Literature

1. H.S.M. Coxeter *Introduction to Geometry.*
2. R. Courant, H. Robbins *What is Mathematics?*
3. M. Kordos, L.W. Szczerba *Geometry for Teachers.*
4. M. Stark *Analytic Geometry.*
5. S.J. Zetel *Triangle Geometry.*
6. A. Tarski *What is elementary geometry*, in: The Axiomatic Method, North Holland, 1959.

Language of Tarski's Euclidean geometry theory

The language of Tarski's Euclidean geometry theory consists of:

1. Individual variables $A, B, C, \ldots$ denoting points,
2. two predicates (relational symbols):

   ternary "$\mathcal{B}$",

   quaternary "$\equiv$".

The formula $\mathcal{B}(ABC)$ is read: "B lies between A and C", and the formula $AB \equiv CD$ means that the segment AB is congruent to the segment CD.

Axiomatics of dimensionless absolute geometry

- A1 $AB \equiv BA$
- A2 $AB \equiv PQ \wedge AB \equiv RS \rightarrow PQ \equiv RS$
- A3 $AB \equiv CC \rightarrow A = B$
- A4 $\exists X(\mathcal{B}(QAX) \wedge AX \equiv BC)$ (segment construction axiom)
- A5 $(A \neq B \wedge \mathcal{B}(ABC) \wedge \mathcal{B}(A'B'C') \wedge AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D') \rightarrow CD \equiv C'D'$ (five-segment axiom)
- A6 $\mathcal{B}(ABA) \rightarrow A = B$
- A7 $\mathcal{B}(APC) \wedge \mathcal{B}(BQC) \rightarrow \exists X(\mathcal{B}(PXB) \wedge \mathcal{B}(QXA))$ (Pasch's axiom)

Dimension axioms

- A8 $\exists A, B, C(\neg\mathcal{B}(ABC) \wedge \neg\mathcal{B}(BCA) \wedge \neg\mathcal{B}(CAB))$ (lower dimension axiom)
- A9 $(P \neq Q \wedge AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ) \rightarrow (\mathcal{B}(ABC) \vee \mathcal{B}(BCA) \vee \mathcal{B}(CAB))$ (upper dimension axiom)

Euclid's axiom and its equivalent form

A10 $\mathcal{B}(ADT) \wedge \mathcal{B}(BDC) \wedge A \neq D \rightarrow \exists X, Y(\mathcal{B}(ABX) \wedge \mathcal{B}(ACY) \wedge \mathcal{B}(XTY))$

A10′ $\mathcal{B}(ABC) \wedge \mathcal{B}(CDE) \wedge \mathcal{B}(EFA) \wedge AB \equiv BC \wedge CD \equiv DE \wedge EF \equiv FA \rightarrow FA \equiv BD$

Continuity axiom

A11 $\forall \mathcal{X}, \mathcal{Y}\{\exists A \forall X, Y[X \in \mathcal{X} \wedge Y \in \mathcal{Y} \rightarrow \mathcal{B}(AXY)] \rightarrow \exists B \forall X, Y[X \in \mathcal{X} \wedge Y \in \mathcal{Y} \rightarrow \mathcal{B}(XBY)]\}$

Models of Tarski's theory (*Euclidean planes*) are structures of the form: $\langle \mathcal{P}, \mathcal{B}, \equiv \rangle$, where $\mathcal{P}$ is a set of points and $\mathcal{B}$, $\equiv$ are relations defined on it.

Cartesian Euclidean plane

Definition 1. Let $\mathbf{F} = \langle F, +, \cdot, 0, 1, \leq \rangle$ be an ordered Pythagorean field. The Cartesian Euclidean plane over the field $\mathbf{F}$ is the structure $\mathcal{C}_2(\mathbf{F}) = \langle F^2, \equiv_{\mathbf{F}}, \mathcal{B}_{\mathbf{F}} \rangle$.

Representation theorem

Theorem 1. *Every model of axioms* A1, …, A10 *is isomorphic to a Cartesian Euclidean plane over an ordered Pythagorean field* $\mathbf{F}$*. If* A11 *is added, then* $\mathbf{F}$ *is the field of real numbers* $\mathbb{R}$ *(the theory is categorical).*

Notation and abbreviations

Def. $Col(ABC) :\leftrightarrow (\mathcal{B}(ABC) \vee \mathcal{B}(BCA) \vee \mathcal{B}(CAB))$

Def. $(A_1, A_2, \ldots, A_n) \equiv (A'_1, A'_2, \ldots, A'_n) :\leftrightarrow \forall_{i,j=1}^{n}(A_iA_j \equiv A'_iA'_j)$

Def. $\mathrm{AFS}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} :\leftrightarrow (\mathcal{B}(ABC) \wedge \mathcal{B}(A'B'C') \wedge AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D')$

A5: $A \neq B \wedge \mathrm{AFS}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} \rightarrow CD \equiv C'D'$

Def. $\mathrm{FS}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} :\leftrightarrow (Col(ABC) \wedge Col(A'B'C') \wedge AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D')$

Theorem 2. $A \neq B \wedge \mathrm{FS}\begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \end{pmatrix} \rightarrow CD \equiv C'D'$

Similarities and isometries

Definition 2. A similarity is any automorphism of the structure $\langle \mathcal{P}, \mathcal{B}, \equiv \rangle$, i.e. a bijection of the set of points $\mathcal{P}$ preserving the relations $\mathcal{B}$ and $\equiv$.

Definition 3. An isometry is a mapping $\varphi : \mathcal{P} \to \mathcal{P}$ satisfying: $\varphi(X)\varphi(Y) \equiv XY$.

Theorem 3 (Theorem 2)**.** $\mathcal{B}(ABC) \wedge (ABC) \equiv (A'B'C') \to \mathcal{B}(A'B'C')$.

Theorem 4 (Theorem 3)**.** *Every isometry is a similarity.*

Definitions and notation

1. Line through points $A \neq B$: $l(A,B) := \{X \mid Col(ABX)\}$.
2. Ray with origin A through B: $h(A,B) := \{X \mid \mathcal{B}(ABX) \vee \mathcal{B}(AXB)\}$.
3. Perpendicular bisector of segment AB ($A \neq B$): $s(A,B) := \{X \mid AX \equiv BX\}$.
4. Midpoint: $\mathbf{M}(A,B) = X \leftrightarrow \mathcal{B}(AXB) \wedge AX \equiv XB$.
5. Central symmetry: $\mathbf{S}_O(A) = A' :\leftrightarrow \mathbf{M}(A,A') = O$.

Def. A transformation φ is called an involution if $\varphi \circ \varphi = id$ and $\varphi \neq id$.

Thm. $\mathbf{S}_O$ is an involutive isometry.

Definitions and notation (cont.)

6. Points A, C, B form a right angle:

   $\mathcal{R}(ACB) :\leftrightarrow AB \equiv A\mathbf{S}_C(B)$.
7. $a \perp b :\leftrightarrow$

   $\exists A, C, B(\mathcal{R}(ACB) \wedge C \in a, b \wedge A \in a \wedge B \in b \wedge C \neq A, B)$.
8. Orthogonal projection of point A onto line l:

   $\mathbf{R}_l(A) = A' :\leftrightarrow \forall X \in l(\mathcal{R}(AA'X))$.
9. Axial symmetry:

   $\mathbf{S}_l(A) = A' :\leftrightarrow \mathbf{M}(A,A') = \mathbf{R}_l(A)$.

Thm. $\mathbf{S}_l$ is an involutive isometry.

Thm. Let $A \in a, b$. Then:

$a \perp b \leftrightarrow \mathbf{S}_A = \mathbf{S}_a \circ \mathbf{S}_b \leftrightarrow \mathbf{S}_b \circ \mathbf{S}_a = \mathbf{S}_a \circ \mathbf{S}_b \neq id$.

Definitions and notation (cont.)

10. $a \parallel b :\leftrightarrow a = b \vee a \cap b = \emptyset$.

Thm. $a \parallel b \leftrightarrow \exists c(a, b \perp c)$.

11. A bound vector is an ordered pair of points.

    Congruence of bound vectors: $(A, B) \equiv (C, D) :\leftrightarrow \mathbf{M}(A, D) = \mathbf{M}(B, C)$.

    A free vector $\overrightarrow{AB}$ is the equivalence class of the congruence relation with representative (A, B).

Angles

12. A (directed) angle is an ordered pair of rays with a common origin.

    Congruence relation of angles:

    $\sphericalangle ABC \equiv \sphericalangle DEF \leftrightarrow \exists A', C', D', F'(A' \in h(B, A) \wedge C' \in h(B, C) \wedge D' \in h(E, D) \wedge F' \in h(E, F) \wedge A', C' \neq B \wedge (A'BC') \equiv (D'EF'))$.

    Equivalence classes of the congruence relation of angles are free angles.

- Directed angles are congruent if there exists a composition of two axial symmetries that maps one onto the other.

  Equivalence classes of the congruence relation of directed angles are free directed angles.

Theorem 5 (Theorem 7 — rigidity for a line)**.** *If an isometry fixes two distinct points, then it also fixes all points collinear with them.*

Theorem 6 (Rigidity for the plane)**.** *If an isometry fixes three noncollinear points, then it fixes all points of the Euclidean plane (it is the identity).*

Corollary 1. *An isometry fixing two distinct points is either the identity or an axial symmetry.*

Theorem 7. *If $\neg Col(ABC)$ and $(ABC) \equiv (A'B'C')$, then there exists a composition of two or three axial symmetries that maps A, B, C respectively onto A', B', C'.*

Theorem 8 (On perfect homogeneity)**.** *If $\neg Col(ABC)$ and $(ABC) \equiv (A'B'C')$, then there exists exactly one isometry φ such that $\varphi(A) = A'$, $\varphi(B) = B'$, and $\varphi(C) = C'$.*

Corollary 2. *Every isometry of the plane is a composition of two or three axial symmetries.*

Theorem 9. *The composition of two axial symmetries is not an axial symmetry.*

Theorem 10 (Reduction theorem). *If the lines a, b, c belong to the same pencil (i.e. are parallel or intersect in one point), then there exists a line d belonging to the same pencil such that $\mathbf{S}_d = \mathbf{S}_c \circ \mathbf{S}_b \circ \mathbf{S}_a$.*

Definition 4. A composition of two axial symmetries is called an *even isometry*, and a composition of three axial symmetries is called an *odd isometry.*

Definition 5. The composition $\mathbf{S}_l \circ \mathbf{S}_k$ is called a *translation* if $l \parallel k$, and a *rotation* if $l \cap k \neq \emptyset$.

Corollary 3. *In the representation $\mathbf{S}_b \circ \mathbf{S}_a$ of a rotation or a translation, one of the lines can be chosen arbitrarily from the corresponding pencil.*

Remark 1. If $\varphi = \mathbf{S}_b \circ \mathbf{S}_a$ is a translation, $A \in a$, $B = \mathbf{R}_b(A)$, then for any point X we have $\overrightarrow{X\varphi(X)} = 2\overrightarrow{AB} =: \overrightarrow{u}$. If $\psi = \mathbf{S}_b \circ \mathbf{S}_a$ is a rotation and β is the measure of the directed angle from a to b, then for any $X \neq O$ we have $|\sphericalangle XO\psi(X)| = 2\beta =: \alpha$. We use the notation $\varphi = \mathbf{T}_{\overrightarrow{u}}$ and $\psi = \mathbf{R}_O^{\alpha}$.

Definition 6. The composition $\mathbf{S}_b \circ \mathbf{S}_A$ of a central symmetry with an axial symmetry is called a *glide reflection*.

Theorem 11. *Every odd isometry is a glide reflection. If $A \in b$, then $\mathbf{S}_b \circ \mathbf{S}_A$ is an axial symmetry.*

Remark 2. If $A \in l \perp b$ and $B \in l, b$, then $\mathbf{S}_b \circ \mathbf{S}_A = \mathbf{T}_{2\overrightarrow{AB}} \circ \mathbf{S}_l = \mathbf{S}_l \circ \mathbf{T}_{2\overrightarrow{AB}}$.

Corollary 4. *The group of isometries of the Euclidean plane is bi-involutive, i.e. every isometry is a composition of two involutions.*

Theorem 12. *For any isometry φ, point A, and line a the following holds:*

1. $\varphi \circ \mathbf{S}_a \circ \varphi^{-1} = \mathbf{S}_{\varphi(a)}$,
2. $\varphi \circ \mathbf{S}_A \circ \varphi^{-1} = \mathbf{S}_{\varphi(A)}$.