Mathematics. Multivariable Calculus

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Contour diagrams

Definition

The curve, on the plain $z = 0$, $f(x, y) = c$ is called a ***contour line*** or a ***level curve***.

P - the plane $z = c$

The level curve at the level $c = Graph(f) \cap P$

We can represent a function as a ***contour diagram***: the family of contour lines for some chosen $c's$.

Examples. Any weather or topographical map.

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What one can read from a contour diagram?

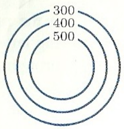


Figure 12.36: Mountain peak

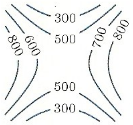


Figure 12.37: Pass between two mountains

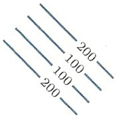


Figure 12.38: Long valley

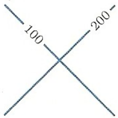


Figure 12.39: Impossible contour lines

Contour diagram and graphs

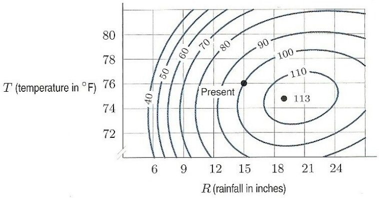
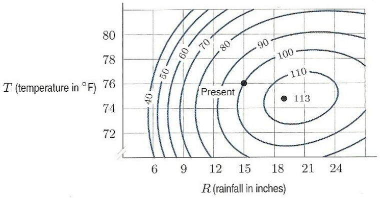


Figure 12.40: Corn production, C, as a function of rainfall and temperature

The contour diagram is created form the graph by joining all the points of the same level and dropping the curve into the plane $z = 0$.

Contour diagram and graphs



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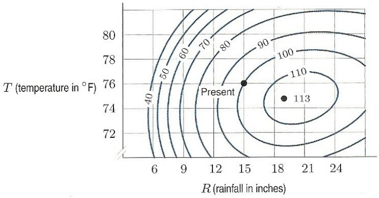

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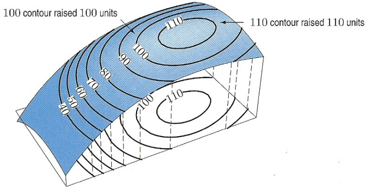

Figure 12.41: Getting the graph of the corn yield function from the contour diagram

The contour diagram is created form the graph by joining all the points of the same level and dropping the curve into the plane $z = 0$.

The graph is created from the contour diagram by lifting each contour above the plane to a height equal to its value.

Finding contours algebraically

Suppose the surface has equation $z = f(x, y)$. The equation for the contour ar height c is given by: $f(x, y) = c$.

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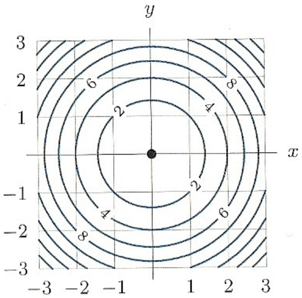

Figure 12.42: Contour diagram for $f(x, y) = x^2 + y^2$ (even values of c only)

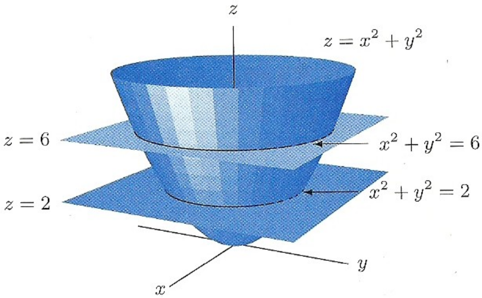

Figure 12.43: The graph of $f(x, y) = x^2 + y^2$

Exercise 2.1

Draw a contour diagram for $f(x, y) = \sqrt{x^2 + y^2}$ and relate it to the graph of f.

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Solution:

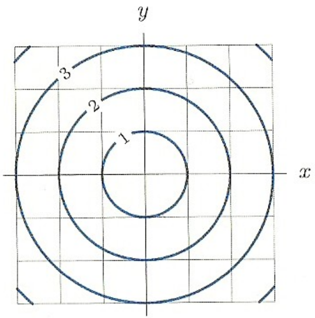


Figure 12.44: A contour diagram for $f(x,y) = \sqrt{x^2+y^2}$

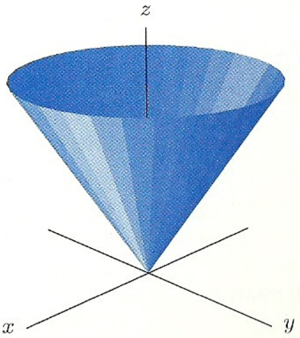


Figure 12.45: The graph of $f(x,y) = \sqrt{x^2+y^2}$

Exercise 2.2

Match Tables (a)–(d) with the contour diagrams (I)–(IV).

(a)

$y\backslash x$	-1	0	1
-1	2	1	2
0	1	0	1
1	2	1	2

(b)

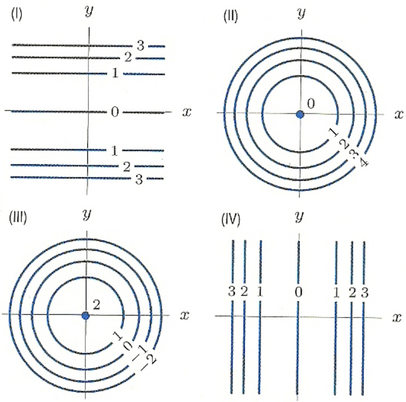
$y\backslash x$	-1	0	1
-1	0	1	0
0	1	2	1
1	0	1	0

(c)

$y\backslash x$	-1	0	1
-1	2	0	2
0	2	0	2
1	2	0	2

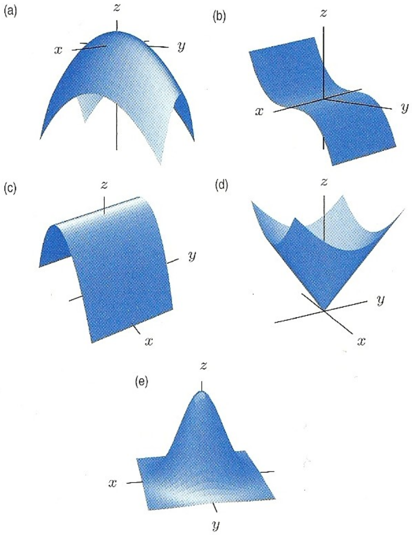
(d)

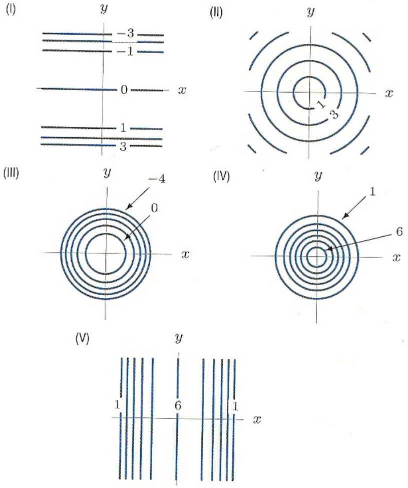
$y\backslash x$	-1	0	1
-1	2	2	2
0	0	0	0
1	2	2	2



Exercise 2.3

Match the surfaces (a)–(e) with the contour diagrams (I)–(V).





Exercise 2.4

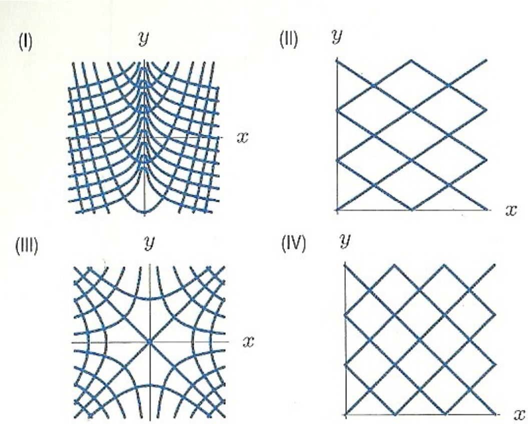
Match the pairs of functions (a)–(d) with the contour diagrams (I)–(IV). In each case, show which contours represent f and which represent g.

(a) $f(x, y) = x + y$,
$g(x, y) = x - y$;

(b) $f(x, y) = 2x + 3y$,
$g(x, y) = 2x - 3y$;

(c) $f(x, y) = x^2 - y$,
$g(x, y) = 2y + \ln|x|$;

(d) $f(x, y) = x^2 - y^2$,
$g(x, y) = xy$.



Exercise 2.5

Match the functions (a)–(d) with the shapes of their level curves (I)–(IV). Sketch each contour diagram.

(a) $f(x, y) = x^2$;	(I) Lines;
(b) $f(x, y) = x^2 + 2y^2$;	(II) Parabolas;
(c) $f(x, y) = y - x^2$;	(III) Hyperbolas;
(d) $f(x, y) = x^2 - y^2$.	(IV) Ellipses.

Exercise 2.6

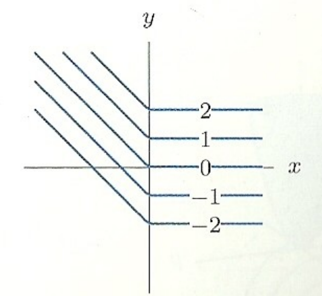
(a) Sketch level curves of $f(x, y) = \sqrt{x^2 + y^2} + x$ for $f = 1, 2, 3$.

(b) For what calues of c can level curves $f = c$ be drawn?

(c) Sketch a contour diagram for f.

Exercise 2.7

The below figure is the contour diagram of $f(x, y)$



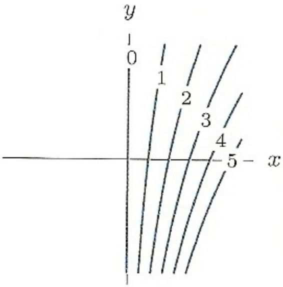
Sketch the contour diagram of each od the following functions.

(a) $3f(x, y)$; (b) $f(x, y) - 10$;

(c) $f(x - 2, y - 2)$; (d) $f(-x, y)$.

Exercise 2.8

The below figure part of the contour diagram of $f(x,y)$.



Complete the diagram for $x < 0$ if

(a) $f(-x,y) = f(x,y)$;   (b) $f(-x,y) = -f(x,y)$.

Exercise 2.9

(a) Draw the contour diagram of $f(x, y) = g(x - y)$ if

(i) $g(t) = t^2$;

(ii) $g(t) = \sin t$;

(iii) $g(t) = \ln|t|$.

(b) What can you say about the level curves of a functions of the form $f(x, y) = g(x - y)$ where $g(t)$ is a one-variable function?

A family of level surfaces

Definition

A ***level surface***, or ***level set***, of a functions of three variables, $f(x, y, z)$, is a surface of the form

$$f(x, y, z) = c,$$

where c is a constant.

Example

The temperature, in ${}^oC$, at a point (x, y, z) is given by $T = f(x, y, z) = x^2 + y^2 + z^2$. The level surfaces of f:

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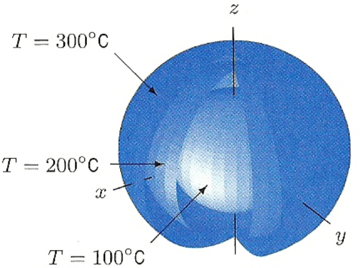

I: Level surfaces of $T = f(x, y, z) = x^2 + y^2 + z^2$, each one having a constant temperature

Exercise 2.10

What do the level surfaces of $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = z - y$ look like?

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Solution:

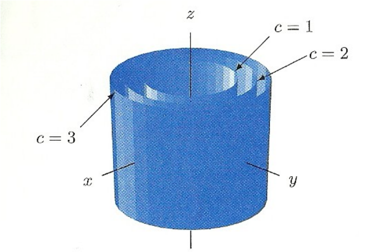

Figure 12.65: Level surfaces of $f(x, y, z) = x^2 + y^2$

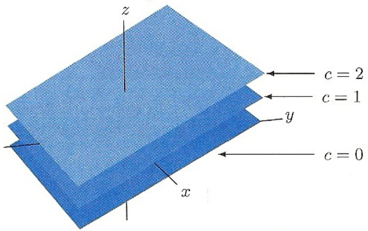

Figure 12.66: Level surfaces of $g(x, y, z) = z - y$

Exercise 2.11

What do the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$ look like?

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Solution:

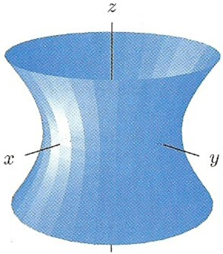


Figure 12.70: Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

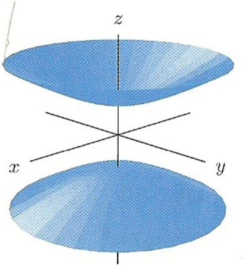


Figure 12.71: Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

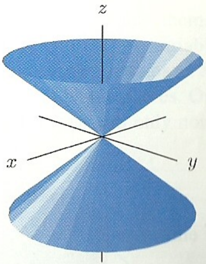


Figure 12.72: Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

A catalog of surfaces

For later reference, here is a small catalog of the surfaces we have encountered.

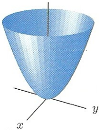


Figure 12.67: Elliptical paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

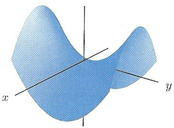


Figure 12.68: Hyperbolic paraboloid $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

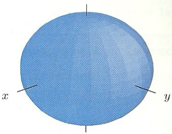
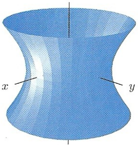


Figure 12.69: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



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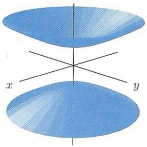

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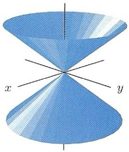

Figure 12.72: Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

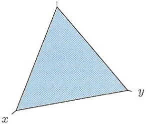

Figure 12.73: Plane $ax + by + cz = d$

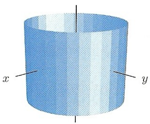

Figure 12.74: Cylindrical surface $x^2 + y^2 = a^2$

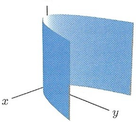

Figure 12.75: Parabolic cylinder $y = ax^2$