Mathematics. Multivariable Calculus

Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn

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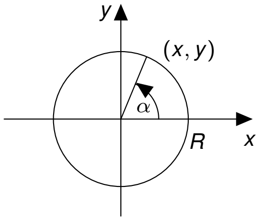
Polar coordinates



Let $A := \{(x, y); x^2 + y^2 \leq R\}$. Then, to compute $\int_A f\, dA$ we can use polar coordinates

$$\begin{cases} x = r\cos\alpha \\ y = r\sin\alpha \end{cases} \qquad r \in [0, R], \quad \alpha \in [0, 2\pi), \quad dA = r dr d\alpha,$$

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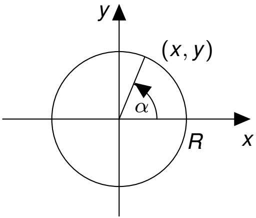

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Thus,

$$\int_A f(x, y)\, dA = \int_0^{2\pi} \int_0^R f(x, y)\, r\, dr\, d\alpha.$$

Polar coordinates

In particular,

- If $A := \{(x, y) \in R^2;\ \ (x-a)^2 + (y-b)^2 = R^2\}$, then we use the following change of coordinates:

$$\begin{cases} x = a + r\cos\alpha \\ y = b + r\sin\alpha \end{cases} \qquad r \in [0, R], \quad \alpha \in [0, 2\pi), \quad dA = r\,dr\,d\alpha,$$

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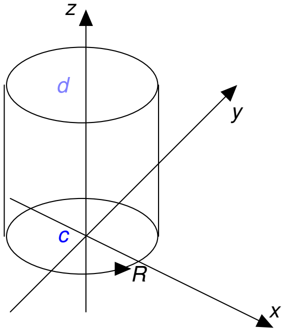
$$\begin{cases} x = a\,r\cos\alpha \\ y = b\,r\sin\alpha \end{cases} \qquad r \in [0, 1], \quad \alpha \in [0, 2\pi), \quad dA = abr\,dr\,d\alpha,$$

Cylindrycal coordinates

Let the set A will be a cylinder:



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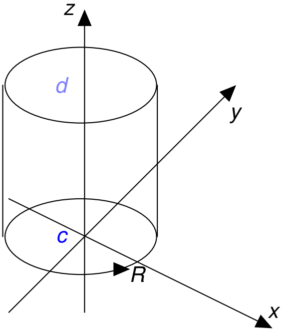

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$$\begin{cases} x = r\cos\alpha \\ y = r\sin\alpha \\ z = z \qquad\qquad r \in [0, R], \quad \alpha \in [0, 2\pi), \quad z \in [c, d] \quad dA = r dr d\alpha dz, \end{cases}$$

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- If base of a cylider is a circle with a center at (a, b), then we use the following change of coordinates:

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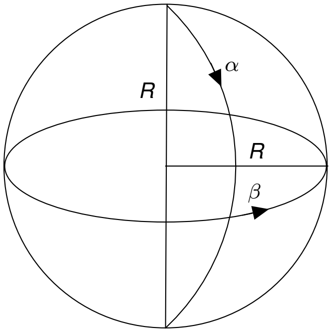
Spherical coordinates



$$r \in [0, R]$$
$$\alpha \in [0, \pi]$$
$$\beta \in [0, 2\pi)$$

Let $A := \{(x, y); x^2 + y^2 + z^2 \leq R\}$. Then, to compute $\int_A f \; dA$ we can use spherical coordinates

$$\begin{cases} x = r \sin\alpha \cos\beta \\ y = r \sin\alpha \sin\beta \\ z = r \cos\alpha \end{cases} \qquad dA = r^2 \sin\alpha dr d\alpha d\beta,$$

The velocity vector

Definition

The velocity vector of a moving object is a vector $\overrightarrow{v}$ such that:

1. *The magnitude of $\overrightarrow{v}$ is the speed of the object.*
2. *The direction of $\overrightarrow{v}$ is the direction of motion.*

Thus the speed of the object is $||\overrightarrow{v}||$ and the velocity vector is tangent to the object's path.

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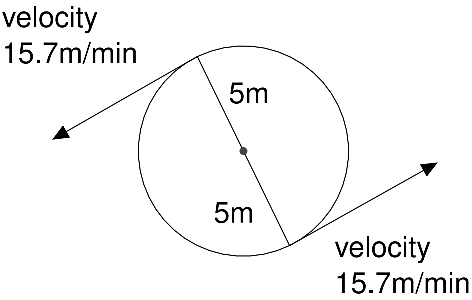
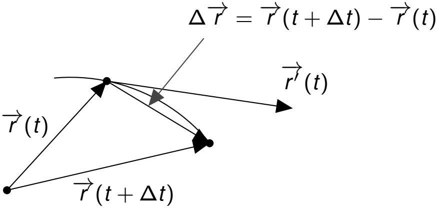
Exercise 6.7 A child is sitting on a ferris wheel of diameter 10 meters, making one revolution every 2 minutes. Find the speed of the child and draw velocity vectors at two different times.



Solution. One revolution around a circle of radius 5 is a distance of 10π, so the child's speed is $\frac{10\pi}{2} = 5\pi \approx 15.7 m/min$. The direction of motion is tangent to the circle, and hence perpendicular to the radius at that point.

Computing the velocity



Definition

The ***velocity vector****,*

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t},$$

whenever the limit exists. We use the notation $\vec{v} = \frac{d\vec{r}}{dt} = \vec{r}'(t)$.

The components of the velocity vector

If we represent a curve parametrically by $x = f(t), y = g(t)$, $z = h(t)$, then we can write its position vector as:

$$\overrightarrow{r}(t) = [f(t), g(t), h(t)].$$

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In this situation,

$$\frac{\Delta \overrightarrow{r}(t + \Delta t) - \overrightarrow{r}(t)}{\Delta t} =$$

$$= \Big[\frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t}\Big].$$

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In the limit as Δt goes to zero we can see that the components of the velocity vector of a particle moving in space with position vector $\overrightarrow{r}(t)$ at time t are given by

$$\overrightarrow{v}(t) = [f'(t), g'(t), h'(t)].$$

Exercise 8.8

Find the components of the velocity vector for the child on the ferris wheel in Exercise 6.7 using a coordinate system which has its origin at the center of the ferris wheel and which makes the rotation counterclockwise.

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$$\vec{r}(t) = [5\cos(\omega t), 5\sin(\omega t)],$$

where ω is chosen to make the period 2 minutes,

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Thus, the motion is described by the equation

$$\overrightarrow{r}(t) = [5\cos(\pi t), 5\sin(\pi t)],$$

where t is in minutes.

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To see that the direction is correct we compute that

$$\overrightarrow{v} \cdot \overrightarrow{r} = 0$$

So the velocity vector, $\overrightarrow{v}$, is perpendicular to $\overrightarrow{r}$ and hence tangent to the circle.

The acceleration vector

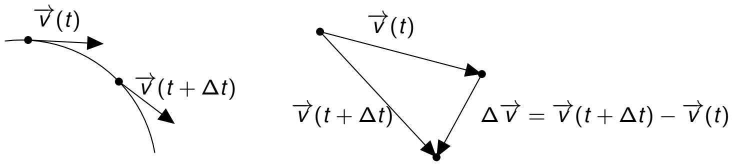

Figure. Computing the difference between two velocity vectors

Definition

The acceleration vector *of an object moving with velocity $\vec{v}(t)$ at time t is*

$$\vec{a}(t) = \lim_{\Delta t \to 0} = \frac{\Delta\vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t},$$

if a limit exists. We use the notation $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \vec{r}\,''(t)$.

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we can express the acceleration in components. Remind, that the velocity vector is given by

$$\overrightarrow{v}(t) = [f'(t), g'(t), h'(t)].$$

So, from the definition of the acceleration vector, we can compute that

$$\overrightarrow{a}(t) = [f''(t), g''(t), h''(t)].$$

Exercise 8.9

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Solution. The child's position vector is given by

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and the velocity vector is

$$\vec{v}(t) = \Big[-5\pi\sin(\pi t), 5\pi\cos(\pi t)\Big].$$

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Thus, the acceleration vector is

$$\vec{a}(t) = \Big[-5\pi^2\cos(\pi t), -5\pi^2\sin(\pi t)\Big].$$

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Notice, that $\vec{a}(t) = -\pi^2\vec{r}(t)$. Thus, the acceleration vector is a multiple of $\vec{r}(t)$ and points toward the origin.

Fact

Uniform circular motion: For a particle whose motion is described by

$$\overrightarrow{r}(t) = [R\cos(\omega t), R\sin(\omega t)]$$

1. *Motion is in the circle of radius R with period* $\frac{2\pi}{\omega}$.
2. *Velocity,* $\overrightarrow{v}(t)$, *is tangent to the circle and speed is constant* $||\overrightarrow{v}(t)|| = \omega R$.
3. *Accelaration,* $\overrightarrow{a}(t)$, *points toward the center of the circle with* $||\overrightarrow{a}(t)|| = ||\overrightarrow{v}(t)||^2/R$.

Exercise 8.10

Consider the motion given by the vector equation

$$\overrightarrow{r}(t) = \overrightarrow{r_0} + f(t)\overrightarrow{v_0},$$

where $\overrightarrow{r_0} = [2, 6, 0]$, $f(t) = t^3 + t$ and $\overrightarrow{v_0} = [4, 3, 1]$. Show that this is straight line motion in the direction of the vector $[4, 3, 1]$ and relate the acceleration vector to the velocity vector.

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Fact

Motion in a straight line: For a particle whose motion is described by

$$\vec{r}(t) = \vec{r_0} + f(t)\vec{v_0}$$

1. *Motion is along the straight line through the point with position vector $\vec{r_0}$ parallel to $\vec{v_0}$.*
2. *Velocity, $\vec{v}$, and acceleration, $\vec{a}$, are parallel to the line.*

The length of curve

The speed of a particle is the magnitude of its velocity vector:

$$\text{Speed} = ||\overrightarrow{v}|| = \sqrt{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right) + \left(\frac{dz}{dt}\right)}.$$

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Theorem

If the curve C is given parametrically for $a \leq t \leq b$ by smooth functions and if the velocity vector $\overrightarrow{v}$ is not $\overrightarrow{0}$ for $a < t < b$, then

$$\textit{Length of } C = \int_a^b ||\overrightarrow{v}(t)|| dt.$$

Exercise 8.11

Find the circumference of the ellipse given by the parametric equations

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Solution.

$$\text{Circumference} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)} dt =$$

$$\int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\; dt = 9.69.$$