Mathematics. Multivariable Calculus

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Example

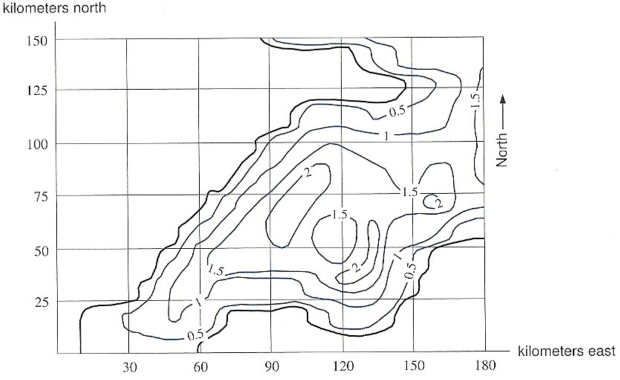

Figure 16.1: Population density of foxes in southwestern England

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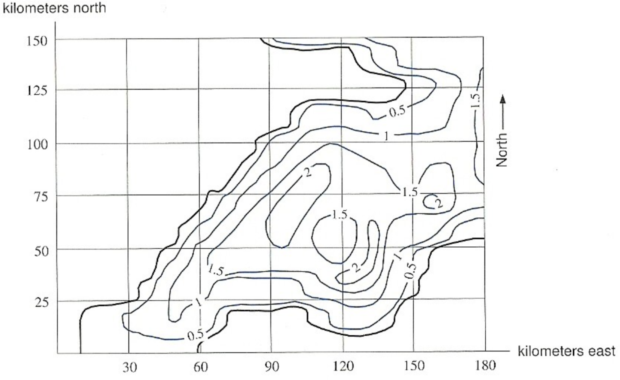

Figure 16.1: Population density of foxes in southwestern England

Estimate population = 18 000 foxes

Lower estimate = 4 000; Upper estimate = 35 000

$\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$
$L_{ij} = \inf\{f(x, y) : \ x \in [x_{i-1}, x_i), \ y \in [y_{j-1}, y_j)\}$
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Definition

Suppose the function f is continuous on $R = [a,b] \times [c,d]$. The ***definite integral*** *(**double integral**) of f over R is the quantity*

$$\int_R f\, dA = \lim_{\Delta x \to 0,\, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x_i \Delta y_j$$

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Suppose the function f is bounded on R. The ***definite integral*** *(**double integral**) of f over R is the quantity*

$$\int_R f\, dA = \sup_{\mathcal{P}} \sum_{i,j} L_{ij} \Delta x_i \Delta y_j = \inf_{\mathcal{P}} \sum_{i,j} M_{ij} \Delta x_i \Delta y_j$$

Interpretation if the Double Integral as Volume

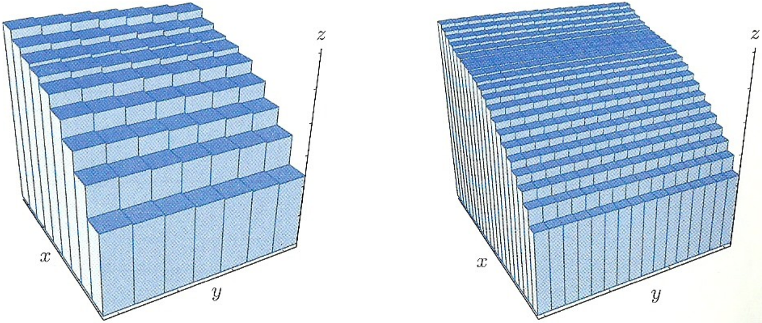

Figure 16.3: Approximating volume under a graph with finer and finer Riemann sums

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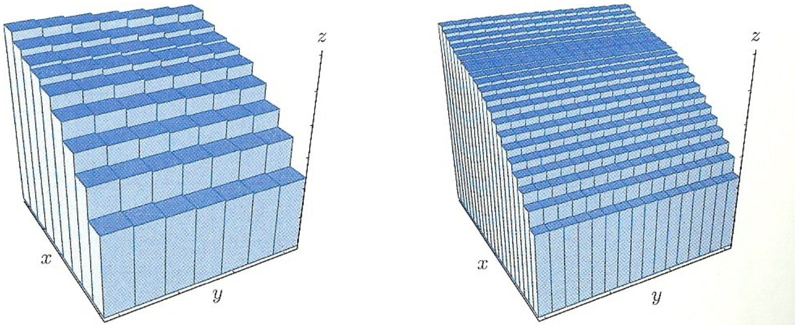

Figure 16.3: Approximating volume under a graph with finer and finer Riemann sums

Fact

If x, y, z, represent length and f is non-negative, then

$$\int_R f\, dA = \text{ volume under graph of } f \text{ above region } R.$$

Interpretation if the Double Integral as Average Value

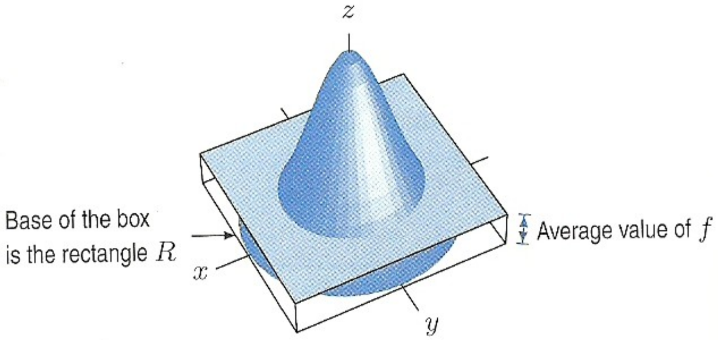

Figure 16.5: Volume and average value

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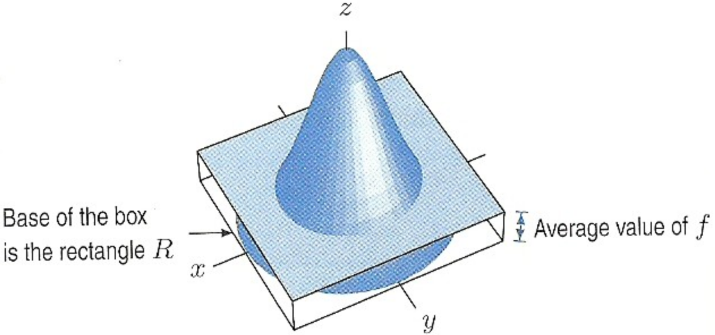

Figure 16.5: Volume and average value

Fact

$$\frac{1}{\textit{Area of } R} \int_R f \, dA = \textit{ Average value of } f \textit{ on the region } R.$$

Exercise 7.1

Decide (without calculation) whether the integrals are positive, negative, or zero. Let D be the region inside the unite circle centered at the origin, let R be the right half of D and let B be the bottom half of D.

1. $\int_D dA$;
2. $\int_B dA$;
3. $\int_R 5x\,dA$;
4. $\int_B 5x\,dA$;
5. $\int_D 5x\,dA$;
6. $\int_D (y^3 + y^5)\,dA$;
7. $\int_B (y^3 + y^5)\,dA$;
8. $\int_R (y^3 + y^5)\,dA$;
9. $\int_B (y - y^3)\,dA$;
10. $\int_D (y - y^3)\,dA$;
11. $\int_D \sin y\,dA$;
12. $\int_D \cos y\,dA$;
13. $\int_D e^x\,dA$;
14. $\int_D xe^x\,dA$;
15. $\int_D xy^2\,dA$;
16. $\int_B x\cos y\,dA$.

Exercise 7.2

Let $f(x, y)$ be a function of x and y which is independent of y, that is, $f(x, y) = g(x)$ for some one variable function g.

(a) What does the graph of f look like?

(b) Let R be the rectangle $a \leq x \leq b$, $c \leq y \leq d$. By interpreting the integral as a volume, and using your answer to part (a), express $\int_R f\,dA$ in terms of a one-variable integral.

Exercise 7.3

It is known that

$$|a+b| \leq |a| + |b|,$$

for any numbers a and b.

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Use this to explain why

$$\left| \int_R f \, dA \right| \leq \int_R |f| \, dA.$$

Iterated Integral

Fact

If R is the rectangle $a \leq x \leq b$, $c \leq y \leq d$ and f is a continuous function on R, then the double integral of f over R is equal to the ***iterated integral***

$$\int_R f\, dA = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

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Exercise 7.4

A building is 8 miters wide and 16 meters long. It has a flat roof that is 12 meters high at one corner, and 10 meters high at each of the adjacent corners. What is the volume of the building?

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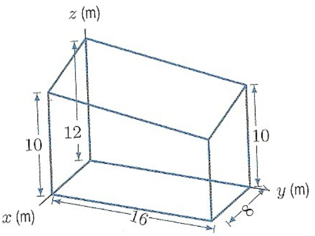

Figure 16.12: A slant-roofed building

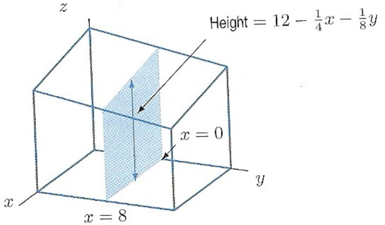

Figure 16.13: Cross-section of a building

Solution of Exercise 7.4

$$f(x, y) = 12 - \frac{x}{4} - \frac{y}{8}$$

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Iterated Integrals Over Non-Rectangular Regions

The density at the point (x, y) of a triangular metal plate, as shown below, is $\delta(x, y)$.

Express its mass as an iterated integral.

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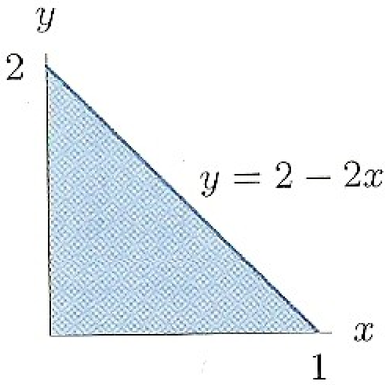


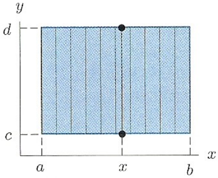
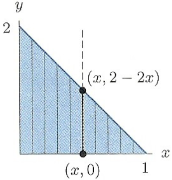


Figure 16.15: Integrating over a rectangle using vertical strips



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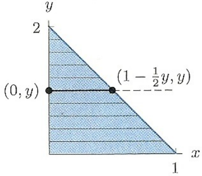
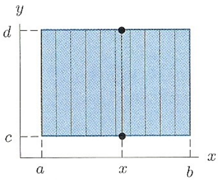


Figure 16.17: Integrating over a triangle using horizontal strips

$$\text{Mass } = \int_0^1 \int_0^{2-2x} \delta(x, y)\, dy\, dx$$



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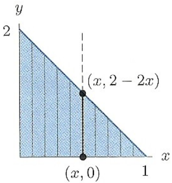

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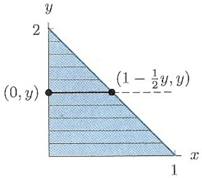

Figure 16.17: Integrating over a triangle using horizontal strips

$$\text{Mass } = \int_0^1 \int_0^{2-2x} \delta(x,y)\,dydx = \int_0^2 \int_0^{1-y/2} \delta(x,y)\,dxdy.$$

Exercise 7.5

Find the mass M of a metal plate R bounded by $y = x$ and $y = x^2$, with density given by $\delta(x, y) = 1 + xy$ kg/meter^2.

Solution:

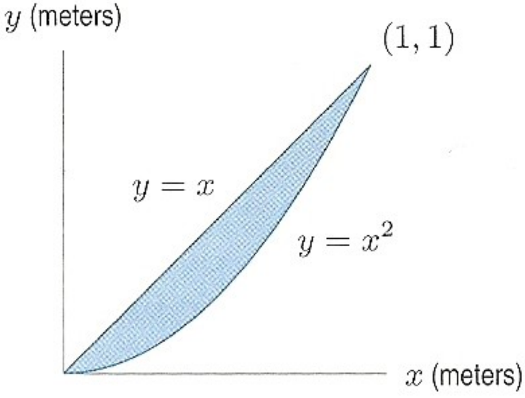

Figure 16.18: A metal plate with density $\delta(x, y)$

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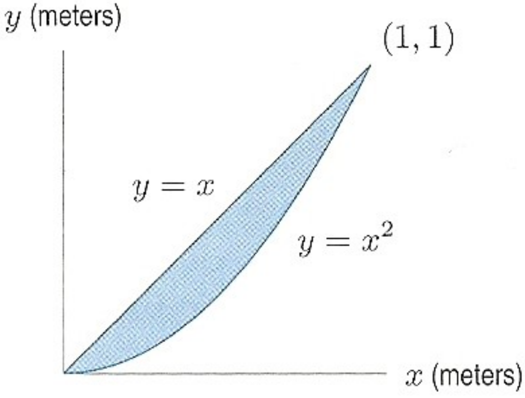

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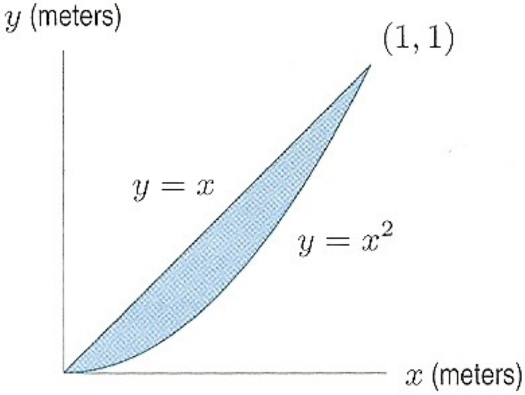

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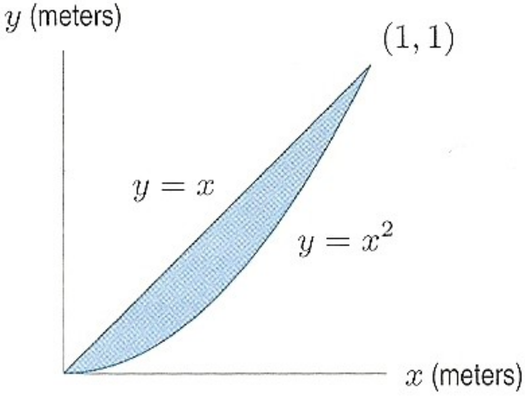

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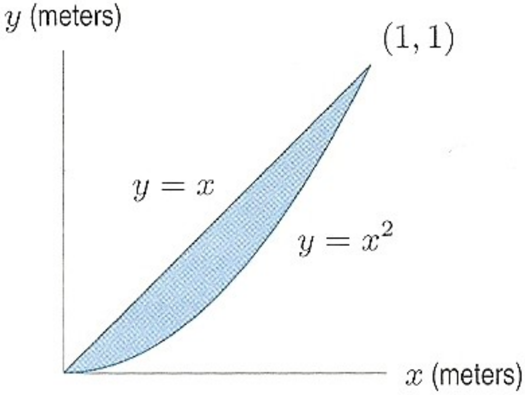

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Exercise 7.6

A city occupies a semicircular region of radius 3 km bordering on the ocean. Find the average distance *d* from points in the city to the ocean.

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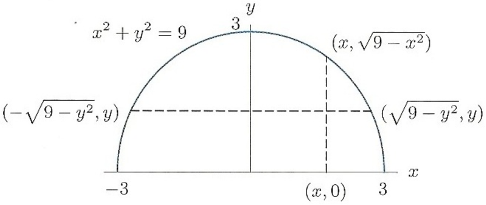

Figure 16.19: The city by the ocean showing a typical vertical strip and a typical horizontal strip

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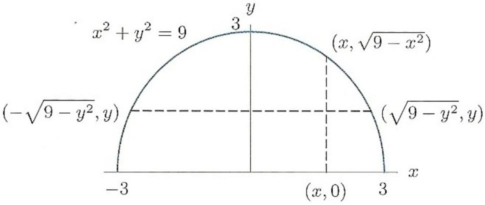

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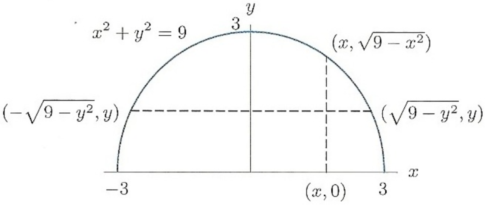

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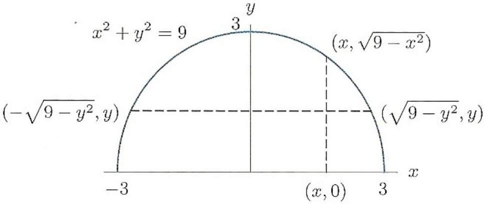

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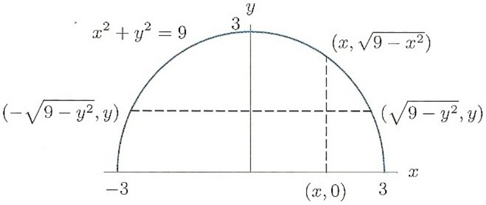

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Polar Coordinates

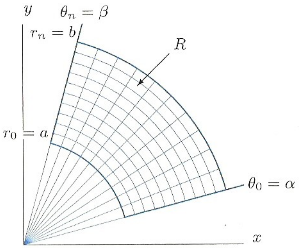

Figure 16.29: Dividing up a region using a polar grid

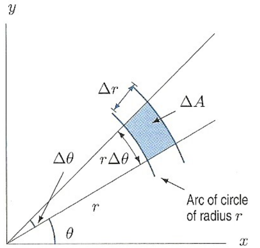

Figure 16.30: Calculating area ΔA in polar coordinates

When computing integrals in polar coordinates use

$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \end{cases},$$

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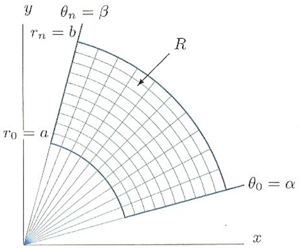

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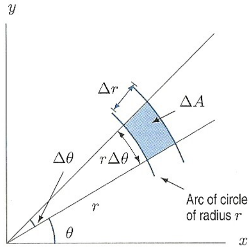

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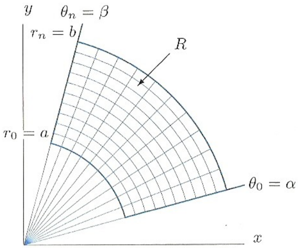

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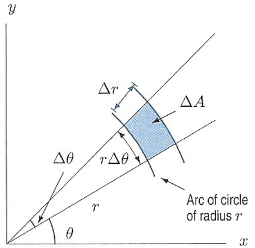

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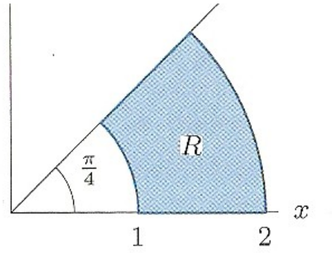
Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown below.



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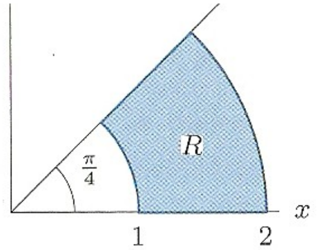

Solution:

$$f(x(r,\theta), y(r,\theta)) = 1/(x^2 + y^2)^{3/2}$$

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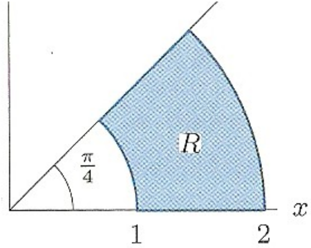
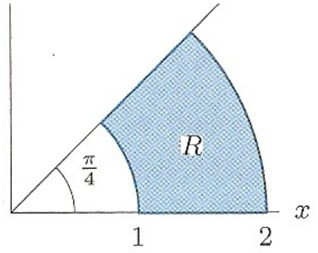


Solution:

$$f(x(r, \theta), y(r, \theta)) = 1/(x^2 + y^2)^{3/2} = 1/(r^2)^{3/2}$$

Exercise 7.7

Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown below.

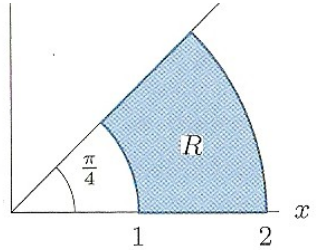


Solution:

$$f(x(r,\theta), y(r,\theta)) = 1/(x^2 + y^2)^{3/2} = 1/(r^2)^{3/2} = 1/r^3.$$

Exercise 7.7

Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown below.



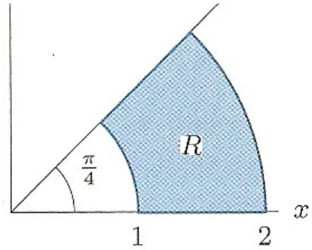
Solution:

$$f(x(r,\theta), y(r,\theta)) = 1/(x^2 + y^2)^{3/2} = 1/(r^2)^{3/2} = 1/r^3.$$

$$\int_R f dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} r dr d\theta$$

Exercise 7.7

Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown below.



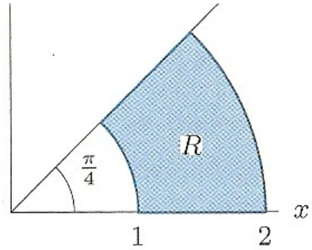
Solution:

$$f(x(r,\theta), y(r,\theta)) = 1/(x^2 + y^2)^{3/2} = 1/(r^2)^{3/2} = 1/r^3.$$

$$\int_R f dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} r dr d\theta = \int_0^{\pi/4} \int_1^2 \frac{1}{r^2} dr d\theta$$

Exercise 7.7

Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown below.



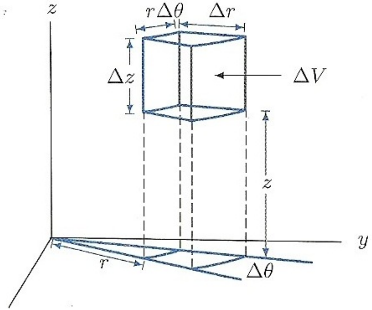
Solution:

$$f(x(r,\theta), y(r,\theta)) = 1/(x^2 + y^2)^{3/2} = 1/(r^2)^{3/2} = 1/r^3.$$

$$\int_R f dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} r dr d\theta = \int_0^{\pi/4} \int_1^2 \frac{1}{r^2} dr d\theta = \frac{\pi}{8}.$$

Cylindrical Coordinates

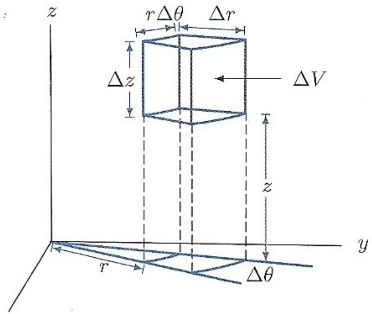
When computing integrals in cylindrical coordinates use



$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \\ z = z \end{cases},$$

Cylindrical Coordinates

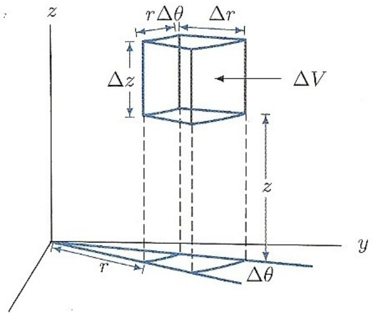
When computing integrals in cylindrical coordinates use



$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \quad , \quad x^2 + y^2 = r^2 \\ z = z \end{cases}$$

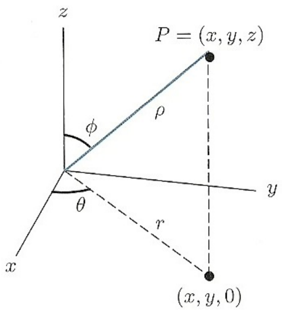
Cylindrical Coordinates

When computing integrals in cylindrical coordinates use

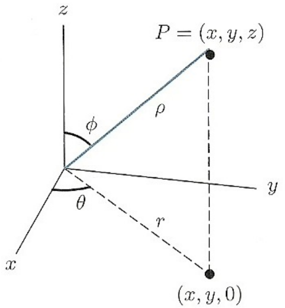


$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \\ z = z \end{cases}, \quad x^2 + y^2 = r^2, \quad dV = r\,dr\,d\theta\,dz.$$

Spherical coordinates



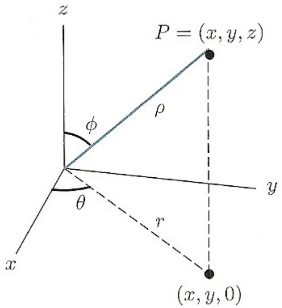
Spherical coordinates



When computing integrals in cylindrical coordinates use

$$\begin{cases} x = & \rho \sin\phi \cos\theta \\ y = & \rho \sin\phi \cos\theta \\ z = & \rho \cos\phi \end{cases},$$

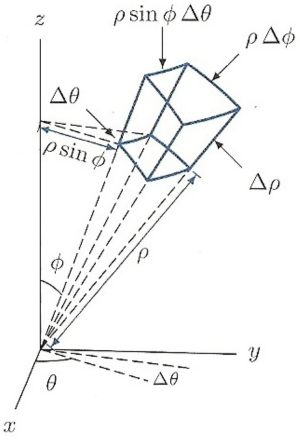
Spherical coordinates



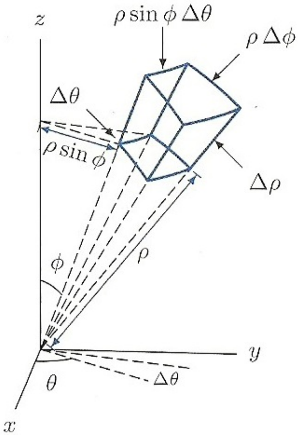
When computing integrals in cylindrical coordinates use

$$\begin{cases} x = & \rho \sin\phi \cos\theta \\ y = & \rho \sin\phi \cos\theta \\ z = & \rho \cos\phi \end{cases} , \quad x^2 + y^2 + z^2 = \rho^2.$$

Volume element in spherical coordinates



Volume element in spherical coordinates



$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$