List of geometry problems. No. 1

1. Using Tarski's axioms A1, A2 prove that the relation of congruence of segments (unordered pairs of points) is an equivalence relation.

2. Using A3 and A4 prove: (a) $AA \equiv BB$; (b) $\mathcal{B}(ABB)$.

3. Using the previous exercise and axioms A6 and A7 prove:

   (a) $\mathcal{B}(ABC) \Rightarrow \mathcal{B}(CBA)$;
   (b) $\mathcal{B}(ABC) \wedge \mathcal{B}(BAC) \Rightarrow A = B$;
   (c) $\mathcal{B}(ABD) \wedge \mathcal{B}(BCD) \Rightarrow \mathcal{B}(ABC)$.

4. Using A3 and A5: $A \neq B \wedge \mathrm{AFS}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} \to CD \equiv C'D'$ prove:

   (a) $\mathcal{B}(ABC) \wedge \mathcal{B}(A'B'C') \wedge AB \equiv A'B' \wedge BC \equiv B'C' \Rightarrow AC \equiv A'C'$ (composition of segments);
   (b) $Q \neq A \Rightarrow \exists X!(\mathcal{B}(QAX) \wedge AX \equiv BC)$ (uniqueness of laying off a segment);

5. Treating any n-gon as the set of its vertices, define in the language of Tarski (also using the abbreviation $Col(ABC)$): a) a right triangle; b) an isosceles right triangle; c) a convex quadrilateral; d) a rhombus; e) a parallelogram; f) a rectangle; g) a square; h) a trapezoid; i) any regular n-gon.

6. Using only the relation $\mathcal{B}$, define the triangle ABC as a set of points (consisting of vertices, boundary and interior).

7. Determine what kind of isometries are: a) $\mathbf{R}_A^\alpha \circ \mathbf{R}_B^\beta$ depending on the measures α, β and the position of points A, B;
   b) $\mathbf{T}_{\vec{u}} \circ \mathbf{R}_A^\alpha$; c) $\mathbf{S}_C \circ \mathbf{S}_B \circ \mathbf{S}_A$; d) $\mathbf{S}_k \circ \mathbf{S}_l \circ \mathbf{S}_k$; e) $\mathbf{R}_A^\alpha \circ \mathbf{S}_l$; f) $\mathbf{S}_m \circ \mathbf{S}_l \circ \mathbf{S}_k \circ \mathbf{S}_m \circ \mathbf{S}_l \circ \mathbf{S}_k$.

8. $A \in a, b$; $a \perp b$; $\vec{u} \parallel a$. Reduce to the simplest form:
   a) $\mathbf{S}_a \circ \mathbf{T}_{\vec{u}} \circ \mathbf{S}_b$; b) $\mathbf{S}_A \circ \mathbf{S}_b \circ \mathbf{T}_{\vec{u}}$; c) $\mathbf{S}_A \circ \mathbf{T}_{\vec{u}} \circ \mathbf{S}_b$; d) $\mathbf{S}_a \circ \mathbf{R}_A^{90^\circ} \circ \mathbf{S}_b$; e) $\mathbf{S}_A \circ \mathbf{R}_A^{90^\circ} \circ \mathbf{S}_b$; f) $\mathbf{S}_A \circ \mathbf{S}_a^{\vec{u}}$.

9. Which of the sets of isometries are subgroups and which are normal subgroups of the group of plane isometries:
   a) rotations, b) translations, c) rotations and translations, d) rotations fixing a given point A, e) isometries fixing a given point A, f) translations and central symmetries, g) isometries fixing a given line a.
   Determine which transformations make up the sets in items e) and g).

10. Characterize the groups generated by the given sets of isometries by listing all their transformations:
    a) $\{\mathbf{T}_{\vec{u}}\}$ where $\vec{u} \neq \theta$; b) $\{\mathbf{T}_{\vec{u}}, \mathbf{S}_A\}$; c) $\{\mathbf{S}_a, \mathbf{S}_b\}$ where $a \parallel b$; d) $\{\mathbf{S}_a, \mathbf{S}_b\}$ where $a \nparallel b$; determine under which assumption this group is infinite; e) $\{\mathbf{S}_A, \mathbf{S}_B\}$ where $A \neq B$; f) $\{\mathbf{S}_l^{\vec{u}}\}$; g) $\{\mathbf{S}_A, \mathbf{S}_b\}$ where $A \notin b$.

11. Prove that if a plane figure has exactly two distinct axes of symmetry, then they are perpendicular.

12. Prove that if a plane figure has two distinct centers of symmetry, then it has infinitely many of them.

13. Given two points inside an angle. Construct a parallelogram in which these points are opposite vertices and the remaining vertices lie on the arms of the angle.

14. Given an equilateral triangle and a point M inside this triangle. Determine points A and B lying on the sides of the triangle such that M is the midpoint of segment AB. Determine the number of solutions depending on the position of point M.

15. Point $M \neq C$ lies on the bisector of the exterior angle at vertex C of triangle ABC. Prove that $|MA| + |MB| > |CA| + |CB|$.

16. Point K lies on side AB of an acute triangle ABC. Determine points: L on side BC and M on side CA such that triangle KLM has the smallest perimeter.

17. Equilateral triangles ABC and BDE are oriented counterclockwise. Prove that point B and the midpoints of segments AE and CD form an equilateral triangle (or coincide).

Hint. In problems 13, 14 use central symmetry, in problems 15, 16 use axial symmetries, and in problem 17 use a rotation by 60°.