List of Geometry Problems. No. 2

1. Assume that $A \neq B$. What transformation is the composition:

   a) $\mathbf{J}_B^2 \circ \mathbf{J}_A^{\frac{1}{2}}$, b) $\mathbf{J}_B^{-2} \circ \mathbf{J}_A^{\frac{1}{2}}$, c) $\mathbf{J}_B^4 \circ \mathbf{J}_A^{\frac{1}{2}}$, d) $\mathbf{J}_A^2 \circ \mathbf{T}_{\overrightarrow{u}}$, e) $\mathbf{J}_B^2 \circ \mathbf{S}_A$, f) $\mathbf{S}_A \circ \mathbf{J}_B^2 \circ \mathbf{S}_A$?

2. In an arbitrary triangle ABC inscribe a square so that two vertices of the square lie on the side AB, and the other two lie on the sides BC and AC. Hint: Use a homothety.

3. Points A', B', C', D' are respectively the midpoints of the sides AB, BC, CD, DA of a square. What type of similarity maps $ABCD$ onto $A'B'C'D'$? Determine its center and scale. Express it as a composition of an isometry and a homothety.

4. Points M and N are respectively the midpoints of the sides AB and CD of the rectangle $ABCD$. For what ratio $|AB| : |BC|$ are the rectangles $ABCD$ and $ADNM$ similar? What type of similarity maps $ABCD$ onto $ADNM$? Write it as a composition of an isometry and a homothety.

5. The altitude CH drawn from the right angle vertex of the right triangle ABC divides the triangle into two triangles ACH and CBH similar to it. For each pair of triangles determine the type of similarity, find its center and the rotation angle in the case of spiral similarity, and the axis in the case of dilative symmetry.

6. The line a intersects the plane Π at an angle of 45°. Determine what transformations the following are:
   a) $\mathbf{S}_a \circ \mathbf{S}_\Pi$; b) $\mathbf{S}_\Pi \circ \mathbf{S}_a \circ \mathbf{S}_\Pi$; c) $\mathbf{S}_a \circ \mathbf{S}_\Pi \circ \mathbf{S}_a$; d) $\mathbf{S}_a \circ \mathbf{S}_\Pi \circ \mathbf{S}_a \circ \mathbf{S}_\Pi$.

7. Let φ be an arbitrary isometry of space. What transformations are the following:
   a) $\varphi \circ \mathbf{S}_\Pi \circ \varphi^{-1}$; b) $\varphi \circ \mathbf{T}_{\overline{u}} \circ \mathbf{R}_l^\alpha \circ \varphi^{-1}$.

8. How should the lines a, b be chosen so that the composition $\mathbf{S}_a \circ \mathbf{S}_b$ is:
   a) a translation; b) an axial symmetry; c) a rotation?

9. Describe what transformation the composition $\mathbf{S}_\Pi \circ \mathbf{S}_\Phi \circ \mathbf{S}_\Psi \circ \mathbf{S}_\Pi \circ \mathbf{S}_\Phi \circ \mathbf{S}_\Psi$ is, when

   (a) the planes Π, Φ, Ψ are coaxial,

   (b) $|\Pi \cap \Phi \cap \Psi| = 1$ (they have exactly one common point),

   (c) $\Pi \cap \Phi \cap \Psi = \emptyset$, but Π, Φ, Ψ are not coaxial.

10. Lines a, b contain the edges passing through the vertex B of a cube, lines a', b' contain respectively the opposite edges, $\Phi := Pl(a, b)$, $\Psi := Pl(a', b')$, $\Omega := Pl(b, b')$. Determine what transformations the following are:
    a) $\mathbf{S}_a \circ \mathbf{S}_{a'}$; b) $\mathbf{S}_b \circ \mathbf{S}_{a'}$; c) $\mathbf{S}_a \circ \mathbf{S}_\Omega$; d) $\mathbf{S}_b \circ \mathbf{S}_a \circ \mathbf{S}_B$; e) $\mathbf{S}_\Psi \circ \mathbf{S}_\Omega \circ \mathbf{S}_\Phi$ f) $\mathbf{S}_{b'} \circ \mathbf{S}_\Omega \circ \mathbf{S}_\Phi$.

    Describe the groups generated by: a) $\{\mathbf{S}_a, \mathbf{S}_\Phi\}$, b) $\{\mathbf{S}_B, \mathbf{S}_\Phi\}$, c) $\{\mathbf{S}_a, \mathbf{S}_B\}$, d) $\{\mathbf{S}_a, \mathbf{S}_B, \mathbf{S}_\Phi\}$.

11. Can the composition of a rotation and a central symmetry have as the set of fixed points:
    (1) $\emptyset$; (2) a oneâ€“element set; (3) a line; (4) a plane; (5) the entire space?

12. Determine the sets of invariant lines and planes for all types of isometries.

13. Describe the groups generated by:

    (a) $\{\mathbf{S}_\Psi, \mathbf{S}_\Phi\}$, when the planes Φ, Ψ intersect at an angle of 30°.

    (b) $\{\mathbf{S}_a, \mathbf{S}_b\}$, when the lines a, b intersect at an angle of 45°.

    (c) $\{\mathbf{S}_\Pi, \mathbf{S}_\Psi, \mathbf{S}_\Phi\}$, when $\Pi \perp \Phi, \Psi$ and the planes Φ, Ψ intersect at an angle of 60°.

    The groups described in (a) and (c) are the full groups of proper isometries of certain solids, and the group in (b) is the rotation group of a certain solid. Indicate these solids.

14. Describe the group of proper isometries of a regular tetrahedron.