4. Let A be a symmetric orthogonal matrix of order 3. Show that:

   (a) if $A \neq -J$ and $\det A = -1$, then A is a reflection in a plane,

   (b) if $A \neq J$ and $\det A = 1$, then A is a half-turn (rotation by π, i.e. reflection in a line).

   (c) What are the eigenvalues of these matrices?

5. Using the properties of the group of isometries of space, prove that every orthogonal matrix of order 3 is a product of two symmetric orthogonal matrices.

6. Show that the radius of the Euler (nine-point) circle is equal to half the radius of the circumcircle of a triangle, and that its center coincides with the midpoint of the segment joining the orthocenter and the circumcenter.

7. Show that the Euler circle of an isosceles triangle is tangent to one of its sides, and that the Euler circle of a right triangle passes through one of its vertices.

8. Construct an isosceles triangle ABC given its base AB and the radius of its Euler circle $r_9 = \frac{1}{3}|AB|$.

9. A circle ω with a chord CD (not a diameter) is given. Construct a right triangle ABC whose Euler circle is ω, and one of whose sides contains the chord CD.

10. A triangle ABC is given and points $C' = \mathbf{J}_B^2(A)$, $B' = \mathbf{M}(A, C)$. In what ratio does the line $C'B'$ divide the side BC?

11. A triangle ABC is given and points $C' = \mathbf{S}_B(A)$, $B' = \mathbf{J}_C^{-2}(A)$. Lines BB', CC' intersect at point P. In what ratio does the line AP divide the side BC?

12. Prove that the lines joining the points of tangency of the incircle of any triangle with the opposite vertices are concurrent.

13. Prove the following theorem on angle bisectors: If D is the intersection point of the bisector of angle $\angle ACB$ with side AB of triangle ABC, and D' is the intersection point of the external angle bisector at vertex C with line AB, then $\frac{|AD|}{|BD|} = \frac{|AD'|}{|BD'|} = \frac{|AC|}{|BC|}$.

14. Prove that the feet of the internal angle bisectors at vertices A and B of triangle ABC and the intersection point of the external angle bisector at vertex C with the line $l(A, B)$ lie on one line (or the external angle bisector is parallel to $l(A, B)$).