Analytic form of isometries and similarities

Theorem 1. *Let O be a fixed point of the Euclidean plane (space). Every isometry φ is a composition of an isometry ψ fixing O and a translation.*

$$\varphi = \mathbf{T}_{\overrightarrow{OO'}} \circ \psi, \text{ where } O' = \varphi(O).$$

- Denote $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in the case of the plane
- $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the case of space.
- Then every isometry can be written in the form

  $$X \mapsto AX + B$$

  where A is an orthogonal matrix (i.e. $AA^T = J$) of degree 2 or 3, respectively,

Analytic form of isometries and similarities

- and a similarity

  $$X \mapsto kAX + B$$

  where $k \in \mathbb{R}_+$.
- A similarity (isometry) is even if additionally $\det A = 1$, and odd if $\det A = -1$.
- The isometry

  $$X \mapsto AX$$

  is an involution if the orthogonal matrix A is symmetric.

Cor. 1. *Every isometry of the form $X \mapsto AX$ is:*

1. *a rotation if* $\det A = 1$,
2. *a rotary reflection if* $\det A = -1$.

Most important examples

- Rotation matrix by angle α:

  $$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

- Reflection matrices with respect to Ox and Oy:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Reflection matrix with respect to a line inclined to the positive semi-axis Ox at angle α:

$$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

Description of plane isometries and similarities using complex numbers

$z, a, b \in \mathbb{C}$ (are complex numbers), z is the complex coordinate of point Z,

- even isometries: $\varphi(z) = az + b$ where $|a| = 1$,
- odd isometries: $\varphi(z) = a\overline{z} + b$ where $|a| = 1$,
- even similarities: $\varphi(z) = az + b$ where $a \neq 0$,
- odd similarities: $\varphi(z) = a\overline{z} + b$ where $a \neq 0$,
- The analytic form of a transformation φ with a fixed point P is obtained by applying to a transformation ψ with fixed point O an inner automorphism determined by the translation $\mathbf{T}_{\overrightarrow{OP}}$

$$\varphi = \mathbf{T}_{\overrightarrow{OP}} \circ \psi \circ \mathbf{T}_{\overrightarrow{PO}}.$$

- For example:

$$\varphi(z) = i(z - z_0) + z_0$$

is the analytic form of the rotation $\mathbf{R}_{z_0}^{90^\circ}$.

Special points in a triangle

Theorem 2. *In any triangle ABC, the following triples of lines are concurrent:*

1. *medians,*
2. *perpendicular bisectors of sides,*
3. *altitudes,*
4. *internal angle bisectors,*

Standard notation for points and lengths of special segments in triangle ABC:

- G - centroid, i.e. the intersection point of medians,
- O - circumcenter,
- H - orthocenter, i.e. the intersection point of altitudes,
- I - incenter,
- a, b, c - lengths of sides opposite vertices A, B, C,
- $p := \frac{1}{2}(a+b+c)$ - semiperimeter,
- R - circumradius, r - inradius.

Def. 1. The line $l(O,G)$ passing through the centroid and the circumcenter is called the *Euler line* (for an equilateral triangle it is any line passing through $O = G$).

Theorem 3. $H = \mathbf{J}_G^{-2}(O)$.

Theorem 4 (Circumcenter of the nine-point circle)**.** *The midpoints of the sides of a triangle, the feet of the altitudes, and the midpoints of the segments joining the orthocenter with the vertices lie on one circle.*

Def. 2. The circle described in Th.4 is called the *Euler circle* (or *Feuerbach circle*) (or *nine-point circle*).

Oriented length and ratio of division

Def. 3. If $l(A,B) \parallel l(C,D)$, then the oriented (with respect to AB) length of segment CD is defined as:

$$|CD|_{AB} := \begin{cases} |CD| & \text{if vectors } \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ have the same direction} \\ -|CD| & \text{if vectors } \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ have opposite directions} \end{cases}$$

Def. 4. The oriented ratio of parallel segments AB and CD is the number $\dfrac{AB}{CD} := \dfrac{|AB|}{|CD|_{AB}}$. The number $\dfrac{AM}{MB}$ is called the oriented ratio in which point $M \neq B$ divides segment AB.

Remark 1. For any $A \neq B$ and any $x \in \mathbb{R} \setminus \{-1\}$, there exists exactly one point M such that $\dfrac{AM}{MB} = x$.

Ceva's and Menelaus' theorems

Theorem 5 (Ceva)**.** *Let A', B', C' be points different from the vertices of triangle ABC, lying respectively on the lines of sides BC, CA, and AB. The lines AA', BB', and CC' are concurrent (i.e. have a common point or are parallel) if and only if*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Theorem 6 (Menelaus)**.** *Let A', B', C' be points different from the vertices of triangle ABC, lying respectively on the lines of sides BC, CA, and AB. These points lie on one line if and only if*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = -1.$$