Mathematics. Multivariable Calculus

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The Chain Rule for $z = f(x, y)$, $x = x(t)$, $y = y(t)$

Consider the local linearization

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y. \tag{1}$$

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$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{2}$$

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Taking in (2) the limit as $\Delta t \longrightarrow 0$, we get the following result:

Fact

If $f(x, y)$, $x(t)$ and $y(t)$ are differentiable and if $z = f(x, y)$, $x = x(t)$ and $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{3}$$

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One can easily extend the above formula to arbitrary number of variables of any of appearing functions.

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Suppose that $z = f(x, y) = x \sin y$, where $x = t^2$ and $y = 2t + 1$. Let $z = g(t)$. Compute $g'(t)$ directly and using the chain rule.

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$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (\sin y)(2t) + (x\cos y)(2) =$$

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Exercise 5.3

Let (x, y) be Cartesian coordinates and (r, φ) be polar coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$.

For any function z of two variables express $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$ in terms of polar coordinates.

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Solution:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \varphi}\right)^2 .$$

The Second-Order Partial Derivatives of $z = f(x, y)$

Definition

$$\frac{\partial^2 z}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx} =: (f_x)_x,$$

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Interpretation of the Second-Order Derivatives

The string is 1 meter long and at time t seconds, the point x meters from one end is displaced

$$f(x,t) = a\sin(\pi x)\sin(\varphi t)$$

meters from its rest position, where $a > 0$ is amplitude, and $\varphi > 0$ is frequency.

We have

$$f_{xx} = -a\pi^2 \sin(\pi x)\sin(\varphi t),\ f_{xx}(0.3,1) \approx -0.01 < 0$$

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$$f_{tt} = a\varphi^2 \sin(\pi x)\sin(\varphi t),\ f_{tt}(0.3, 1) = -7200 < 0.$$

Interpretation of the Second-Order Derivatives

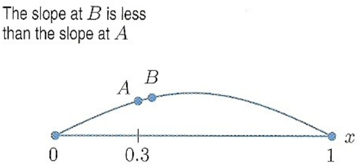

Figure 14.51: Interpretation of $f_{xx}(0.3, 1) < 0$: The concavity of the string at $t = 1$

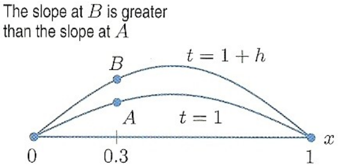

Figure 14.52: Interpretation of $f_{xt}(0.3, 1) > 0$: The slope of one point on the string at two different times

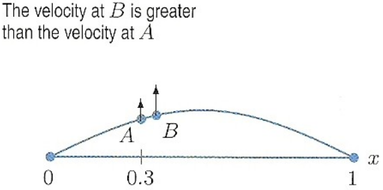

Figure 14.53: Interpretation of $f_{tx}(0.3, 1) > 0$: The velocity of different points on the string at $t = 1$

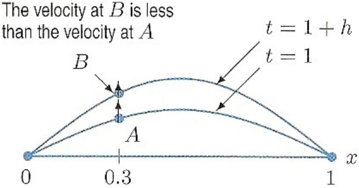

Figure 14.54: Interpretation of $f_{tt}(0.3, 1) < 0$: Negative acceleration. The velocity of one point on the string at two different times

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Observe that

$$f_{xy}(x, y) = 2y + 6xe^y = f_{yx}(x, y).$$

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$f_y(x, y) = 3xy^2\cos(x^2 + y^3)$

Exercise 5.5

The same as in Ex. 5.4 for $f(x, y) = x\sin(x^2 + y^3)$.

Solution:

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Observe that again $f_{xy} = f_{yx}$.

Equality of Mixed Partial Derivatives

Fact

If one of second-order partial derivatives f_{xy} or f_{yx} exists at point (a, b) and is continuous at that point, then the second one also exists and is continuous at (a, b) and

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Taylor Polynomials

Fact (Taylor Polynomial of Degree 1 near (a, b))

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

$f_x(a, b)(x - a) + f_y(a, b)(y - b) = df$ at (a, b)

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Fact (Taylor Polynomial of Degree 2 near (a, b))

$$\begin{aligned} f(x, y) &\approx Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)+ \\ &+\tfrac{f_{xx}(a,b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \tfrac{f_{yy}(a,b)}{2}(y - b)^2. \end{aligned}$$

$\frac{f_{xx}(a,b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a,b)}{2}(y - b)^2 = \frac{1}{2}d^2 f$ at (a, b)

Taylor Polynomials

Fact (Taylor Polynomial of Degree n near (a, b))

$$f(x, y) \approx T_n(x, y) = f(a, b) + df(a, b) + \frac{d^2 f(a, b)}{2} + \cdots + \frac{d^n f(a, b)}{n!}.$$

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Moreover we can evaluate the error $R_n(x, y) = f(x, y) - T_n(x, y)$:

Fact

Lagrange form of the error

$$R_n(x, y) = \frac{d^{n+1} f(a + \theta \Delta x, b + \theta \Delta y)}{(n+1)!}$$

for some $\theta \in (0, 1)$.

Exercise 5.6

Find the quadratic Taylor polynomials about $(0, 0)$ for the following functions

1. $(x - y + 1)^2$;
2. $(y - 1)(x + 1)^2$;
3. $e^{-2x^2 - y^2}$;
4. $1/(1 + 2x - y)$;
5. $e^x \cos y$;
6. $\cos(x + 3y)$;
7. $\ln(1 + x^2 - y)$;
7. $\sin(2x) + \cos y$.

Differentiability

Definition

A function f is ***differentiable at the point*** (a, b) *if there exists a linear map* $A : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a, b) - A(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

Fact

If a function is differentiable at a point, then is continuous at this point.

Differentiability

Fact

If a function is differentiable at a point, then both partial derivatives exist there.

Differentiability

Fact

If a function is differentiable at a point, then both partial derivatives exist there.

Ex 1. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$. Prove that f is not differentiable at the origin.

Differentiability

Fact

Having both partial derivatives at a point does not guarantee that a function is differentiable there.

Differentiability

Fact

Having both partial derivatives at a point does not guarantee that a function is differentiable there.

Ex 2. Consider the function $f(x, y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$. Show that the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist, but that f is not differentiable at $(0, 0)$.

Differentiability

Fact

A function f is differentiable at (a, b) if there exist both partial derivatives $f_x(a, b)$ and $f_y(a, b)$ and are continuous on a small disc centered at the point (a, b). Moreover, if $\vec{u} = [u_1, u_2]$, then

$$A\vec{u} = [f_x(a, b), f_y(a, b)] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

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Ex 3. Show that the the function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

Differentiability in General

Let now $f : D \longrightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_m)$, $f_k : D \longrightarrow \mathbb{R}$, $k = 1, \dots, m$.

Definition

*A function f is **differentiable at the point** $\vec{a} \in D$ if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{\vec{u} \to 0} \frac{||f(\vec{a} + \vec{u}) - f(a, b) - A\vec{u}||}{|\vec{u}|} = 0.$$

Differentiability and Partial Derivatives

Fact

A function f is differentiable at $\vec{a}$ if, and only if there exist all partial derivatives $\frac{\partial f_k}{\partial x_l}(\vec{a})$, $k = 1, \ldots, m$, $l = 1, \ldots, n$, and all are continuous around $\vec{a}$. Moreover the linear map A is represented by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & \cdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Exercise 5.7

In Exercises 1–10, list the points in the xy-plane, if any, at which the function $z = f(x, y)$ is not differentiable.

1. $z = -\sqrt{x^2 + y^2}$

2. $z = \sqrt{(x+1)^2 + y^2}$

3. $z = e^{-(x^2+y^2)}$

4. $z = |x| + |y|$

5. $z = x^{1/3} + y^2$

6. $z = |x+2| - |y-3|$

7. $z = (\sin x)(\cos |y|)$

8. $z = |x-3|^2 + y^3$

9. $z = 4 + \sqrt{(x-1)^2 + (y-2)^2}$

10. $z = 1 + \left((x-1)^2 + (y-2)^2\right)^2$

Exercise 5.8

For the functions f in Problems 11–14 answer the following questions. Justify your answers.

(a) Use a computer to draw a contour diagram for f.

(b) Is f differentiable at all points $(x, y) \neq (0, 0)$?

(c) Do the partial derivatives f_x and f_y exist and are they continuous at all points $(x, y) \neq (0, 0)$?

(d) Is f differentiable at $(0, 0)$?

(e) Do the partial derivatives f_x and f_y exist and are they continuous at $(0, 0)$?

11. $$f(x, y) = \begin{cases} \dfrac{x}{y} + \dfrac{y}{x}, & x \neq 0 \text{ and } y \neq 0, \\ 0, & x = 0 \text{ or } y = 0. \end{cases}$$

12. $$f(x, y) = \begin{cases} \dfrac{2xy}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

13. $$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

14. $$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$