Mathematics. Multivariable Calculus

Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn

May 20, 2014

Parametric equations in three dimentions

A curve in the plane may be parametrized by a pair of equations of the form $x = f(t), y = g(t)$. To describe a motion in 3-dimensional space parametrically, we need a third equation giving z in terms of t.

Parametric equations in three dimentions

A curve in the plane may be parametrized by a pair of equations of the form $x = f(t), y = g(t)$. To describe a motion in 3-dimensional space parametrically, we need a third equation giving z in terms of t.

Exercise 8.1 Find the parametric equations for the curve $y = x^2$.

Parametric equations in three dimentions

A curve in the plane may be parametrized by a pair of equations of the form $x = f(t), y = g(t)$. To describe a motion in 3-dimensional space parametrically, we need a third equation giving z in terms of t.

Exercise 8.1 Find the parametric equations for the curve $y = x^2$.

Solution. A possible parametrization in two dimensions is $x = t, y = t^2$. Since the curve is in the xy-plane, the z-coordinate is zero, so the parametrization in three dimensions is

$$x = t, y = t^2, z = 0.$$

Exercise 8.2

Find parametrical equations for the particle that starts at $(0, 3, 0)$ and moves around a circle in the *yz*-plane in the clockwise direction.

Exercise 8.2

Find parametrical equations for the particle that starts at $(0, 3, 0)$ and moves around a circle in the yz-plane in the clockwise direction.

Solution. Since the motion is in the yz-plane, we have $x = 0$ at all times t. Looking at the yz-plane from the positive x-direction we see motion around a circle of radius 3 in the clocwise direction. Thus

$$x = 0, y = 3\cos t, z = -3\sin t.$$

Exercise 8.3

Describe in words the motion given parametrically by

$$x = \cos t, y = \sin t, z = t.$$

Parametric equations of a line

Definition

Parametric equations of a line through the point (x_0, y_0, z_0) *and parallel to the vector* $\overrightarrow{r} = [a, b, c]$ *are*

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Exercise 8.4

(a) Describe in the words the curve given by the parametric equations $x = 3 + t, y = 2t, z = 1 - t$.

(b) Find parametric equations for the line through the points $(1, 2, -1)$ and $(3, 3, 4)$.

Exercise 8.4

(a) Describe in the words the curve given by the parametric equations $x = 3 + t, y = 2t, z = 1 - t$.

(b) Find parametric equations for the line through the points $(1, 2, -1)$ and $(3, 3, 4)$.

Solution. (a) The curve is a line through the point $(3, 0, 1)$ and parallel to the vector $\overrightarrow{u} = [1, 2, -1]$.

Exercise 8.4

(a) Describe in the words the curve given by the parametric equations $x = 3 + t, y = 2t, z = 1 - t$.
(b) Find parametric equations for the line through the points $(1, 2, -1)$ and $(3, 3, 4)$.

Solution. (a) The curve is a line through the point $(3, 0, 1)$ and parallel to the vector $\overrightarrow{u} = [1, 2, -1]$.

(b) The line is parallel to the vector $\overrightarrow{PQ} = [2, 1, 5]$. Thus, using the point P, the parametric equations are

$$x = 1 + 2t, y = 2 + t, z = -1 + 5t.$$

Exercise 8.4

(a) Describe in the words the curve given by the parametric equations $x = 3 + t, y = 2t, z = 1 - t$.

(b) Find parametric equations for the line through the points $(1, 2, -1)$ and $(3, 3, 4)$.

Solution. (a) The curve is a line through the point $(3, 0, 1)$ and parallel to the vector $\overrightarrow{u} = [1, 2, -1]$.

(b) The line is parallel to the vector $\overrightarrow{PQ} = [2, 1, 5]$. Thus, using the point P, the parametric equations are

$$x = 1 + 2t, y = 2 + t, z = -1 + 5t.$$

Using the point Q gives the equations

$$x = 3 + 2t, y = 3 + t, z = 4 + 5t,$$

which represent the same line.

Position vector

A point in the plane with coordinates (x, y) can be represented by the position vector $\overrightarrow{r} = [x, y]$.

Position vector

A point in the plane with coordinates (x, y) can be represented by the position vector $\overrightarrow{r} = [x, y]$.

Exercise 8.5 Use vectors to give a parametrization for the circle of radius $\frac{1}{2}$ at the point $(-1, 2)$.

Position vector

A point in the plane with coordinates (x, y) can be represented by the position vector $\overrightarrow{r} = [x, y]$.

Exercise 8.5 Use vectors to give a parametrization for the circle of radius $\frac{1}{2}$ at the point $(-1, 2)$.

Solution. The circle of radius 1 centered at the origin is parametrized by the vector-valued function

$$\overrightarrow{r_0} = [\cos t, \sin t], \quad 0 \leq t \leq 2\pi.$$

Position vector

A point in the plane with coordinates (x, y) can be represented by the position vector $\overrightarrow{r} = [x, y]$.

Exercise 8.5 Use vectors to give a parametrization for the circle of radius $\frac{1}{2}$ at the point $(-1, 2)$.

Solution. The circle of radius 1 centered at the origin is parametrized by the vector-valued function

$$\overrightarrow{r_0} = [\cos t, \sin t], \quad 0 \leq t \leq 2\pi.$$

Thus, the parametrization is

$$\overrightarrow{r} = \left[-1 + \frac{1}{2}\cos t, 2 + \frac{1}{2}\sin t\right] \quad 0 \leq t \leq 2\pi,$$

Position vector

A point in the plane with coordinates (x, y) can be represented by the position vector $\overrightarrow{r} = [x, y]$.

Exercise 8.5 Use vectors to give a parametrization for the circle of radius $\frac{1}{2}$ at the point $(-1, 2)$.

Solution. The circle of radius 1 centered at the origin is parametrized by the vector-valued function

$$\overrightarrow{r_0} = [\cos t, \sin t], \quad 0 \le t \le 2\pi.$$

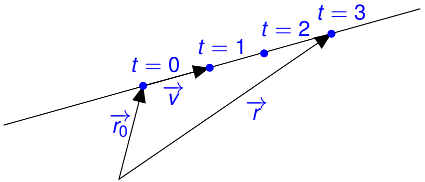
Thus, the parametrization is

$$\overrightarrow{r} = \left[-1 + \frac{1}{2}\cos t, 2 + \frac{1}{2}\sin t \right] \quad 0 \le t \le 2\pi,$$

or equivalently,

$$x = -1 + \frac{1}{2}\cos t, \quad y = 2 + \frac{1}{2}\sin t, \quad 0 \le t \le 2\pi.$$

Paramatric equation of a line



Definition

The line through the point with position vector $\overrightarrow{r_0} = [x_0, y_0, z_0]$ *in the direction of the vector* $\overrightarrow{v} = [a, b, c]$ *has parametric equation*

$$\overrightarrow{r}(t) = \overrightarrow{r_0} + t\overrightarrow{v}.$$

Exercise 8.6

1. Find parametric equations for the line passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.

Exercise 8.6

1. Find parametric equations for the line passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.
   Solution.
   $$\overrightarrow{r}(t) = [2, -1, 3] + t[-3, 6, 1].$$

Exercise 8.6

1. Find parametric equations for the line passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.
   Solution.
   $$\overrightarrow{r}(t) = [2, -1, 3] + t[-3, 6, 1].$$

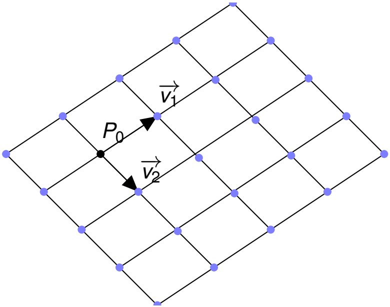
2. Represent the line segment from $(2, -1, 3)$ to $(-1, 5, 4)$ parametrically.

Exercise 8.6

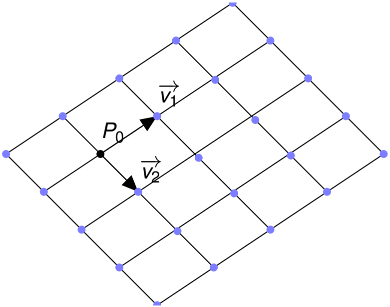
1. Find parametric equations for the line passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.
   Solution.
   $$\overrightarrow{r}(t) = [2, -1, 3] + t[-3, 6, 1].$$

2. Represent the line segment from $(2, -1, 3)$ to $(-1, 5, 4)$ parametrically.
   Solution.
   $$\overrightarrow{r}(t) = [2, -1, 3] + t[-3, 6, 1], \; 0 \leq t \leq 1.$$

Parameterizing planes



Parameterizing planes



Definition

Parametric equation for the plane

The plain through the point with position vector $\vec{r_0}$ and containing the two nonparallel vectors $\vec{v_1}$ and $\vec{v_2}$ has parametric equation

$$\vec{r}(s,t) = \vec{r_0} + s\vec{v_1} + t\vec{v_2}$$

Parameterizing planes

If $\overrightarrow{r_0} = [x_0, y_0, z_0]$, and $\overrightarrow{v_1} = [a_1, a_2, a_3]$, and $\overrightarrow{v_2} = [b_1, b_2, b_3]$, then the parametric equations of the plane can be written in the form:

$$x = x_0 + sa_1 + tb_1, \quad y = y_0 + sa_2 + tb_2, \quad z = z_0 + sa_3 + tb_3.$$

Parameterizing planes

If $\overrightarrow{r_0} = [x_0, y_0, z_0]$, and $\overrightarrow{v_1} = [a_1, a_2, a_3]$, and $\overrightarrow{v_2} = [b_1, b_2, b_3]$, then the parametric equations of the plane can be written in the form:

$$x = x_0 + sa_1 + tb_1, \quad y = y_0 + sa_2 + tb_2, \quad z = z_0 + sa_3 + tb_3.$$

Exercises 8.7 Write a parametric equation for the plane through the point $(2, -1, 3)$ and containing the vectors $\overrightarrow{v_1} = [2, 3, -1]$ and $\overrightarrow{v_2} = [1, -4, 5]$.

Parameterizing planes

If $\overrightarrow{r_0} = [x_0, y_0, z_0]$, and $\overrightarrow{v_1} = [a_1, a_2, a_3]$, and $\overrightarrow{v_2} = [b_1, b_2, b_3]$, then the parametric equations of the plane can be written in the form:

$$x = x_0 + sa_1 + tb_1, \quad y = y_0 + sa_2 + tb_2, \quad z = z_0 + sa_3 + tb_3.$$

Exercises 8.7 Write a parametric equation for the plane through the point $(2, -1, 3)$ and containing the vectors $\overrightarrow{v_1} = [2, 3, -1]$ and $\overrightarrow{v_2} = [1, -4, 5]$.

Solution. The parametric equation is

$$x = 2 + 2s + t, \ \ y = -1 + 3s - 4t, \ \ z = 3 - s + 5t.$$