Mathematics. Multivariable Calculus

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April 3, 2013

Definition

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Illustration

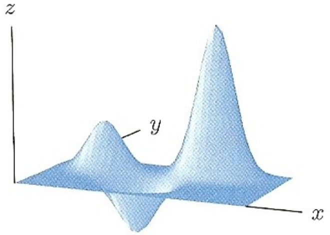

Figure 15.1: Local and global extrema for a function of two variables on $0 \leq x \leq a$, $0 \leq y \leq b$

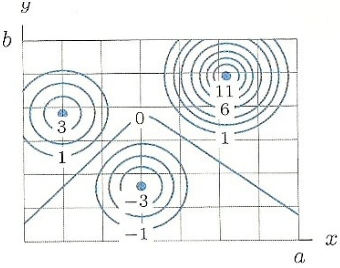

Figure 15.2: Contour map of the function in Figure 15.1

Critical points

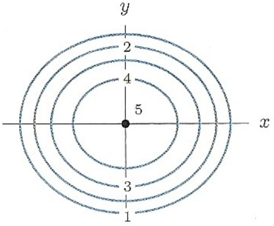

Figure 15.3: Contour diagram around a local maximum of a function

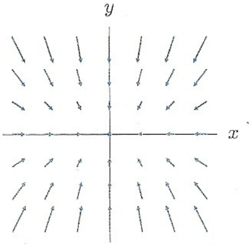

Figure 15.4: Gradients pointing toward the local maximum of the function in Figure 15.3

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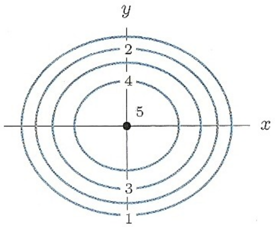

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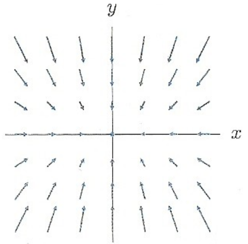

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Let f has a local extremum at the point P. If f is differentiable at P then

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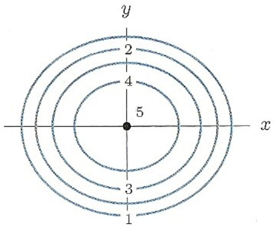

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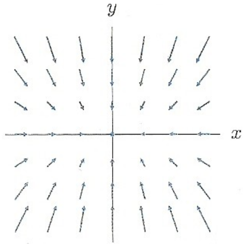

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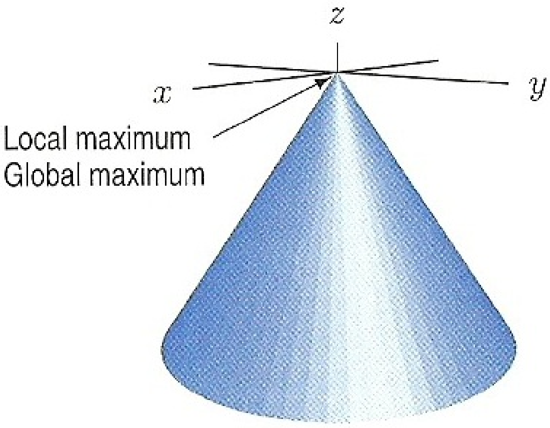

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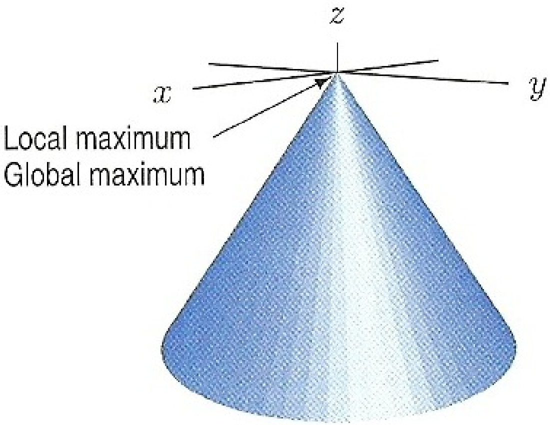

Definition

Points where the gradient is either $\vec{0}$ or undefined are called ***critical points*** *of the function.*

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Find the local extrema of the function $f(x, y) = 2y^3 + 3x^2 - 6xy$.

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$f_{min}(1, 1) = -1$

The graph of $f(x, y) = ax^2 + bxy + cy^2$, $a \neq 0$

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$$D = 4ac - b^2 = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 =$$

$$= 4\left(\tfrac{1}{2}f_{xx}(0,0)\right)\left(\tfrac{1}{2}f_{yy}(0,0)\right) - (f_{xy}(0,0))^2$$

Figure 15.13: Concave up: $D > 0$ and $a > 0$

Figure 15.14: Concave down: $D > 0$ and $a < 0$

Figure 15.15: Saddle poin $D < 0$

Shift to the point (x_0, y_0)

Second Derivative Test for Functions of Two Variables

Suppose (x_0, y_0) is a point where grad $f(x_0, y_0) = \vec{0}$. Let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
- If $D < 0$, then f has a saddle point at (x_0, y_0).
- If $D = 0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (x_0, y_0).

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Complete the solution of Exercise 6.1.

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Exercise 6.3

For the following functions find the critical points and classify them as local maxima, local minima, saddle points, or none of these:

1. $f(x, y) = x^2 - 2xy + 3y^2 - 8y;$
2. $f(x, y) = 400 - 3x^2 - 4x + 2xy - 5y^2 + 48y;$
3. $f(x, y) = x^3 + y^2 - 3x^2 + 10y + 6;$
4. $f(x, y) = x^3 - 3x + y^3 - 3y;$
5. $f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10;$
6. $f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9;$
7. $f(x, y) = (x + y)(xy + 1);$
8. $f(x, y) = 8xy - \frac{1}{4}(x + y)^4;$
9. $f(x, y) = 5 + 6x - x^2 + xy - y^2;$
10. $f(x, y) = e^{2x^2+y^2}.$

Exercise 6.4

Let

$$f(x, y) = kx^2 + y^2 - 4xy, \quad k \neq 4.$$

Determine the value of k (if any) for which the critical point at $(0, 0)$ is a saddle, a local maximum, a local minimum.

Exercise 6.5

Let

$$f(x, y) = x^3 - 3xy^2.$$

Show that there is one critical point at $(0, 0)$ and that $D = 0$ there. Show that the contour of f consists of three lines intersecting at the origin and that these lines divide the plane into six regions around the origin where f alternates from positive to negative.

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The graph of this function is called a *monkey saddle*.

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Fact (Weierstrass Theorems on Continuous Functions)

A continuous function on a compact region R is bounded and has a global maximum at some point in R and a global minimum at some point in R.

Example

The function

$$f(x, y) = \frac{1}{x^2 + y^2}$$

has neither global minima nor global maxima in the region $R:\ 0 < x^2 + y^2 \leq 1$.

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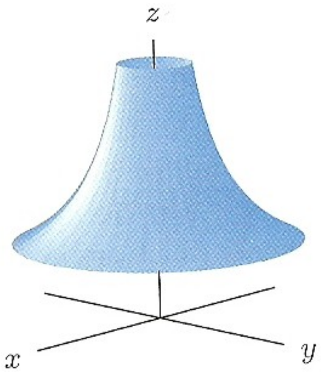

Exercise 6.6

Do the following functions have global maxima or minima?

1. $f(x, y) = x^2 - 2y^2$;
2. $f(x, y) = x^2 y^2$;
3. $f(x, y) = x^3 + y^3$;
4. $f(x, y) = -2x^2 - 7y^2$;
5. $f(x, y) = x^2/2 + 3y^3 + 9y^2 - 3x$.

Exercise 6.7

Find the global minima and global maxima of the below functions on the region $R:\ \max(|x|, |y|) \leq 1$, and say whether it occurs on the boundary of the square. [Hint: Use graphs.]

1. $z = x^2 + y^2$;
2. $z = -x^2 - y^2$;
3. $z = x^2 - y^2$.

Exercise 6.8

An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length and height less or equal to 135cm. Find the dimensions of the suitcase of maximum volume that a passenger may carry under this regulation.

Constrained Optimization: Lagrange Multipliers

Optimize an *objective function* $f(x, y)$ subject to a *constraint* $g(x, y) = c$.

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Conditinal local ang global minima are defined analogously.

Lagrange Multipliers; Illustration

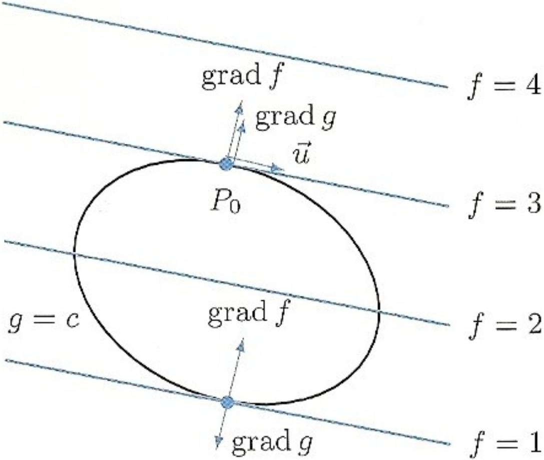

Figure 15.28: Maximum and minimum values of $f(x, y)$ on $g(x, y) = c$ are at points where grad f is parallel to grad g

Lagrange Multipliers

Fact

If a differentiable function, f, has an extremum subject to a differentiable constraint $g = c$ at a point P_0, then either P_0 satisfies the equation

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*The number λ is called the **Lagrange multiplier**.*

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The objective function: $f(x, y) = x + y$, the constraint: $g(x, y) = x^2 + y^2 = 4$

$\nabla f(x, y) = [1, 1]$, $\nabla g(x,) = [2x, 2y]$

$1 = 2\lambda x$, $1 = 2\lambda y$,$x = y$

Exercise 6.9

Find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 4$.

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$f_{min}(-\sqrt{2}, -\sqrt{2}) = -2\sqrt{2}$, $f_{max}(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$

Meaning of the Lagrange Multiplier

$f(x_0(c), y_0(c))$

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We have $[f_x, f_y] = [\lambda g_x, \lambda g_y]$ at the optimum point, so

$$\frac{df}{dc} = \lambda \left(\frac{\partial g}{\partial x} \cdot \frac{dx_0}{dc} + \frac{\partial g}{\partial y} \cdot \frac{dy_0}{dc} \right) = \lambda \frac{dg}{dc}.$$

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Fact

The value of λ is the rate of change of the optimum value of f as c increases (where $(g(x, y) = c)$. If the optimum value of f is written as $f(x_0(c), y_0(c))$, then

$$\frac{d}{dc} f((x_0(c), y_0(c)) = \lambda.$$

Exercise 6.10

In Exercises 1–15, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint, if such values exist.

1. $f(x, y) = x + y, \quad x^2 + y^2 = 1$
2. $f(x, y) = 3x - 2y, \quad x^2 + 2y^2 = 44$
3. $f(x, y) = xy, \quad 4x^2 + y^2 = 8$
4. $f(x, y) = x^2 + y^2, \quad x^4 + y^4 = 2$
5. $f(x, y) = x^2 + y^2, \quad 4x - 2y = 15$
6. $f(x, y) = x^2 + y, \quad x^2 - y^2 = 1$
7. $f(x, y) = x^2 - xy + y^2, \quad x^2 - y^2 = 1$
8. $f(x, y, z) = x + 3y + 5z, \quad x^2 + y^2 + z^2 = 1$
9. $f(x, y, z) = x^2 - 2y + 2z^2, \quad x^2 + y^2 + z^2 = 1$
10. $f(x, y, z) = 2x + y + 4z, \quad x^2 + y + z^2 = 16$
11. $f(x, y) = x^2 + 2y^2, \quad x^2 + y^2 \leq 4$
12. $f(x, y) = x + 3y, \quad x^2 + y^2 \leq 2$
13. $f(x, y) = xy, \quad x^2 + 2y^2 \leq 1$
14. $f(x, y) = x^3 - y^2, \quad x^2 + y^2 \leq 1$
15. $f(x, y) = x^3 + y, \quad x + y \geq 1$