Mathematics. Multivariable Calculus

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What is the difference in the behaviour of the following functions near the point $(0,0)$?

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0); \end{cases} \qquad g(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}$$

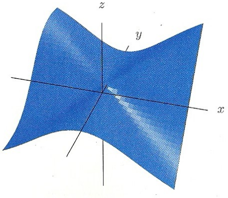

Figure 12.80: Graph of $z = x^2y/(x^2+y^2)$

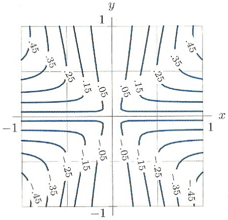

Figure 12.81: Contour diagram of $z = x^2y/(x^2+y^2)$

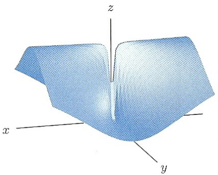

Figure 12.82: Graph of $z = x^2/(x^2+y^2)$

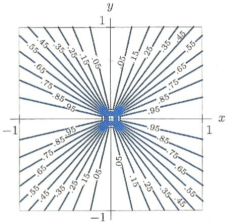

Figure 12.83: Contour diagram of $z = x^2/(x^2+y^2)$

Definitions

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Definition

The function f has a ***limit*** L at the point (a, b), written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $f(x, y)$ is as close to L as we please whenever the distance from the point (x, y) to the point (a, b) is sufficiently small, but not zero.

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$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

A function is ***continuous on a region*** D in the xy-plane if it is continuous at each point of D.

The basic theorem

Recall that an ***elementary function*** is a function built up of a finite combination of constant functions, field operations (addition, multiplication, division, and root extractions–the elementary operations)–and algebraic, exponential, and logarithmic functions and their inverses under repeated compositions. Among the simplest elementary functions are the logarithm, exponential function (including the hyperbolic functions), power function, and trigonometric functions.

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We can easily extend the above definition to multi-variable functions.

Theorem

All elementary functions are continuous at all points where they are defined.

Exercise 3.1

Are the following functions continuous at all points in the given regions?

1. $\frac{1}{x^2+y^2}$ on the square $-1 \le x \le 1$, $-1 \le y \le 1$;
2. $\frac{1}{x^2+y^2}$ on the square $1 \le x \le 2$, $1 \le y \le 2$;
3. $\frac{y}{x^2+2}$ on the disk $x^2 + y^2 \le 1$;
4. $\frac{e^{\sin x}}{\cos y}$ on the rectangle $-\pi/2 \le x \le \pi/2$, $0 \le y \le \pi/4$;
5. $\tan(xy)$ on the square $-2 \le x \le 2$, $-2 \le y \le 2$;
6. $\sqrt{2x - y}$ on the disk $x^2 + y^2 \le 4$.

Exercise 3.2

Find the limits of the following functions as $(x, y) \longrightarrow 0$.

1. $f(x, y) = e^{-x-y}$;
2. $g(x, y) = x^2 + y^2$;
3. $h(x, y) = \frac{x}{x^2+1}$;
4. $i(x, y) = \frac{x+y}{2+\sin y}$;
5. $j(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$.

Definition

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For all points at which the limits exist, we define the ***partial derivatives of*** f ***at the point*** (a, b) by

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h},$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}.$$

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If we let a and b vary, we have the ***partial derivative functions*** $f_x(x, y)$ and $f_y(x, y)$.

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If we let a and b vary, we have the ***partial derivative functions*** $f_x(x, y)$ and $f_y(x, y)$.

- $f_x(a, b)$ is a **rate of change of** f **with respect to** x at the point (a, b);
- $f_y(a, b)$ is a **rate of change of** f **with respect to** y at the point (a, b).

Alternative Notations

If $z = f(x, y)$, we can write

$$f_x(x, y) = \frac{\partial z}{\partial x} \quad \text{and} \quad f_y(x, y) = \frac{\partial z}{\partial y},$$

$$f_x(a, b) = \frac{\partial z}{\partial x} \Big|_{(a,b)} \quad \text{and} \quad f_y(a, b) = \frac{\partial z}{\partial y} \Big|_{(a,b)}.$$

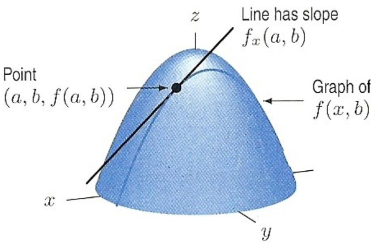
Visualizing Partial Derivatives on a Graph

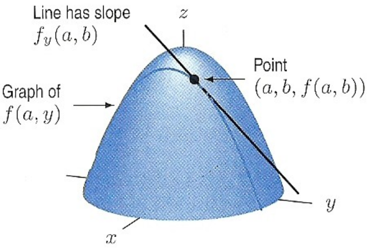
The partial derivative $f_x(a, b)$ is the slope of the tangent line to the curve

$$Graph(f) \cap \{y = b\}$$

at $x = a$.

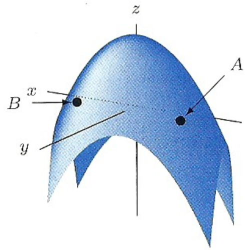
Analogically for $f_y(a, b)$.





Exercise 3.3

The surface $z = f(x, y)$ is shown on the following figure. The points A and B are on the plane $z = 0$.

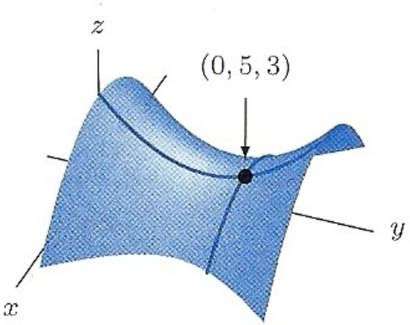


(a) What is the sign of
  (i) $f_x(A)$?
  (ii) $f_y(A)$?

(b) The point P in the plane $z = 0$ moves along a straight line from A to B. How does the sign of $f_x(P)$ change? How does the sign of $f_y(P)$ change?

Exercise 3.4

The following figure shows the saddle-shaped surface $z = f(x, y)$.

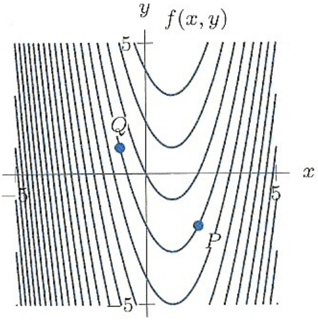


(a) What is the sign of $f_x(0, 5)$?

(b) What is the sign of $f_y(0, 5)$?

Exercise 3.5

The following figure shows contours of $f(x, y)$ with values of f on the contours omitted.



If $f_x(P) > 0$, find the sign of

(a) $f_y(P)$; (b) $f_y(Q)$; (c) $f_x(Q)$.

Differentiate with the respect to one variable, regarding the others variables as constants!

Example. Find $f_x(3, 2)$ and $f_y(1, 1)$ algebraically, where

$$f(x, y) = \frac{x^2}{y + 1}.$$

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Example. Find $f_x(3,2)$ and $f_y(1,1)$ algebraically, where

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Solution 1.

$f(x,2) = \frac{x^2}{3}$, thus $f'(x,2) = \frac{2x}{3}$, hence $f_x(3,2) = 2$.

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$f(1,y) = \frac{1}{y+1}$, thus $f'(1,y) = -\frac{1}{(y+1)^2}$, hence $f_y(1,1) = -\frac{1}{4}$.

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Solution 2.

$f_x(x,y) = \frac{2x}{y+1}$, thus $f_x(3,2) = 2$.

Differentiate with the respect to one variable, regarding the others variables as constants!

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Solution:

$f_x(x, y, z) = \frac{2xy^3}{z}$;

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Solution:

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$f_x(1, 2, 1) = 16$;

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$f_x(1, 2, 1) = 16$; $\quad f_y(2, 1, 1) = 12$;

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7. g_x if $g(x, y) = \ln(ye^{xy})$;
8. $\frac{\partial z}{\partial y}\,|_{(1,0,5)}$ if $z = e^{x+2y}\sin y$;
9. $\frac{\partial f}{\partial x}\,|_{(\pi/3,1)}$ if $f(x, y) = x\ln(y\cos x)$.