Mathematics. Multivariable Calculus

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Zooming in to see local linearity

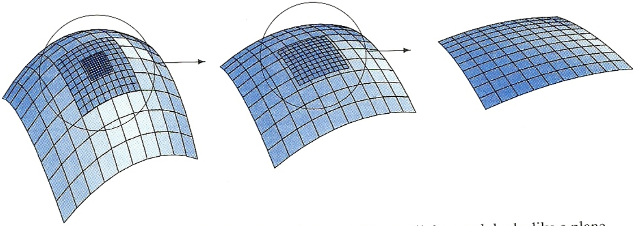

Figure 14.19: Zooming in on the graph of a function of two variables until the graph looks like a plane

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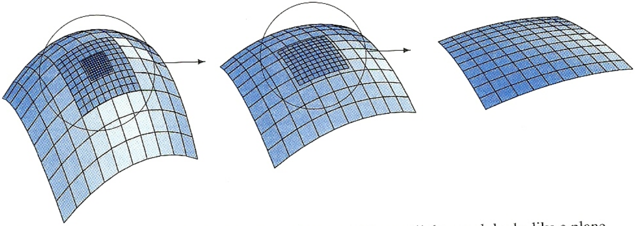

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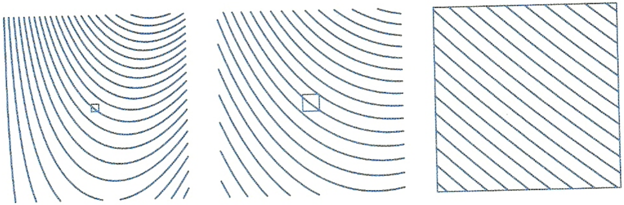

Figure 14.20: Zooming in on a contour diagram until the lines look parallel and equally spaced

Differentiability and the tangent plane

Definition

We say that a n-variable function f of variables $x_1, \ldots, x_n$ is ***differentiable*** at the point $(a_1, \ldots, a_n)$ if there exist all partial derivatives

$$f_{x_k}(a_1, \ldots, a_n), \quad k = 1, \ldots, n.$$

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Fact

Assuming f is differentiable at (a, b), the equation of the tangent plane is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Illustration of differentiability

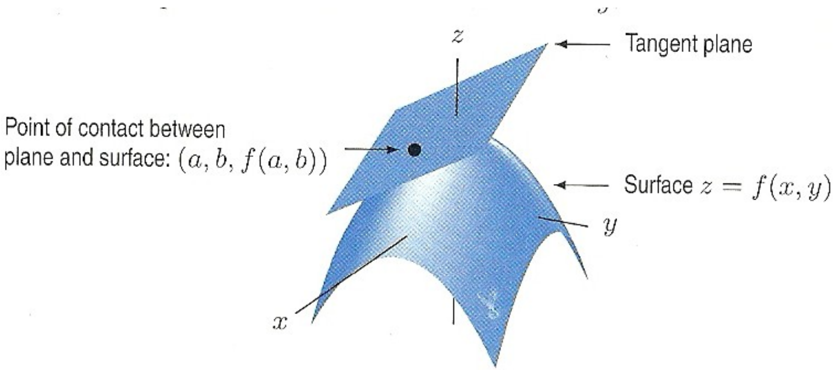

Figure 14.21: The tangent plane to the surface $z = f(x, y)$ at the point (a, b)

Exercise 4.1

Find the equation for the tangent plane to the surface

$$z = x^2 + y^2$$

at the point $(3, 4)$.

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Definition

Provided f is differentiable at (a, b), we can approximate $f(x, y)$:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The right side of this approximation is called the ***local linearization*** of f near the point (a, b).

the Tangent Plane approximation shown graphically

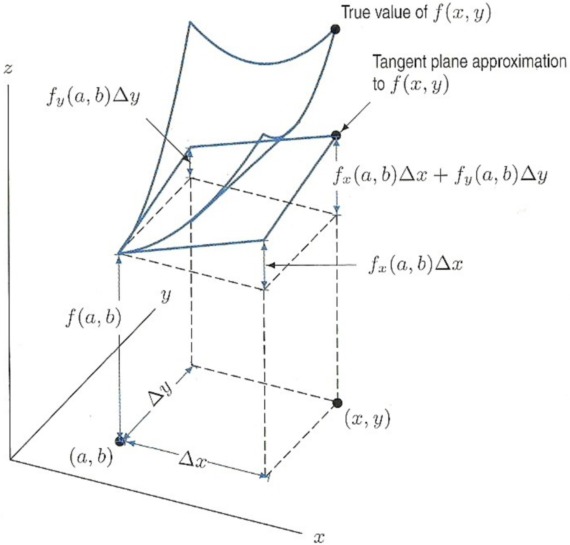

Figure 14.22: Local linearization: Approximating $f(x, y)$ by the z-value from the tangent plane

Exercise 4.2

Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Estimate $f(29/10, 21/5)$ and $f(2, 2)$ using the linearization and compare your answers to the true values.

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The differential at a general point is often written

$$df = f_x dx + f_y dy.$$

The differential in the magnified coordinate system

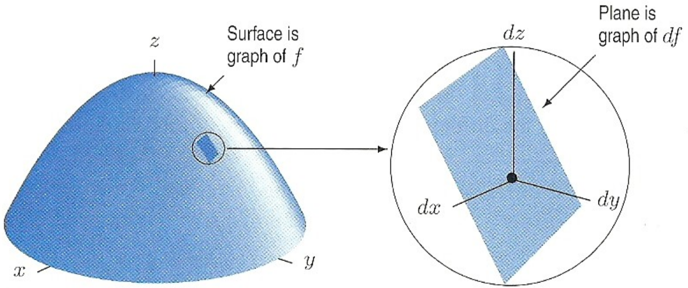

Figure 14.23: The graph of f, with a view through a microscope showing the tangent plane in the magnified coordinate system

Exercise 4.3

Compute the differentials of the following functions:

(a) $f(x, y) = x^2 e^{5y}$, (b) $u = x \sin(xyz) + \ln(x^2 + y^2 + z^2)$,
(c) $g(x, y) = x \cos(2x)$.

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(b)
$$\begin{aligned} du &= \left(\sin(xyz) + xyz\cos(xyz) + \tfrac{2x}{x^2+y^2+z^2}\right) dx + \\ &+ \left(x^2 z\cos(xyz) + \tfrac{2y}{x^2+y^2+z^2}\right) dy + \\ &+ \left(x^2 y\cos(xyz) + \tfrac{2z}{x^2+y^2+z^2}\right) dz. \end{aligned}$$

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(c) $dg = (\cos(2x) - 2x\sin(2x))dx$.

Exercise 4.4

The density ρ (in g/cm^3) of carbon dioxide gas CO_2 depends upon its temperature T (in Kelvins) and pressure P (in atmospheres). The ideal gas model for CO_2 gives what is called the state equation

$$\rho = r\frac{P}{T},$$

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$-\frac{rP}{T^2} < 0$ - if the pressure is kept constant increasing the temperature expands the gas, hence decreases its density;

$\frac{r}{T} > 0$ - if the temperature is kept constant increasing the pressure compresses the gas and therefore increases its density.

Definition

*If $\vec{u} = [u_1, u_2]$ is a unit vector, we define the **directional derivatives**, $f_{\vec{u}}$, in the direction of $\vec{u}$ at a point (a, b) by*

$$f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

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If $\vec{v} \neq 0$ is not a unit vector then we take $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ and define

$$f_{\vec{v}} = f_{\vec{u}}.$$

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Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of the vector $\vec{v} = [1, 1]$.

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$$
\begin{aligned}
f_{\vec{v}}(1,0) = f_{\vec{u}}(1,0) &= \lim_{h\to 0} \frac{f(1+h/\sqrt{2}, h/\sqrt{2}) - f(1,0)}{h} = \\
&= \lim_{h\to 0} \frac{(1+h/\sqrt{2})^2 + (h/\sqrt{2})^2 - 1}{h} = \\
&= \lim_{h\to 0} \frac{h\sqrt{2}+h^2}{h} = \lim_{h\to 0}(\sqrt{2}+h) = \sqrt{2}.
\end{aligned}
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Definition

*The **gradient vector** of a differentiable function $f(x,y)$ at the point (a,b) is*

$$\nabla f(a,b) = [f_x(a,b), f_y(a,b)].$$

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We can think about the gradient as the result of applying the vector operator

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]$$

to the function f. The above operator is called "del".

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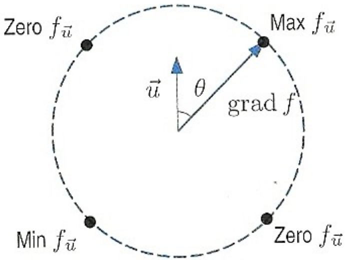


: Values of the directional derivative at different angles to the gradient

Geometric Properties of the Gradient Vector in the Space

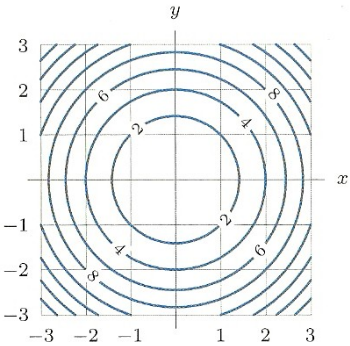
Fact

If f is differentiable function at the point (a, b) and $\nabla f(a, b) \neq 0$, then:

- The direction of $\nabla f(a, b)$ is
  - Perpendicular to the contour of f through (a, b)
  - In the direction of increasing f
- The magnitude of the gradient vector, $|\nabla g|$, is
  - The maximum rate of change of f at that point
  - Large when the contour are close together and small when they are far apart.

Exercise 4.7

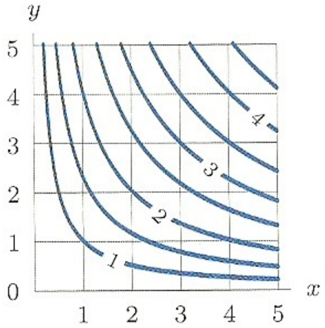
Use the below contour diagram of f to decide if the specified directional derivative is positive, negative, or approximately zero.



1. At point $(-2, 2)$, in dir. $[1, 0]$.
2. At point $(0, -2)$, in dir. $[0, 1]$.
3. At point $(-1, 1)$, in dir. $[1, 1]$.
4. At point $(-1, 1)$, in dir. $[-1, 1]$.
5. At point $(0, -2)$, in dir. $[1, 2]$.
6. At point $(0, -2)$, in dir. $[1, -2]$.

Exercise 4.8

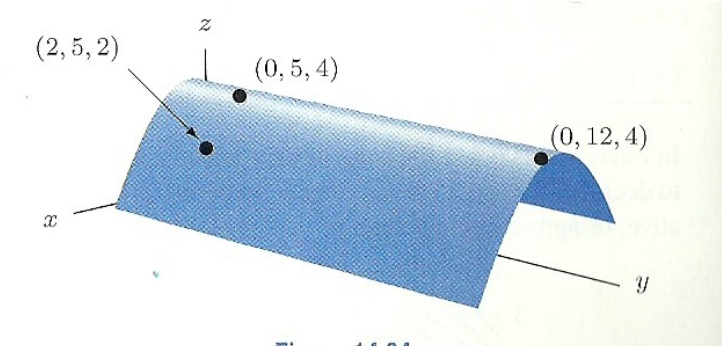
Use the below contour diagram of f to estimate the directional derivative of $f(x, y)$ in the given directions and points.



1. At point $(1, 1)$, in dir. $[1, 0]$.
2. At point $(1, 1)$, in dir. $[0, 1]$.
3. At point $(1, 1)$, in dir. $[1, 1]$.
4. At point $(4, 1)$, in dir. $[1, 1]$.
5. At point $(3, 3)$, in dir. $[-2, 1]$.
6. At point $(4, 1)$, in dir. $[-2, 1]$.

Exerciese 4.9

The surface $z = g(x, y)$ is in the below figure. What is the sign of each of the following directional derivatives?



1. $g_{\vec{u}}(2, 5)$ where $\vec{u} = [1, -1]/\sqrt{2}$;
2. $g_{\vec{u}}(2, 5)$ where $\vec{u} = [1, 1]/\sqrt{2}$.

The Gradient Vector in the Plane

Definition

$$\nabla f(a, b, c) = [f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)] \, .$$

The Gradient Vector in the Plane

Definition

$$\nabla f(a,b,c) = [f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)]\,.$$

Fact

If f is differentiable at (a,b,c) and $\vec{u}$ is a unit vector, then

$$f_{\vec{u}}(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}.$$

I, in addition, $\nabla f(a,b,c) \neq 0$, then

- $\nabla f(a,b,c)$ is in the direction of the greatest rate of increase of f
- $\nabla f(a,b,c)$ is perpendicular to the level surface of f at (a,b,c)
- $|\nabla f(a,b,c)|$ is the maximum rate of change of f at (a,b,c).

Example

$$f(x,y,z) = x^2 + y^2, \qquad g(x,y,z) = -x^2 - y^2 - z^2.$$

Describe $\nabla f(0,1,1)$, $\nabla f(1,0,1)$, $\nabla g(0,1,1)$, $\nabla g(1,0,1)$.

Example

$$f(x, y, z) = x^2 + y^2, \qquad g(x, y, z) = -x^2 - y^2 - z^2.$$

Describe $\nabla f(0, 1, 1)$, $\nabla f(1, 0, 1)$, $\nabla g(0, 1, 1)$, $\nabla g(1, 0, 1)$.

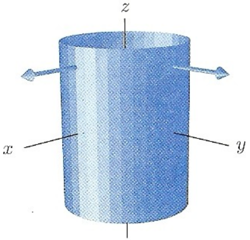


Figure 14.39: The level surface $f(x, y, z) = x^2 + y^2 = 1$ with two gradient vectors

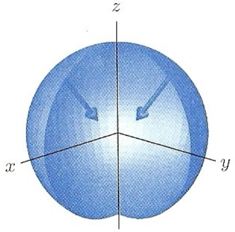


Figure 14.40: The level surface $g(x, y, z) = -x^2 - y^2 - z^2 = -2$ with two gradient vectors