AN INTRODUCTION TO SOME PROBLEMS OF SYMPLECTIC TOPOLOGY

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Abstract. We give a short introduction to some open problems in symplectic topology, including existence of symplectic structure on $M \times S^1$ or on exotic tori and existence of symplectic circle actions on a symplectic manifolds which admit smooth circle actions. Relations between these problems are also explained.

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1. INTRODUCTION

The principal aim of this note is to explain some open questions on symplectic manifolds in a way accessible to non-specialists and students. For this purpose we include an extensive preliminary part where basic notions and facts are described. Last three sections contain a discussion of:

- existence of symplectic forms on closed manifolds;
- existence of symplectic circle actions;
- existence of symplectic structures on exotic tori and a related question on symplectomorphisms of tori.

I omit most of technical details, to enable the reader to follow main route to those problems. Hopefully, this can be read by anybody knowing main facts and notions of elementary differential topology and the elementary part of de Rham theory of differential forms on manifolds. For further reading, detailed proof, enlightening comments and more I recommend a beautiful book by Dusa McDuff and Dietmar Salamon [MS].

This article is based on lectures delivered during Winter School on Topological Methods in Nonlinear Analysis which was organized by Juliusz Schauder Center for Nonlinear Studies at Copernicus University, Toruń, in February 2009. Here I skip most of that introductory part of the lectures, which contained an elementary review of notions which were used later. The background material can be found in many textbooks and it would not be very useful to include it here. Some possible sources are [BG, BT] for introduction to geometry of differential forms and de Rham complex and [M, H] for a comprehensible introduction to differential topology.

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2. PRELIMINARIES

Linear symplectic forms.

A bilinear skew-symmetric form ω on $\mathbb{R}^k$ is called symplectic if it is nondegenerate, i.e., if for some $X \in \mathbb{R}^k$ we have $\omega(X, Y) = 0$ for any Y, then $X = 0$. Any bilinear skew symmmetric form in a base $e_1^*, ..., e_k^*$ of the dual space $(R^{2n})^*$ is equal to $e_1^* \wedge e_2^* + ... + e_{2n-1}^* \wedge e_{2n}^*$, where $2n \leq k$. If such a form ω is non-degenerate, then $2n = k$ and there exists a base $e_1^*, ..., e_n^*, f_1^*, ..., f_n^*$ such that

$$\omega = e_1^* \wedge f_1^* + e_2^* \wedge f_2^* + ... + e_n^* \wedge f_n^*.$$

In other words, for the dual base $e_1, ..., e_n, f_1, ..., f_n$ we get $\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_j^i$ for any $i, j \in \{1, 2, ..., n\}$. Such a base is called symplectic. Any symplectic linear form admits many symplectic bases.

Equivalently, ω on $\mathbb{R}^{2n}$ is non-degenerate if and only if $\omega^n = \omega \wedge ... \wedge \omega$ is nonzero.

If we define J by $Je_i = f_i, Jf_i = -e_i, \ \ i = 1, 2, ..., n$, then $J^2 = -Id$. Thus we have on $\mathbb{R}^{2n}$ a complex linear structure. We have also, for any $v, w \in \mathbb{R}^{2n}$ $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$ if $v \neq 0$. The formula $< v, w >= \omega(v, Jw)$ defines a scalar product in $\mathbb{R}^{2n}$. In such a case we say that J is *compatible* with ω.

In the other direction, if J is a complex structure on $\mathbb{R}^{2n}$ and $<, >$ is a J-invariant scalar product, then $\omega(v, w) = - < v, Jv >$ is a symplectic linear form on $\mathbb{R}^{2n}$ and J is compatible with ω. For any given J there exists a J-invariant scalar product given for example as the averaged form $\frac{1}{2}(< v, w > + < Jv, Jw >)$, where $<, >$ is arbitrary scalar product.

Using symplectic bases, it is easy to see that if J_0 is the standard complex structure, then any other J is induced from J_0 by a linear isomorphism $T, \ J = TJ_0T^{-1}$. Since J_0 is preserved by complex isomorphisms T if and only if T is a complex isomorphism, we can identify J with an element of the quotient $GL(\mathbb{R}, 2n)/GL(\mathbb{C}, n)$. Up to homotopy type this is the quotient of maximal compact subgroups.

Corollary. *The space of all linear symplectic forms on* $\mathbb{R}^{2n}$ *is homeomorphic to* $GL(\mathbb{R}, 2n)/GL(\mathbb{C}, n)$ *and has the homotopy type of* $O(2n)/U(n)$*. If* $n = 2$*, then it is homotopically equivalent to* $S^2 \cup S^2$.

This is also not difficult to deduct from existence of a symplectic base that $\omega^n = \omega \wedge \omega \wedge ... \wedge \omega$ (n times) is a volume form (a non-zero 2n-form on $\mathbb{R}^{2n}$).

Note also that the space of all complex structures compatible with a given symplectic form is large, since one can change a symplectic base by a complex isomorphism preserving the form to get another symplectic base associated with the same form. As above one gets a homeomorphism of that space with $Sp(n)/GL(\mathbb{C}, n) \cap Sp(n)$, where $Sp(n)$ denotes the space of linear isomorphisms preserving the standard symplectic form.

Proposition 2.1. *The space of all complex structures on* $\mathbb{R}^{2n}$ *compatible with a given symplectic form is contractible.*

Symplectic differential forms.

Now we will consider exterior differential 2-forms on smooth manifolds, i.e., smooth sections of the second exterior power of the cotangent bundle. If such a form ω is symplectic, then at any point $x \in M$ we have a symplectic linear form ω_x on T_xM and ω_x smoothly depends on x. By definition, a 2-form ω is non-degenerate at $x \in M$ if for any nonzero vector $X \in T_xM$, the 1-form $\iota_X\omega$ does not vanish, where $(\iota_X\omega)(Y) = \omega(X, Y)$. Moreover, it is assumed that $d\omega = 0$.

Definition 2.2. *A differential 2-form is called* **symplectic** *if it is* **closed** *and* **non-degenerate at every point**.

A smooth complex structure on the tangent vector bundle of a manifold is called *almost complex structure* on M. This means that there is a bundle endomorphism $J : TM \to TM$ such that

(1) $J^2 = -Id$;
(2) $\omega(JV, JW) = \omega(V, W)$ for all U, V;
(3) a symmetric form defined as $g(U, V) = \omega(U, JV)$ is a Riemannian metric on M.

Contractibility of the space of complex structures on T_xM compatible with ω_x implies that there exist almost complex structures compatible with any symplectic form. Namely, construct J first locally using symplectic bases and then combine local structures to a global almost structure using 2.1 to deform one local J to another. In terms of bundles, a ω-compatible J is a section of a bundle with contractible fibre and the argument describes how to construct a section of such bundle. The space of such sections is contractible, thus we have

Proposition 2.3. *If a manifold has a symplectic structure, then it admits an almost complex structure. The space of all almost complex structures compatible with a given symplectic form*

However, only nondegeneracy is used to construct J. Thus existence of an almost complex structure is equivalent to existence of a differential form, not necessarily closed, which is non-degenerate at each point.

Examples.

(1) In $\mathbb{R}^{2n}$ consider coordinates $x_1, x_2, ..., x_n, y_1, ..., y_n$. The formula $\omega = dx_1 \wedge dy_2 + dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n$ defines a symplectic form.
Since ω is invariant with respect to translations, it defines also a symplectic form on the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.
(2) The volume form of a Riemannian surface is symplectic. Any oriented surface is symplectic.
(3) If ω_1, ω_2 are symplectic forms on manifolds M_1, M_2 respectively, then $\omega_1 \times \omega_2 = p_M^*\omega_M + p_N^*\omega_N$, where p_M, p_N are projections, is a symplectic form on $M_1 \times M_2$.
(4) For any manifold M the cotangent bundle T^*M admits a (noncompact) symplectic manifold. A symplectic form is given by a form $d\lambda$, where λ is the tautological 1-form on T^*M given by
$$\lambda_{v^*} = v^* d\pi.$$
Here $\pi : T^*M \to M$ jest the projection of the cotangent bundle and $v^* \in T^*M$ is a point in T^*M. In local coordinates $x_1, ...x_n$ on M we have the formula $\lambda_{v^*}(\frac{\partial}{\partial x_j}) = y_j$, if $v^* = \sum y_j dx_j$.

The following theorem shows some rigidity of symplectic structures.

Theorem 2.4. *[Moser] If ω_t is a smooth path of symplectic forms on M such that the cohomology class $[\omega_t]$ is constant, then there exists an isotopy $\psi_t \in Diff(M)$ satisfying $\psi_0 = Id_M$ and $\psi_t^*\omega_t = \omega_0$.*

The proof is based on so called Moser trick. Since $[\omega_t] = const$, thus there exists σ_t such that $\frac{d}{dt}\omega_t = d\sigma_t$. Consider the 1-parameter family of vector fields X_t defined (uniquely, since ω_t are non-degenerate) by the equation $\sigma_t = -\iota(X_t)\omega_t$. This family defines a path ψ_t of diffeomorphisms by

$$\frac{d}{dt}\psi_t = X_t(\psi_t).$$

Differentiating, with respect to t, the equality $\psi_t^*\omega_t = \omega_0$ we get $\psi_t^* d(\sigma_t + \iota(X_t)\omega_t) = 0$, thus the isotopy ψ_t given by X_t has the required property.

The theorem can be used to prove the following property which shows that there is no local symplectic invariants. This is in contrast with Riemannian geometry, where curvature invariants play a prominent role.

Theorem 2.5. *(Darboux) For any symplectic form ω on M and any point $P \in M$ there exists a local coefficient system $x_1, ..., x_n, y_1, ..., y_n$ around P such that $\omega = dx_1 \wedge dy_2 + dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n$.*

3. SYMPLECTIC FORMS ON CLOSED MANIFOLDS

The existence problem for symplectic structures has a simple answer in the case of open (i.e., non-compact or with non-empty boundary) manifolds. However, the proof of the following theorem is quite difficult (see [MS]).

Theorem 3.1. *[Gromov] If M is an open almost complex manifold, then it admits a symplectic form.*

For closed manifolds the problem whether there is a symplectic form on a given manifold is simple only in dimension 2, where orientability is necessary and sufficient. In dimension 4 there are some answers, see Section 6, and in higher dimensions essentially nothing is known.

Consider a closed symplectic manifold M of dimension $2n$. By Proposition 2.3 M is almost complex. There are non-trivial obstructions to impose an almost complex structure on M. For $2n = 4$ a characterization of closed almost complex manifolds was given by Ehresman and Wu.

Theorem 3.2. *A closed 4-manifold M admits an almost complex structure if and only if there exists a class $c \in H^2(M, \mathbb{Z})$ such that its reduction mod 2 is equal to the second Stiefel - Whitney class of M and $c^2 = 2\chi(M) + 3\sigma(M)$, where χ, σ denote respectively the Euler characteristic and the signature.*

Using refehwu one can check that the connected sum $\#^k\mathbb{C}P^2$ of k copies of $\mathbb{C}P^2$ is almost complex if and only if k is odd. This implies that $\mathbb{C}P^2\#\mathbb{C}P^2$ admits no symplectic structure. To calculate this, let us recall that for $\mathbb{C}P^2$ we have $w_1 \neq 0, \chi = 3, \sigma = 1$. It is not difficult to calculate that $H^2(\#^k\mathbb{C}P^2) \cong \oplus^k H^2(\mathbb{C}P^2)$ and that $(a_1, ..., a_k)^2 = a_1^2 + ... + a_k^2 \in H^4(M, \mathbb{Z}) \cong \mathbb{Z}$. Thus $\chi = k + 2, \sigma = k, w_1 = (1, 1, ..., 1)\ mod\, 2$. Thus for the class $c = (a_1, ..., a_k)$ required by the Ehresmann - Wu theorem all entries should be odd integers. One can show that $c^2 = a_1^2 + ... + a_k^2$ cannot be equal to $\neq 5k + 4$ for even k. For $k = 2$ this boils down to a simple fact that there is no integers a, b such that $a^2 + b^2 = 14$. For $k = 3$ a solution is $c = (3, 3, 1)$.

For closed manifolds another obstruction to existence of symplectic forms arises from the fact that ω^n is a volume form. This implies that any symplectic form determines an orientation of the underlying manifold. However, this can be concluded as well from the almost complex structure, since any complex structure on a vector space V defines uniquely an orientation of V. But for closed manifolds it shows more. Namely, on a connected 2n-dimensional manifold we have $\int_M \omega^n \neq 0$, thus the cohomology class $[\omega]^n = [\omega] \cup ... \cup [\omega]$ as well as $[\omega]$ are nonzero.

Thus we have two basic obstructions to get a symplectic structure.

Proposition. *If a 2n-dimensional manifold M admits a symplectic structure, then M is almost complex and there is a class $u \in H^2(M;\mathbb{Z})$ such that $u^n \neq 0$. In particular, $H^2(M) \neq 0$.*

Examples.

(1) Complex projective space $\mathbb{C}P^n$ is symplectic and the following Fubini - Study form τ gives a symplectic structure.

$$\tau = \frac{1}{2(\sum_\mu \overline{z}_\mu z_\mu} \sum_k \sum_{j \neq k} \overline{z}_j z_j dz_k \wedge d\overline{z}_k - \overline{z}_j z + k dz_j \wedge d\overline{z}_k.$$

where we denote $dz_j = dx_j + idy_j, d\overline{z}_j = dx_j - idy_j$, for $z_j = x_j + iy_j$.

This is an example of a *Kähler* manifold, i.e., a **complex** manifold with a Riemannian metric g such that $\omega(V, W) = g(V, JW)$ is a closed form, where J is the almost complex structure on M provided by its complex structure.

(2) The sphere S^{2n} of dimension $2n$ does not admit any symplectic form for $n > 2$, since $H^2(S^{2n} = 0$. For the same reason $S^3 \times S^1$ is not symplectic. Moreover, $S^2 \times S^4$ is not symplectic because for any $x \in H^2(S^2 \times S^4; \mathbb{Z})$ we have $x^3 = 0$.

4. CONSTRUCTIONS OF SYMPLECTIC MANIFOLDS

The product of two symplectic manifolds is symplectic. Hence, the question whether a fibre bundle with symplectic base and symplectic fibre is symplectic is natural. In general this fail to be true as the following example shows.

Example. Let $S^3 \to S^2$ be the Hopf fibre bundle. It is a bundle with fibre S^1 given by the natural action, by multiplication, of unit (of module 1) complex numbers on unit quaternions. Then $S^3 \times S^1 \to S^3 \to S^2$ is a fibre bundle map with fibre T^2, hence both base and fibre are symplectic, while the total space is not. Moreover, the structure group of this fibre bundle is the symplectomorphism group of the fibre, which is (since we are in dimension 2) the group of volume preserving diffeomorphisms. In fact, the structure group of Hopf fibration is the isometry group of S^1, thus the structure group of $S^3 \times S^1 \to S^2$ is the isometry group of T^2.

A sufficient condition for a fibre bundle $p : M \to B$ with both base and fibre symplectic to have a symplectic total space was given by Thurston [Th]. A condition imposed on bundles was that the structure group is the symplectomorphism group of the fibre. We say in this case that such a bundle is *symplectic*. This is natural, if one expects on M a symplectic form which restricts to a symplectic form on all fibres.

For a point b in the base let i_b denote the inclusion of the fibre $F_b = p^{-1}(b) \subset M$. In a symplectic fibration each fibre has a well defined symplectic form ω_b symplectomorphic to ω_F. But, as the example above shows, some further assumptions are needed.

Theorem 4.1. *[Thurston] Consider a symplectic bundle $p : M \to B$ with closed symplectic base (B, ω_B) and closed symplectic fibre (F, ω_F). If there exists a cohomology class $u \in H^2(M, \mathbb{R})$ such that $i_b^* u = [\omega_b]$, then there exists a symplectic form ω_M on M which is compatible with the fibration, i.e., $i_b^* \omega_M = \omega_b$.*

However, if a symplectic manifolds fibres with a symplectic fibre then one cannot in general expect the base to be symplectic. An example is $\mathbb{C}P^3$ which is fibred over S^4 with fibre S^2. More such examples can be deduced from [R].

We give now two interesting examples of a fibre bundle with symplectic base and fibre.

Example 4.2.

Let $Diff(D^{2n}, S^{2n-1})$ denote the group of diffeomorphisms equal to the identity in a neighborhood of the boundary sphere S^{2n-1}. Then f extends by the identity to a diffeomorphism of any 2n-manifold X if an embedding of D^{2n} into X is given. Consider $f \in Diff(D^{2n}, S^{2n-1})$ not in the identity component. For $X = S^{2n}$ we get again a diffeomorphism f_S which is not isotopic to the identity and it is a classical fact that it correspond to an exotic (2n+1)-sphere $\Sigma_f = D^{2n+1} \cup_{f_S} D^{2n+1}$ (a smooth manifold homeomorphic but not diffeomorphic to the sphere with the standard differential structure). If X is the 2n-torus $\mathbb{T}^{2n} = S^1 \times ... \times S^1$, denote the resulting diffeomorphism by f_T. In this case we will get also an exotic manifold in the following way. Take $\mathbb{T}^{2n} \times [0, 1]$ and glue the ends according to $(x, 0) \sim (f_T(x), 1)$. The resulting manifold depends, up to diffeomorphism, only on the isotopy class of f and it is called the *mapping torus of f_T*. We denote it by $\mathbb{T}(f_T)$. From the fact that f_T is supported in a disk (i.e., is equal to Id outside a disk) it is not difficult to argue that $\mathbb{T}(f_T)$ is obtained from the standard torus $\mathbb{T}^{2n+1}$ by a connected sum with the homotopy sphere Σ_f. This is known that $\mathbb{T}(f_T)$ is homeomorphic but not diffeomorphic to $\mathbb{T}^{2n+1}$, cf. [W], "Fake Tori" chapter. Note also that, by construction, $\mathbb{T}(f_T)$ fibers over S^1 with fibre $\mathbb{T}^{2n}$.

Now $M = \mathbb{T}_f \times S^1$ fibres over $\mathbb{T}^2$ with fiber $\mathbb{T}^{2n}$. The fibration is symplectic if and only if the diffeomorphism f_T is a symplectomorphism. Moreover, if this is the case, then the other assumption of Thurston's theorem is satisfied. This is because f is homotopic (even topologically isotopic) to the identity, thus the fibration is equivalent, up to fibrewise homotopy equivalence, to the product $\mathbb{T}^2 \times \mathbb{T}^{2n}$. So the required cohomology class exists.

Example 4.3.

Let A be a linear map of the torus $\mathbb{T}^2$ given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Consider $\mathbb{T}_A = \mathbb{T}^2 \times [0, 1]/(x, 0) \sim (A(x), 1)$ and a fibration $M = \mathbb{T}_A \times S^1 \to \mathbb{T}^2$. By Theorem 4.1 there exists a symplectic structure on M. However, direct calculations show that first Betti number $b_1 M = \dim H_1(M, \mathbb{R}) = 3$. This implies that there is no Kähler structure on M, since odd Betti numbers of closed Kähler manifolds are always even.

This was first example of a closed symplectic manifold with no Kähler structure. Later many other such examples were constructed. One can say that all known bounds for the topology of closed Kähler manifolds fail in symplectic case (see [TO]).

Let us sketch two other useful constructions of new symplectic manifolds from old ones.

First is the blow up of a manifold. By blow up of a point x of a $2n$-manifold M we mean a compactification of $M - \{x\}$ by $\mathbb{C}P^{n-1}$, where in a chart $U \cong \mathbb{C}^n$ around x (with x corresponding to $(0, ..., 0)$) any complex plane is compactified by the point of $\mathbb{C}P^{n-1}$ which represents this plane. A direct generalization gives a blow up along a submanifold. This is given by compactifying each normal space of a symplectic submanifold as above. Topologically, a manifold obtained by blow up of a point in M is diffeomorphic to a connected sum $M \# \overline{\mathbb{C}P^n}$, where $\overline{\mathbb{C}P^n}$ is $\mathbb{C}P^n$ with the orientation reversed.

Theorem 4.4. *If M is a symplectic 2n-manifold, then a blow up of M is also symplectic.*

However, there is no canonical choice of a symplectic structure on a blown up symplectic manifold. For various data used to perform the operation one can get different (non symplectomorphic) symplectic structures. This is in contrast with the case of differential manifolds, where on a connected sum one can construct a unique, up to a diffeomorphism, differential structure.

Another operation on symplectic manifolds was introduced by Gompf [G] and it is often called Gompf's surgery. Consider two symplectic manifolds M, N and symplectic submanifolds M_0, N_0 of codimension two. Assume that M_0 is symplectomorphic to N_0 and the normal bundle $\nu(M_0)$ is inverse to the normal bundle $\nu(N_0)$. This means that there is a orientation changing linear isomorphism of that bundles, covering the given symplectomorphism $\Phi_0 : M_0 \to N_0$. In terms of Chern classes, $c_1\nu(M_0) = -c_1\nu(N_0)$. Note that first Chern classes classify complex bundles of complex dimension 1. Then one can find a symplectic structure on $(M - \nu_1(M_0)) \cup_f (N - \nu_1(N_0)$, where ν_1 denotes the open unit disc bundle and f is a diffeomorphism of the boundary sphere bundles covering Φ_0.

As an application, Gompf has shown that in any even dimension greater than 2 any finitely presented group is the fundamental group of a closed symplectic manifold. Compare also [IRTU], where some restrictions on the fundamental group were found under assumption that the symplectic structure is *symplectically aspherical,* i.e., the symplectic form vanishes on all spherical homology 2-classes.

5. SYMPLECTIC GROUP ACTIONS

Isomorphisms in the category of symplectic manifolds is a diffeomorphism preserving symplectic forms. Thus a *symplectomorphism* of (M, ω) is a diffeomorphism $f : M \to M$ such that $f^*\omega = \omega$. The group of all symplectomorphism will be denoted by $Symp(M, \omega)$. For a compact manifold we consider the C^1 topology on the group. This is always an infinitely dimensional space, since for any path H_t of smooth functions the path X_t of vector fields defined by $\iota_{X_t}\omega = dH_t$, the associated path of diffeomorphisms preserves the form ω, compare the proof of Theorem 2.4.

If a group G acts smoothly on a symplectic manifold (M, ω), then we say that the action is *symplectic* if ω is $G-$invariant. In particular, $g \in Symp(M, \omega)$ for any $g \in G$. We restrict in this note to the case $G = S^1$.

For a smooth action of S^1 there is a vector field V on M which generate the action, i.e.,the action is the flow of V. It is the image of the unit invariant vector field tangent to S^1 under the differential of the action. The field V is tangent to orbits of the action and its zero set is equal to the fixed point set. If it is a symplectic action, then the form $\iota_V \omega$ is a closed 1-form, as it follows from the formula $L_V \omega = \iota_V d\omega + d\iota_V \omega$ for the Lie derivative $L_V \omega$. If the cohomology class $[\iota_V \omega]$ vanishes, then the action is called hamiltonian and its moment map is defined as a map $H : M \to \mathbb{R}$ such that $dH = \iota_V \omega$. More generally, if we assume that $[\iota_V \omega]$ is an integer class (it is in the image of $H^2(M; \mathbb{Z})$), then there exists a generalized moment map $H : M \to S^1$ such that $H^*\theta = \iota_V \omega$, where θ is the standard invariant 1-form on S^1. Moment maps have nice properties: the set of critical points is equal to the zero set of V, hence to the fixed point set of the action. The moment map is a Morse-Bott function, i.e., it is nondegenerate in the normal bundle of the critical point submanifold.

Certainly, a hamiltonian action must have fixed points, since in this case the moment map is a real valued map on a closed manifold. In dimension 4 a symplectic action on a closed manifold is hamiltonian if and only if it has fixed points. In dimension 6 an example of a non-hamiltonian symplectic action with non-empty set of fixed points was constructed by McDuff.

It is well-known that fixed points of a symplectic action are symplectic manifolds, cf. [GuSt], Lemma 27.1.

Lemma 5.1. *Let G be a compact Lie group. If G acts symplectically on a symplectic manifold M, then the fixed point set M^G is a symplectic submanifold.*

Proof. Let $x \in M^G$. Then, when an invariant Riemannian metric is chosen, G acts on a normal slice via a faithful orthogonal representation. Thus $U \in T_x(M)$ belongs to $T_x(M^G)$ if and only if $g_*U = U$ for every $g \in G$. Moreover, vectors of the form $V - g_*V$ span a subspace of T_xM transversal to M^G. Hence for $U \in T_x(M^G)$ we have $\omega(U, V) = \omega(g_*U, g_*V) = \omega(U, g_*V)$, and therefore $\omega(U, V - g_*V) = 0$ for any $g \in G$ and $V \in T_xM$. So if $\omega(U, W) = 0$ for all $W \in T_xM^G$, then also $\omega(U, W') = 0$ for all $W' \in T_xM$ and this implies $U = 0$. Thus $\omega|M^G$ is symplectic. □

Corollary 5.2. *Let G be a compact Lie group and let H be a closed subgroup of G. If G acts symplectically on a symplectic manifold M, then the set of points with isotropy equal to H is a symplectic manifold.*

An analogous property for almost complex manifolds and actions is straightforward.

Lemma 5.3. *If a compact Lie group G acts smoothly on an almost complex manifold M preserving an almost complex structure J, then the fixed point set M^G is a J-holomorphic submanifold of M.*

Proof. If J is G-invariant, then for $U \in T_x(M^G)$ and any $g \in G$ we have $g_*(JU) = g_*Jg_*^{-1}g_*U = JU$. □

Let us assume now that S^1 acts freely and symplectically on (M, ω), V generates the action and $X = M/S^1$. The 1-form $\iota_V \omega$ is closed and descends to X to a closed nowhere vanishing 1-form. This implies that X fibres over a circle [Ti].

Conversely, if X admits a symplectic fibration over the circle, then $X \times S^1$ admits a symplectic structure. A symplectic fibration over S^1 is the torus $T(f)$ of a symplectomorphism $f\colon M \to M$. Thurston's theorem gives a symplectic structure on $S^1 \times T(f)$, which is a symplectic fibration over $S^1 \times S^1$ with fibre M. It suffices to check that the cohomology class of the symplectic form on M is in the image of the cohomology homomorphism i^*, where $i : M \to T(f)$ is the inclusion. The claim follows from the Mayer–Vietoris exact sequence resulting from a decomposition of S^1 into two intervals (elements in cohomology which are invariant under the gluing map all are in the image of i^*). No other examples of symplectic manifolds of the form $X \times S^1$ are known. See also Section 7.

This extends to the case of a circle action with no fixed points, but then X is in general an orbifold.

6. EXISTENCE QUESTIONS

As we have seen above, there are two basic obstructions to impose a symplectic structure on a closed manifold.

Definition 6.1. *A closed manifold M of dimension $2n$ which is almost complex and has a class $u \in H^2(M, \mathbb{R})$ such that $u^n \neq 0$ will be called* **homotopically symplectic.**

The term *cohomologically symplectic,* or *c-symplectic* is used for a manifold with a class $u \in H^2(M; \mathbb{Z})$ such that $u^n \neq 0$, see e.g. [A]. The term "homotopically symplectic" refers to the homotopy type of the classifying map of the tangent bundle of M, which we consider as a part of the structure of M. An oriented manifold M is almost complex if and only if the classifying map $\tau : M \to BGL(2n, \mathbb{R})$ of its tangent bundle lifts to a map $\tilde{\tau} : M \to BGL(n, \mathbb{C})$ so that $\tau = P\tilde{\tau}$, where $P : BGL(n, \mathbb{C}) \to BGL(2n, \mathbb{R})$ is the forgetful map. This condition depends on the homotopy type of the classifying map.

Question 6.2. *Does any closed, homotopically symplectic manifold admit a symplectic form?*

Obviously the problem depends only on the diffeomorphism type of M. There is a description of symplectic manifolds in topological terms as those manifolds which admit so called topological Lefschetz pencils [G1], but to decide whether a manifold has such a structure is as difficult as to construct a symplectic form.

In dimension 4 the answer to 6.2 is negative.

Example 6.3.

$\#^3\mathbb{C}P^2$ is homotopically symplectic and has no symplectic structure.

That it is a homotopically symplectic manifold we have seen in Section 3. Nonexistence of symplectic structure was proved using Seiberg - Witten invariants of diffeomorphism type. They are defined for closed 4-manifolds via moduli spaces of a differential equation related to the Dirac operator. The invariant is given by a function $SW_M : H^2(M, \mathbb{Z}) \to \mathbb{Z}$ with finite support. See [S]. A powerful theorem providing a much more delicate necessary condition than homotopy symplecticness to existence of a symplectic structure was proved by Taubes [T].

Theorem 6.4. *[Taubes] For any closed symplectic 4-manifold there exists a class $u \in H^2(M, \mathbb{Z})$ such that $SW(u) = \pm 1$.*

The fact that $\#^3\mathbb{C}P^2$ does not satisfy the above condition follows from properties of Seiberg - Witten invariant. Namely, for a connected sum of two closed 4-manifolds with positive b^+, Seiberg - Witten invariant vanish. Here b^+ is the dimension of positive defined part of $H^2(M, \mathbb{R})$ with respect to the intersection form $(u, u') \mapsto (u \cup u')[M]$.

In higher dimensions there is no known example of non-symplectic but homotopically symplectic manifold. In particular it is unknown whether $(\#^3\mathbb{C}P^2) \times S^2$ is symplectic or not.

Seiberg-Witten invariants are defined only in dimension 4. In this dimension they are equivalent to so called Gromov-Witten invariants. The latter was defined in any dimension with a pseudocycle obtained form the moduli space of pseudoholomorphic maps by Michel Gromov in a seminal paper [Gr]. See [MS1] for an exposition of the theory.

We describe now how some related examples can be obtained.

Example 6.5.

Let M and N be two closed simply connected 4-dimensional smooth manifolds such that the following condition holds.

(1) M and N are homeomorphic, but not diffeomorphic.
(2) M is not a symplectic manifold, but N admits a symplectic structure.
(3) The second Stiefel–Whitney class $w_2(M)$ vanishes.
(4) The cohomology $H^*(M, \mathbb{Z})$ is torsion free.

Then $M \times S^2$ is diffeomorphic to $N \times S^2$, hence both are symplectic.

We refer to [W1]. Indeed, under our assumptions the diffeomorphism type is completely determined by the multiplicative structure of the cohomology ring with integer coefficients and the first Pontriagin class. It follows from Theorem 3 in [W1] which can be stated as follows. *The diffeomorphism classes of closed simply connected* 6*-manifolds* M *with torsion free integral cohomology, whose second Stiefel–Whitney class vanishes, correspond bijectively to the isomorphism classes of an algebraic invariant consisting of:*

- *two free abelian groups* $H = H^2(M; \mathbb{Z})$ *and* $G = H^3(M; \mathbb{Z})$*,*
- *a symmetric trilinear map* $\mu : H \times H \times H \to \mathbb{Z}$ *given by the cup product,*
- *a homomorphism* $p_1 : H \to \mathbb{Z}$ *determined by the first Pontriagin class* p_1*.*

Note that $p_1(M \times S^2)$ is inherited from M and p_1 is a topological invariant for closed 4-manifolds. Thus $p_1(M \times S^2) = p_1(N \times S^2)$, $w_2(M \times S^2) = w_2(M \times S^2) = 0$ and thus $M \times S^2$ is diffeomorphic to $N \times S^2$.

Some examples of pairs (M, N) as required above are obtained by applying to symplectic 4-manifolds constructions such as logarithmic transformation or knot surgery. To detect both non-diffeomorphism and non-symplecticness one uses Taubes' theorem. (see 12.4 in [Sc] or [P]). An explicit example is the Barlow surface which is non-symplectic and homeomorphic to $\mathbb{C}P^2$ blown up in 8 points. There exists also a non-symplectic manifold homeomorphic to K3 surface.

7. CIRCLE ACTIONS: SMOOTH VERSUS SYMPLECTIC

A more specific existence question is when does exist a symplectic structure on the product of a manifold by the circle. As it was explained in Section 5, if a closed manifold M with a free S^1 action admits a invariant symplectic form, then $X = M/S^1$ fibres over the circle. In the other direction, for the product $X \times S^1$, a fibration of X over a circle enables us to construct a symplectic form on M.

Question 7.1. *Let X be a closed manifold. Is it true that if $X \times S^1$ is symplectic, then X fibres over S^1?*

For X of dimension 3 this question was posed by Taubes and answered positively, after a series of partial results of many authors, by Friedl and Vidussi [FV]. Their proof uses Seiberg -Witten invariants and it does not extend to higher dimensions.

Remark. Questions 6.2,7.1 cannot simultaneously have positive answers in higher dimensions. Namely, there are manifolds which do not fibre over the circle, but their products with the circle are homologically symplectic, e.g. the connected sum of two copies of tori $T^{2k+1} \# T^{2k+1}$.

A more general conjecture for dimension 4 was stated by Scott Baldridge in [B].

Conjecture 7.2. *Every closed 4-manifold that admits a symplectic form and a smooth circle action also admits a symplectic circle action (with respect to a possibly different symplectic form).*

In the same paper Baldgidge gave a partial answer.

Theorem 7.3. [B] *If M is a closed symplectic 4-manifold with a circle action such that the fixed point set is non-empty, then there exists a symplectic circle action on M.*

It seems unlikely that this continue to be true in higher dimensions, but one can ask the following question: *under what condition a closed symplectic manifold with a smooth circle action does admit a symplectic circle action?*

There are examples of smooth circle actions on symplectic manifolds which have non-symplectic sets of fixed points or non-symplectic sets of points with a given isotropy. By Lemma 5.1 any such action is not symplectic with respect to any symplectic structure.

Example 7.4.

Let M, N be a pair of 4-manifolds described in Section 6. Then $M \times S^2 \times ... \times S^2$ is symplectic (since it is diffeomorphic to $N \times S^2 \times ... \times S^2$) and there is an action, given by the standard action on each copy of S^2, having a sum disjoint copies of M as the fixed point set.

More examples can be found in [HPT].

8. SYMPLECTOMORPHISMS AND EXOTIC TORI

It is known that for any $m \geq 5$, there are *exotic tori,* i.e., smooth manifolds $\mathcal{T}^m$ which are homeomorphic but not diffeomorphic to the standard torus $\mathbb{T}^m$.

Question 8.1. *Given a symplectic manifold $\mathcal{T}^{2n}$ homeomorphic to $\mathbb{T}^{2n}, n > 2$, is $\mathcal{T}^{2n}$ diffeomorphic to $\mathcal{T}^{2n}$?*

This is motivated by the same question posed by Benson and Gordon in [BeG] for Kähler manifolds. It has positive answer, a proof that there are no Kähler structures on exotic tori can be obtained from the Albanese map $M \to T^k$ by showing that for a manifold homeomorphic to a torus the map is a homotopy equivalence. This implies that it is in fact a diffeomorphism. More general results are given in [BC, C].

Let us look on Example 4.2 from that point of view. This leads to the following.

Question 8.2. *Given an exotic sphere Σ_f of dimension $2n-1$, is there a symplectic structure on $\mathcal{T} = (\mathbb{T}^{2n-1}\#\Sigma_f) \times S^1$?*

As we have seen in Section 4, the answer were positive when there exists a diffeomorphism $f \in Diff(D^{2n-2}, S^{2n-3})$ such that Σ_f is exotic and the diffeomorphism f_T obtained from f is isotopic to a symplectomorphism. Thus we come to the following question.

Question 8.3. *Given a symplectomorphism $f : \mathbb{T}^{2n-2} \to \mathbb{T}^{2n-2}$ supported in an embedded disc, is f smoothly isotopic to the identity?*

A similar problem whether a symplectomorphism of a torus which acts trivially on homology is isotopic to the identity was mentioned in [MS], p. 328.

One can also ask under what assumptions a diffeomorphism of $\mathbb{T}^{2n}$ is isotopic to a symplectomorphism. We describe examples such there is no symplectomorphisms in the isotopy class [HT].

Let $\pi_0(\mathrm{Diff}_+(M))$ denote the group of isotopy classes of orientation preserving diffeomorphisms of a smooth oriented manifold M. Assume now that M is $2n$-dimensional and admits almost complex structures, and let $\mathbb{J}M$ denote the set of homotopy classes of such structures, compatible with the given orientation. Any diffeomorphism f acts on the set of all almost complex structures by the rule

$$f_*J = df\,J df^{-1},$$

where $df : TM \to TM$ denotes the differential of f. This action clearly descends to the action of $\pi_0(\mathrm{Diff}_+(M))$ on $\mathbb{J}M$.

We show that there exist diffeomorphisms $f : \mathbb{T}^{8k} \to \mathbb{T}^{8k}$ supported in a disc which do not preserve the homotopy class $[J_0] \in \mathbb{J}M$ of the standard complex structure. Therefore, they cannot be isotopic to symplectomorphisms with respect to the standard symplectic structure ω_0. Indeed, any symplectomorphism carries any almost complex structure compatible with a symplectic form to another almost complex structure compatible with the same symplectic form, but the space of all such almost complex structures is contractible.

Let us a sketch the proof [HT] that such f exist. There is a necessary homotopic condition on a diffeomorphism to preserve the homotopy class of J_0.

Theorem 8.4. *Let $f \in \mathrm{Diff}(\mathbb{T}^{4n})$ be supported in a disc $D^{4n} \subset \mathbb{T}^{4n}$. If f preserves J_0, then the differential df restricted to its support disc D^{4n} gives in $\pi_{4n}GL(4n, \mathbb{R})$ the trivial homotopy class.*

To detect nontriviality of df we apply the generalized $\hat{a}$ genus (with values in $KO(*) \cong \mathbb{Z}_2$. It is well known that there are exotic spheres such that $\hat{a}(\mathbb{T}(f_T)) \neq 0$. We prove that for such f_T we have $[df] \neq 0$. Thus there are f which do not preserve the homotopy class of J_0.

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