Theorem 1. *A transformation φ is a similarity if and only if there exists $\lambda \in \mathbb{R}_+$ such that $|\varphi(A)\varphi(B)| = \lambda|AB|$ for any points A, B.*

Theorem 2. *If two similarities φ, ψ coincide at three non-collinear points, then $\varphi = \psi$.*

Dilations

Def. 1. A *dilation* is a transformation that maps every line to a line parallel to it.

Remark 1. Dilations form a group of transformations.

Theorem 3. *A dilation has exactly one fixed point or is a translation.*

Def. 2. A dilation with a fixed point is called a *homothety.*

Theorem 4. *A homothety is a similarity.*

Def. 3. Let φ be a homothety with fixed point O and similarity ratio λ. The scale of the homothety is defined as the number $s := -\lambda$ if for $X \neq O$ we have $\mathcal{B}(XO\varphi(X))$, and as $s := \lambda$ otherwise. We use the notation $\mathbf{J}_O^s$.

Theorem 5 (Perfect homogeneity theorem)**.** *If $\neg Col(ABC)$ and*

$$\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|} = \frac{|C'A'|}{|CA|},$$

then there exists exactly one similarity φ such that $\varphi(A) = A'$, $\varphi(B) = B'$, and $\varphi(C) = C'$.

Cor. 1. *Every similarity is a composition of a homothety and an isometry.*

Def. 4. The composition of a homothety with an even or odd isometry is called an even or odd similarity, respectively.

Theorem 6. *Let $A \neq B$, and let similarities φ, ψ be both even or both odd, with $\varphi(A) = \psi(A)$ and $\varphi(B) = \psi(B)$. Then $\varphi = \psi$.*

Def. 5. A *spiral similarity* is the composition $\mathbf{R}_O^\alpha \circ \mathbf{J}_O^s$ of a homothety with a rotation having the same center, and a *dilative symmetry* is the composition $\mathbf{S}_a \circ \mathbf{J}_O^s$ of a homothety with a reflection whose axis passes through the center of the homothety.

Remark 2. In Definition 5 the order of transformations can be changed: $\mathbf{R}_O^\alpha \circ \mathbf{J}_O^s = \mathbf{J}_O^s \circ \mathbf{R}_O^\alpha$ and $\mathbf{S}_a \circ \mathbf{J}_O^s = \mathbf{J}_O^s \circ \mathbf{S}_a$ for $O \in a$.

Theorem 7. *Every non-isometric similarity has exactly one fixed point.*

Theorem 8. *Every non-isometric even similarity is a spiral similarity, and every odd one is a dilative symmetry.*

Theorem 9. *Isometries form a normal subgroup of the group of similarities.*

Half-planes and planes in dimension-free geometry

Def. 6. We say that points A, C lie:

1. on opposite sides of line b: $\mathcal{B}(AbC) :\leftrightarrow A, C \notin b \wedge \exists T(T \in b \wedge \mathcal{B}(ATC))$
2. on the same side of line b: $A \simeq_b C :\leftrightarrow \exists T(\mathcal{B}(AbT) \wedge \mathcal{B}(CbT))$.

Def. 7. 1. The half-plane determined by point A and line b is the set: $Hp(A, b) := \{X : \ X \simeq_b A\}$.

2. The plane determined by point A and line b is the set: $Pl(A, b) := b \cup Hp(A, b) \cup \{X : \ \mathcal{B}(XbA)\}$.
3. Notation: capital Greek letters, $Pl(a, b), Pl(A, B, C)$.

Upper dimension axiom for 3-dimensional space

Def. 8. Points $A_1, A_2, \ldots, A_n$ are *coplanar*: $Cp(A_1, A_2, \ldots, A_n) :\leftrightarrow \exists \Phi(A_1, A_2, \ldots, A_n \in \Phi)$

(A9') (Upper dimension axiom for 3-dimensional geometry) $P \neq Q \wedge \forall_{i=1}^{4}(A_i P \equiv A_i Q) \rightarrow Cp(A_1, A_2, A_3, A_4)$

Def. 9. The plane determined in (A9') is called the perpendicular bisector plane of segment PQ and is denoted by $Ps(P, Q)$.

Def. 10. Reflection in a plane and orthogonal projection onto a plane:

$$\mathbf{S}_\Phi(A) = A' :\leftrightarrow A = A' \vee \Phi = Ps(A, A').$$

$$\mathbf{R}_\Phi(A) = \mathbf{M}(A, \mathbf{S}_\Phi(A)).$$

Parallelism and Perpendicularity

- $a \parallel b :\leftrightarrow a = b \vee \exists \Phi(a, b \in \Phi \wedge a \cap b = \emptyset)$.
- Lines a, b are called *skew* if there is no plane Φ such that $a, b \subset \Phi$.
- $a \parallel \Phi :\leftrightarrow \exists b(a \parallel b \subset \Phi)$.
- $a \parallel b \rightarrow \forall \Phi(b \subset \Phi \rightarrow a \parallel \Phi)$.
- $a \perp \Phi :\leftrightarrow \exists A, B(a = l(A, B) \wedge \Phi = Ps(A, B))$.
- $a \perp \Phi \leftrightarrow \exists b, c \subset \Phi(b \neq c \wedge a \perp b, c) \leftrightarrow \forall b \subset \Phi(a \cap b \neq \emptyset \rightarrow a \perp b)$.
- $\Phi \perp \Omega :\leftrightarrow \exists a(\Phi \perp a \subset \Omega)$.
- $a \perp \Phi \rightarrow \forall \Omega(a \subset \Omega \rightarrow \Omega \perp \Phi)$.
- Any two skew lines a, b have a common perpendicular ($c \perp a, b$).

Theorem 10 (Rigidity for space).
1. *If $\neg Cp(A_1, A_2, A_3, A_4)$ and $(A_1A_2A_3A_4) \equiv (B_1B_2B_3B_4)$, then there exists exactly one isometry ϕ such that $\forall_{i=1}^{4} \phi(A_i) = B_i$.*
2. *$\mathbf{S}_\Phi$ is the only non-identity isometry that preserves the given three non-collinear points A, B, C ($\Phi = Pl(ABC)$).*

Theorem 11. *Every isometry of space is a composition of at most four reflections in planes.*

Theorem 12 (Reduction theorem). *If planes Φ_1, Φ_2, Φ_3 are parallel or have a common line, then there exists a plane Ω such that $\mathbf{S}_{\Phi_3} \circ \mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1} = \mathbf{S}_\Omega$.*

Classification of Euclidean space isometries

Def. 11. A composition of an even number of plane reflections is called a *proper motion*, and a composition of an odd number is called an *improper isometry*.

Def. 12. The composition $\mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1}$ is called a *translation* if $\Phi_1 \parallel \Phi_2$, and a *rotation* if $\Phi_1 \nparallel \Phi_2$. If additionally $\Phi_1 \perp \Phi_2$, the rotation is called a *half-turn* or a *reflection in a line*. Rotations are denoted $\mathbf{R}_l^\alpha$, translations $\mathbf{T}_{\overrightarrow{u}}$, half-turns $\mathbf{S}_l$.

Remark 3. The axis l and angle α do not uniquely determine the rotation.

Cor. 2. *In the representation $\mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1}$ of a given rotation or translation, one of the planes can be any plane from the corresponding bundle of planes.*

Cor. 3. $\mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1} = \mathbf{S}_{\Phi_1} \circ \mathbf{S}_{\Phi_2}$ *if and only if* $\Phi_1 \perp \Phi_2$.

Def. 13.
1. A *glide reflection* is the composition $\mathbf{T}_{\overrightarrow{u}} \circ \mathbf{S}_\Phi$, where $\overrightarrow{u} \parallel \Phi$.
2. A *rotary reflection* is the composition $\mathbf{R}_l^\alpha \circ \mathbf{S}_\Phi$, where $l \perp \Phi$.

   If $\mathbf{R}_l^\alpha$ is a half-turn (i.e., $\mathbf{R}_l^\alpha = \mathbf{S}_l$), then the rotary reflection is called a *central symmetry* and denoted $\mathbf{S}_O$, where $\{O\} := l \cap \Phi$.
3. A *screw motion* is the composition $\mathbf{R}_l^\alpha \circ \mathbf{T}_{\overrightarrow{u}}$, where $\overrightarrow{u} \parallel l$.

Cor. 4. $\mathbf{S}_{\Phi_3} \circ \mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1}$ *is: a glide reflection if $\Phi_1 \parallel \Phi_2 \perp \Phi_3$;a rotary reflection if $\Phi_1, \Phi_2 \perp \Phi_3$ and $\Phi_1 \nparallel \Phi_2$; a central symmetry if Φ_1, Φ_2, Φ_3 are mutually perpendicular.*

A screw motion is a composition $\mathbf{S}_{\Phi_4} \circ \mathbf{S}_{\Phi_3} \circ \mathbf{S}_{\Phi_2} \circ \mathbf{S}_{\Phi_1}$, where $\mathbf{S}_{\Phi_4}, \mathbf{S}_{\Phi_3} \perp \mathbf{S}_{\Phi_2} \parallel \mathbf{S}_{\Phi_1}$ and $\mathbf{S}_{\Phi_4} \nparallel \mathbf{S}_{\Phi_3}$.

Theorem 13. *Every improper isometry is either a rotary reflection or a glide reflection (including planar or central symmetries).*

Cor. 5. *Every improper isometry with a fixed point is a rotary reflection.*

Lemma 14. *Every composition of two half-turns is a screw motion.*

Theorem 15. *Every proper motion is a screw motion (including identity, translation, or rotation).*

Cor. 6. 1. *Every proper motion with a fixed point is a rotation.*

2. *The composition of two rotations about intersecting lines is a rotation.*

Cor. 7. *The group of space isometries is bi-involutive, i.e., every isometry is a composition of two involutions (reflections in planes, lines, or points).*

Remark 4. Every similarity of space is a composition of a homothety with an isometry. However, $\mathbf{S}_O = \mathbf{J}_O^{-1}$ is an improper isometry, so even and odd similarities cannot be defined in the same way as in the plane. An even similarity is the composition of an even isometry with a homothety of positive scale.

Def. 14. If $O \in l$, the composition $\mathbf{R}_l^\alpha \circ \mathbf{J}_O^s$ is called a *spiral similarity.*

Theorem 16. *Every non-isometric similarity of space is a spiral similarity (even if $s > 0$, odd if $s < 0$).*