Mathematics. Multivariable Calculus

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Functions of several variables

Examples of quantities depending on more that on variable:

1. the amount of food grown - the amount of rain, the temperature, the amount of fertilizer used;
2. the rate of a chemical reaction - the temperature, the pressure of the environment in which it proceeds;
3. the strength of gravitational attraction between two bodies - their masses, their distance apart;
4. the rate of fallout from a volcanic explosion - the distance from the volcano, the time since the explosion;
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Some ways of representing a function of two variables:

1. graphically (by contour diagrams for instance);
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A graphical way of representing a function:

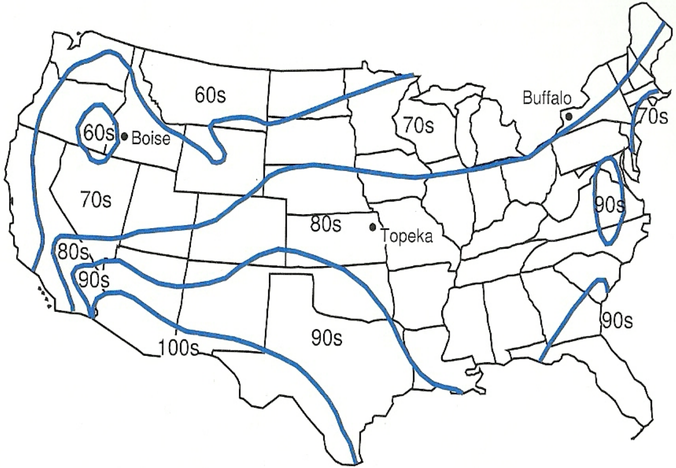
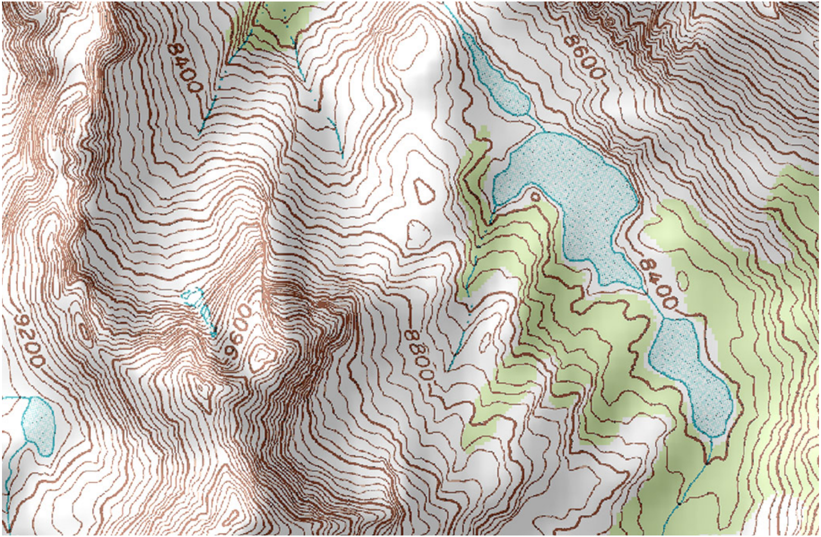


Figure 12.1: Weather map showing predicted high temperatures, T, on a summer day

Another graphical way of representing a function:



A numerical way of representing a function:

Table 12.1 *Quantity of beef bought (pounds/household/week)*

		Price of beef, ($/lb)			
		3.00	3.50	4.00	4.50
Household income per year, I (1000)	20	2.65	2.59	2.51	2.43
	40	4.14	4.05	3.94	3.88
	60	5.11	5.00	4.97	4.84
	80	5.35	5.29	5.19	5.07
	100	5.79	5.77	5.60	5.53

Algebraical ways of representing a function:

1. the strength of gravitational attraction between two bodies -

$$F(m_1, m_2, r) = G\frac{m_1 m_2}{r^2};$$

2. the distance from the origin on the plane -

$$d(x, y) = \sqrt{x^2 + y^2};$$

3. the distance from the origin in the space -

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2};$$

4. the Manhattan distance on the plane -

$$d((x_1, y_1), (x_2, y_2)) =$$

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$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

Graphs od functions of two variables

Let $D \subset \mathbb{R}^2$ and $f : D \longrightarrow \mathbb{R}$

Definition

*The **graph** of a function f of two variables is a set of all points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y) \in D$ and $z = f(x, y)$:*

$$Graph(f) = \{(x, y, z) \in \mathbb{R}^3 : \quad (x, y) \in D, \;\; z = f(x, y)\}.$$

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A wire-frame picture of the graph

Table 12.4 *Table of values of* $f(x,y) = x^2 + y^2$

x \ y	−3	−2	−1	0	1	2	3
−3	18	13	10	9	10	13	18
−2	13	8	5	4	5	8	13
−1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

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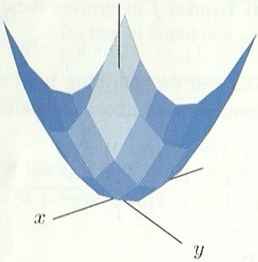

Figure 12.11: Wire frame picture of $f(x, y) = x^2 + y^2$ for $-3 \leq x \leq 3, -3 \leq y \leq 3$

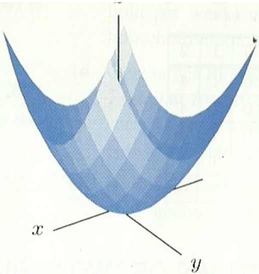

Figure 12.12: Wire frame picture of $f(x, y) = x^2 + y^2$ with more points plotted

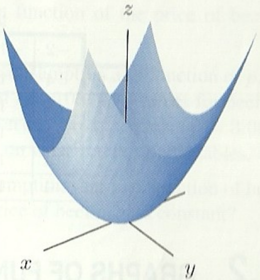

Figure 12.13: Graph of $f(x, y) = x^2 + y^2$ for $-3 \leq x \leq 3, -3 \leq y \leq 3$

A new graph from old one

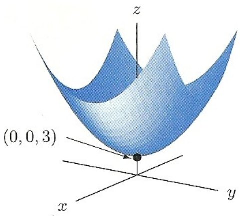

Figure 12.14: Graph of $g(x, y) = x^2 + y^2 + 3$

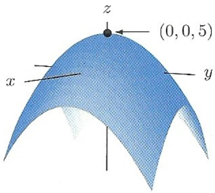

Figure 12.15: Graph of $h(x, y) = 5 - x^2 - y^2$

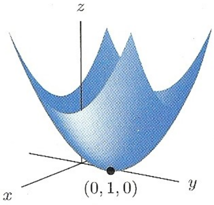

Figure 12.16: Graph of $k(x, y) = x^2 + (y - 1)^2$

$$f(x, y) = x^2 + y^2,$$
$$g(x, y) = f(x, y) + 3,$$
$$h(x, y) = -f(x, y) + 5,$$
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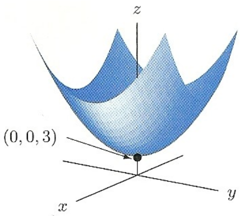

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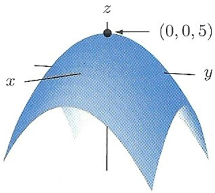

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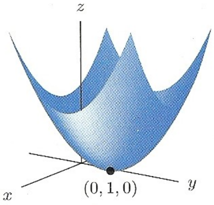

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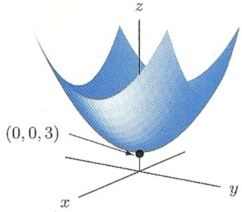

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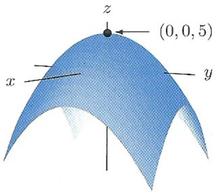

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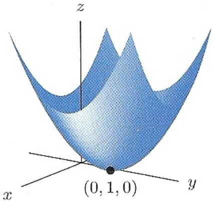

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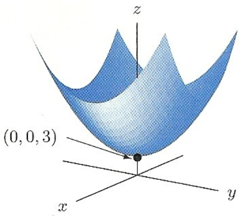

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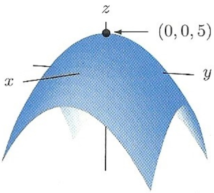

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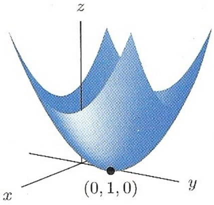

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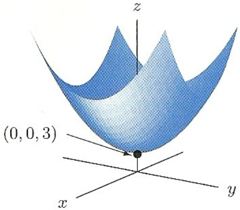

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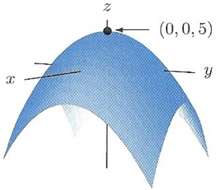

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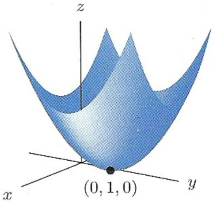

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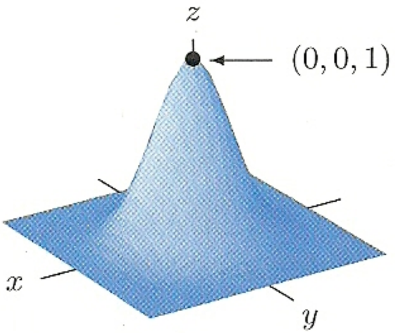

Figure 12.17: Graph of $G(x, y) = e^{-(x^2+y^2)}$

Cross-sections and the graph of a function

Let an x be fixed.

$$f(x,\cdot) : (\{x\} \times R) \cap D \longrightarrow \mathbb{R}$$

Definition

The function $f(x,\cdot)$ is called a ***cross-section*** *of f with fixed x.*

The graph of a cross-section is also called a cross-section.

A cross-section at $x = c$ = $Graph(f) \cap P$,

where P = the plane $x = c$.

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Example of a family of cross-sections

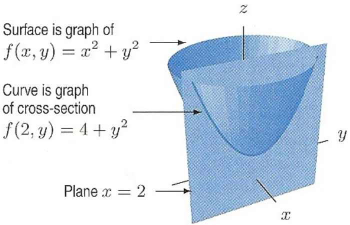

Figure 12.18: Cross-section of the surface $z = f(x, y)$ by the plane $x = 2$

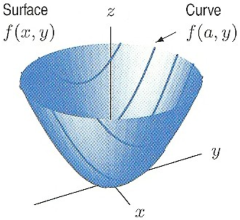

Figure 12.19: The curves $z = f(a, y)$ with a constant: cross-sections with x fixed

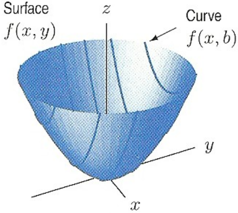

Figure 12.20: The curves $z = f(x, b)$ with b constant: cross-sections with y fixed

Exercise 1.2

Describe the cross-sections of the function $g(x, y) = x^2 - y^2$ with y fixed and then with x fixed.

Use these cross-section to describe the shape of the graph of g.

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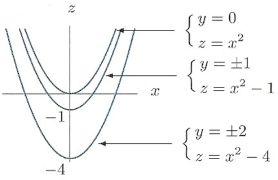

Figure 12.21: Cross-sections of $g(x, y) = x^2 - y^2$ with y fixed

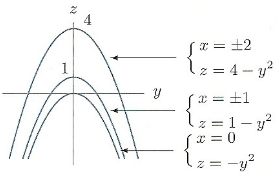

Figure 12.22: Cross-sections of $g(x, y) = x^2 - y^2$ with x fixed

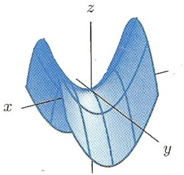

Figure 12.23: Graph of $g(x, y) = x^2 - y^2$ showing cross sections

One variable is missing

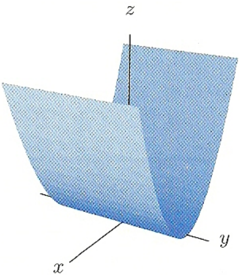

Figure 12.25: A parabolic cylinder $z = x^2$

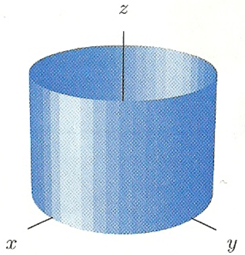

Figure 12.26: Circular cylinder $x^2 + y^2 = 1$

Graph the equation $x^2 + y^2 = 1$ in 3-space.

Although the equation $x^2 + y^2 = 1$ does not represent a function, the surface representing it can be graphed by the method used for $z = x^2$. The graph of $x^2 + y^2 = 1$ in the xy-plane is a circle. Since z does not appear in the equation, the intersection of the surface with any horizontal plane will be the same circle $x^2 + y^2 = 1$. Thus, the surface is the cylinder shown in Figure 12.26.

Exercise 1.3

Without a calculator or computer, match the functions with their graphs in Figure 12.27.

(a) $z = 2 + x^2 + y^2$;

(b) $z = 2 - x^2 - y^2$;

(c) $z = 2(x^2 + y^2)$;

(d) $z = 2 + 2x - y$;

(e) $z = 2$.

Exercise 1.3

Without a calculator or computer, match the functions with their graphs in Figure 12.27.

(a) $z = 2 + x^2 + y^2$;

(b) $z = 2 - x^2 - y^2$;

(c) $z = 2(x^2 + y^2)$;

(d) $z = 2 + 2x - y$;

(e) $z = 2$.

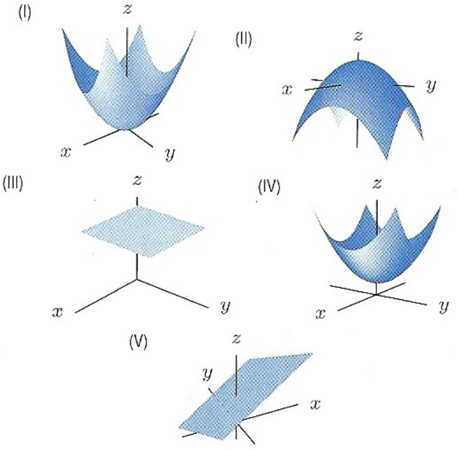

Figure 12.27

Exercise 1.4

Without a calculator or computer, match the functions with their graphs in Figure 12.28.

(a) $z = \frac{1}{x^2+y^2}$;

(b) $z = -e^{-x^2-y^2}$;

(c) $z = x + 2y + 3$;

(d) $z = -y^2$;

(e) $z = x^3 - \sin y$.

Exercise 1.4

Without a calculator or computer, match the functions with their graphs in Figure 12.28.

(a) $z = \frac{1}{x^2+y^2}$;

(b) $z = -e^{-x^2-y^2}$;

(c) $z = x + 2y + 3$;

(d) $z = -y^2$;

(e) $z = x^3 - \sin y$.

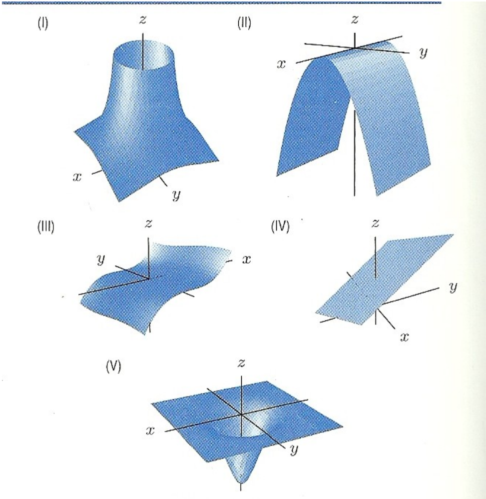

Figure 12.28

Exercise 1.5

(a) $z = xye^{-(x^2+y^2)}$;

(b) $z = \cos\left(\sqrt{x^2 + y^2}\right)$;

(c) $z = \sin y$;

(d) $z = -\frac{1}{x^2+y^2}$;

(e) $z = \cos^2 x \cos^2 y$;

(f) $z = \frac{\sin(x^2+y^2)}{x^2+y^2}$;

(g) $z = \cos(xy)$;

(h) $z = |xy|$;

(i) $z = (2x^2 + y^2)e^{1-x^2-y^2}$.

Exercise 1.5

(a) $z = xye^{-(x^2+y^2)}$;

(b) $z = \cos\left(\sqrt{x^2+y^2}\right)$;

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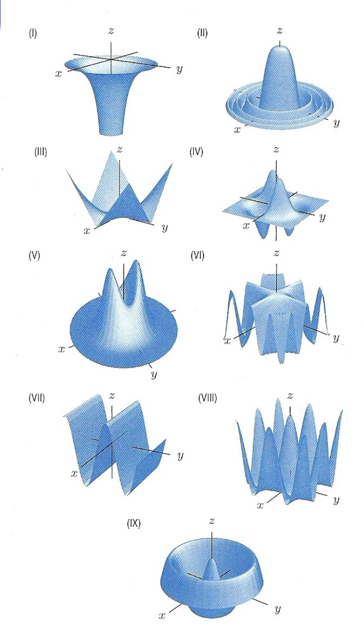

Exercise 1.6

Sketch a graph of the surface and briefly describe it in words:

a) $z = 3$; b) $x^2 + y^2 + z^2 = 9$; c) $z = x^2 + y^2 + 4$;

d) $z = 5 - x^2 - y^2$; e) $z = y^2$; f) $2x + 4y + 3z = 12$;

g) $x^2 + y^2 = 4$; h) $x^2 + z^2 = 4$.