CONSTRUCTIONS OF CONTACT FORMS ON PRODUCTS AND PIECEWISE FIBERED MANIFOLDS

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Abstract. We study constructions of contact forms on closed manifolds. A notion of strong symplectic fold structure is defined and we prove that there is a contact form on $M \times X$ provided that M admits such a structure and X is contact. This result is extended to fibrations satisfying certain natural conditions. Then, generalizing the open book construction, we describe decompositions of manifolds into fibered pieces which allow to construct contact forms.

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1. Introduction

In this paper we study constructions of contact forms on closed orientable manifolds. An intricate question of contact topology is whether a closed almost contact manifold admits a contact structure. It is solved positively only in dimensions three and five [Ma, G1, CPP]. However, even in low dimensions, this is usually very non-trivial to describe a contact form on a given almost contact manifold.

There are some obvious classes of almost contact manifolds. For instance, the product of a stably almost complex manifold M with a contact manifold X is almost contact. In the present paper this is the class we start with. Our main result, Theorem 3.1, gives a construction of contact forms on $M \times X$ under an additional assumption: we require that M admits a singular form called a strong symplectic fold of convex type (see Section 2 for the definition). Then we extend this result to strong symplectic folds of general type and products replaced by fibrations satisfying some rather natural restrictions. Finally, we compile these results into a description of a class of decompositions into fibered pieces which allow to define a contact form on a manifold admitting such decomposition.

Our main construction is based on a generalization of a formula given in [GS] and uses the Giroux open book form on X. It yields a confoliation on $M \times X$. Then we show that it can be deformed to a contact form using the result of Altschuler and Wu [AW].

Motivations for this research can be described as follows. First of all, this gives some new classes of contact structures on products, fibrations as well as a new approach to some known results (for example, products of a surface by a contact manifold belong to the class when our construction works. Another our starting point is related to the description of contact manifolds in terms of open book decompositions [GM], see Section 2. However there are examples of standard topological constructions, as those of taking products or fibrations, blow-ups or performing surgeries, which usually do not preserve the open book decompositions. Our notion of piecewise fibered contact structure of

Section 6 is an attempt to introduce a structure more flexible then the one given by open books.

To give an example of problems where our constructions can be applied, consider the following technical question. In the description of contact forms by open books we have a fibration $E \to S^1$ with a compact fiber P of the complement E of a tubular neighborhood of a codimension 2 submanifold. To get a contact structure, it is sufficient to give an exact symplectic form on P, convex at the boundary and such that the monodromy of the fibration is an exact diffeomorphism. But in fact this difficult requirement can be replaced a weaker condition. It is enough to have a contact form on the fibration, equal to $e^t\lambda + d\theta$ in a collar of the boundary ($= \partial P \times S^1$,) where λ is the contact form on ∂P and $d\theta$ is the standard orientation 1-form on S^1. Our Proposition 5.4 shows that the latter problem is easier than the usual fillability question. In particular we prove that for some overtwisted form λ on S^3 one can extend the form $e^t\lambda + d\theta$ to $D^4 \times S^1$.

A preliminary version of this results was announced under the title "Contact forms on products" as arxiv:1204.1692 [math.SG]. The present version is considerably extended and the assumption of triviality of the monodromy of the open book on X has been removed.

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2. Preliminaries

We consider closed smooth manifolds equipped with (globally defined) contact forms.

Geiges and Stipsicz [GS] gave a formula which yields a contact form on products $M \times S^1$ for some closed M. Let us describe their construction in a slightly more general setup.

Definition 2.1. *A strong symplectic fold structure of convex type on a compact manifold M is a decomposition $M = W_- \cup_N W_+$, where $N = W_- \cap W_+$ is a hypersurface in $Int\, M$, together with exact symplectic forms $\omega_- = d\gamma_-, \omega_+ = d\gamma_+$ on respectively W_-, W_+, such that the forms satisfy the following convexity conditions on a tubular neighborhood $N \times [-1, 1]$ of N and at ∂M:*

1. *$\gamma_- = e^t\lambda$ on $N \times [-1, 0] = N \times [-1, 1] \cap W_-$ and $\gamma_+(t) = e^{-t}\lambda$ on $N \times [0, 1] = N \times [-1; 1] \cap W_+$, where t is the parameter of $[-1, 1]$ and λ is a contact form on N,*
2. *the closure of every component of $M - N$ containing a component of ∂M is an exact symplectic cobordism (by (1), it is necessarily convex at the $N-$end), either convex or concave at the component of ∂M.*

The hypersurface N is called *the fold locus.*

An obvious example is the double $W \cup (-W)$, where W is a compact manifold with boundary and W admits an exact symplectic form satisfying convexity condition (1) at ∂W. Note that a strong symplectic fold does not determine the orientation, since the orientations given by the symplectic forms on the two parts are opposite.

In our terminology we follow Ana da Silva [dS]. She shows that on any closed stably almost complex manifold there exists a symplectic fold, i.e. a decomposition into two submanifolds as above and a 2-form which is symplectic except for the common boundary of the two parts, where the form has fold singularities. In our definition, the forms $\omega_\pm$ agree only after restriction to N, and they do not give any globally defined smooth form on M.

Theorem 2.2. [GS] *If M^{2m} admits a strong symplectic fold, then $M \times S^1$ is contact.*

Proof. Let $d\phi$ denote the standard orientation form on S^1 and $p : M \times S^1 \to M$ be the projection. If $\omega_\pm = d\gamma_\pm$, then $p^*\gamma_\pm + d\phi$ are contact forms outside $N \times [-1,1] \times S^1$.

Choose smooth functions $f, g : [-1,1] \to \mathbb{R}$ such that:

(1) g is odd, equal to 1 near $t = -1$, equal to -1 near $t = 1$, and it is decreasing from -1 to 1,
(2) f is even, positive, equal to $e^{\pm t}$ near ± 1 and increasing on $[-1,0]$,
(3) $f'g - g'f > 0$ on $[-1;1]$.

Then the formula

$$\alpha = f\lambda + g\,d\phi$$

on $[-1,1] \times N \times S^1$ yields a contact form on $N \times [-1,1] \times S^1$ (with contact form λ on N) which extends those defined above. In fact, it is not difficult to calculate:

$$\alpha \wedge (d\alpha)^n = nf^{n-1}(f'g - fg')dt \wedge \lambda \wedge (d\lambda)^n \wedge d\theta > 0.$$

□

Geiges and Stipsicz apply this formula together with the result of Baykur ([B]) to show that for every closed orientable 4-manifold M the product $M \times S^1$ is contact.

In general, the existence of a strong symplectic fold structure seems to be a difficult question. The following classical result of Eliashberg [E] (cf. also [W] and Ch. 6 of [G2]) is the basic tool to construct some examples. Let us recall that W is the *trace* of a (single) *surgery* of index $k+1$ on M^{2n+1} if W is obtained by attaching a handle of index $k+1$ to $M \times [0,1]$. It means that W is diffeomorphic to $M \times [0,1] \cup_f (D^{k+1} \times D^{2n-k+1})$, where $f : S^k \times D^{2n-k+1} \to M \times \{1\}$ is the attaching map of the handle. In particular, $\partial W = M \cup (-M')$, where $M' = (M - f(S^k \times D^{2n-k+1})) \cup (D^{k+1} \times S^{2n-k})$ is the result of the surgery on M. The product $M \times [a;b]$ endowed with the form $d(e^t\lambda)$ is called the *symplectization* of a contact form λ on M.

Theorem 2.3. *Let λ be a contact form on a $(2n+1)-$dimensional manifold M and let W be the trace of a surgery on M of index $k+1$ with $1 \leq k \leq n$ and $n > 1$. If the almost complex structure on $M \times [0,1]$ determined by λ extends to W, then there exists an exact symplectic form ω on W such that ω is the symplectization of λ near $M \times \{0\}$ as well as the symplectization of a contact form in a collar of M'. In particular, M' admits a contact form. Furthermore, if V is a compact connected almost complex $(2n+2)$-dimensional manifold ($n > 1$) and V admits a Morse function maximal on ∂V such that indices of all critical points are less or equal to $n+1$, then V admits a symplectic structure with convex boundary (the boundary is of contact type). A Morse function with the required* *properties exists if and only if V has the homotopy type of a CW-complex of dimension at most $n+1$.*

Let us call any manifold V having the above properties of *Weinstein type.* Thus the double of a manifold of Weinstein type admits a strong symplectic fold.

Remark 2.4. The contact surgery in dimension 4 requires some additional assumption on framings of the attaching spheres of 2-handles, see [G2], Ch. 6.3,6.4.

Theorem 2.3 together with our result yield the existence of contact forms on products of contact manifolds and strong symplectic folds. The ultimate aim of this paper is to give the same conclusion for spaces endowed with a more general structure than strong symplectic folds.

Our construction of contact forms uses Giroux' description of contact structures in terms of open book decompositions. We need an operation which changes the orientation by reversing the direction of a vector field transversal to the contact distribution and leaves the orientation of the latter fixed. In the dimension $4k+1$ one can change the orientation by multiplying the form by -1, but this operation is not good enough. The main additional property we need is a 1-parameter family connecting the given form with the reversed one such that each form of the family has non-degenerate differential.

Let us recall Giroux' description of contact forms.

Definition 2.5. *An open book decomposition of X is given by*

(1) *a codimension two submanifold $B \subset X$ (called the binding),*
(2) *a tubular neighborhood U of B diffeomorphic to $B \times D^2$,*
(3) *a fibration $\pi : E = X - B \to S^1$ with fiber P (called the page)*

such that the monodromy of the fibration π is equal to the identity in $P \cap U$ and $\pi|U$ can be identified with the standard projection $B \times (D^2 - \{0\}) \to S^1$.

According to [Gi, GM], with any closed contact manifold X one can associate an open book decomposition satisfying the following conditions:

(1) P is exact symplectic, i.e., P has 1-form $\overline{\beta}$ such that $d\overline{\beta}$ is symplectic on P,
(2) a tubular neighborhood U of ∂P is of convex type, which means that in a collar $\partial P \times [0, \epsilon)$ we have $\overline{\beta} = e^{-t}\beta$ with β contact on ∂P.
(3) the monodromy $f : P \to P$ of π is exact, which means that $f^*\overline{\beta} - \overline{\beta} = d\varphi$ for some function $\varphi : P \to \mathbb{R}$.

With such data we can associate a contact form on X in a way we describe in details later in this section. The main theorem of [GM] says that any contact form is homotopic (i.e., there exists a deformation through contact forms) to the form of this type described below, which we shall refer to as the *Giroux contact form.*

Let us write a formula for such form.

If $f : P \to P$ is the monodromy of π, we identify E with the quotient of $P \times [0, R]$ for some fixed R, by the identification $\Phi : (x, 0) \sim (f^{-1}(x), R)$.

On $P \times [0, R]$ we put $\eta_E = \hat{\beta} + d\phi$ where

$$\hat{\beta} = \overline{\beta} + u(\phi)d\varphi \tag{2.6}$$

for some non-decreasing function $u : [0; R] \to [0; 1]$ so that

$$u(\phi) = \begin{cases} 0 & \text{for } \phi \in [0;\varepsilon) \\ 1 & \text{for } \phi \in (R-\varepsilon; R]. \end{cases} \tag{2.7}$$

The form $\hat{\beta}$ descends to $(P \times [0,R])/\sim$ since $\Phi^*(\overline{\beta} + d\varphi + d\phi) = \overline{\beta} + d\phi$ and η_E defines a smooth form on E. Moreover, if dimension of P is $2n$, then $\eta_E \wedge (d\eta_E)^n = d\phi(d\overline{\beta})^n + n\overline{\beta}(d\overline{\beta})^{n-1}u'(\phi)d\phi d\varphi$. As $d\overline{\beta}^n > 0$ on P and for R big enough the derivative $|u'(\phi)|$ can be made arbitrary small, η_E is contact. In addition, $\Phi^*(\overline{\beta}+d\varphi+l\cdot d\phi) = \overline{\beta}+l\cdot d\phi$ for any $l \in \mathbb{R}$, so $\eta_E = \hat{\beta} + l \cdot d\phi$ is well-defined on $(P \times [0,R])/\sim$.

Remark 2.8. As far as we know, such "enlarging the circle" trick has never been used before in this context. When we tried to apply the formulae we could find in the literature (as for example the one described in [G2]), then our main result required an additional assumption. That version of this paper applied essentially only when the fibration $E \to S^1$ was trivial. It was rather unexpected that the simple trick described above enabled us to solve this problem.

In the sequel we will use a deformation of such form to one having the opposite orientation of S^1 in the fibration $E \to S^1$. For this reason we have to consider the family of forms $\eta_E = \overline{\beta} + ud\varphi(\phi) + ld\phi$ depending on $l \in [-a,a]$. Again, Φ preserves each form of this family, hence η_E is well-defined. Then we have $\eta_E \wedge (d\eta_E)^n = ld\phi((d\overline{\beta})^n - n\overline{\beta}(d\overline{\beta})^{n-1}u'(\phi)d\varphi)$ and we get

Proposition 2.9. *For R large enough the form η_E is contact if $l \neq 0$.*

As the monodromy f near the boundary ∂P is the identity, the form $\hat{\beta} + l \cdot d\phi$ ($l \in \mathbb{R}$) is equal to $\beta e^r + l \cdot d\phi$ near the boundary of $B \times D^2$ in polar coordinates (r,ϕ) on D^2. We easily now extend $\hat{\beta} + l \cdot d\phi$ to $B \times D^2$ by the formula

$$\alpha = h_1(r)\beta + l \cdot h_2(r)d\phi,$$

where

$$h_1(r) = \begin{cases} 2 & \text{near } r = 0 \\ e^{1-r} & \text{for } r \in [1; \frac{R}{2\pi}], \end{cases}$$

is strictly decreasing with all derivatives at 0 vanishing,

$$h_2(r) = \begin{cases} r^2 & \text{near } r = 0 \\ 1 & \text{for } r \in [1; \frac{R}{2\pi}] \end{cases}.$$

and $h_1(r)h_2'(r) - h_1'(r)h_2(r) > 0$. As another simple calculation shows, the resulting form is contact on X.

If $l = \pm 1$ and R is big enough, for a suitable choice of u we get $\eta_E = \hat{\beta} \pm d\phi$ contact. They determine opposite orientations and we use this pair of forms in the sequel.

Definition 2.10. *If $\eta_E = \hat{\beta} + d\phi$ is the Giroux contact form in the fibration part E of the given open book decomposition, then by $\hat{\eta}$ we denote the form given by the same open book decomposition and the same form β on the page, but equal to $\hat{\beta} - d\phi$ in E.*

Let us explain constructions related to what we give later on. This provides a generalization of the Giroux construction and it is a simple example of a structure which supports contact forms on more general structures than merely open books. This indicates in what direction our search for flexible structures goes.

We will need the following definition. Let $E \to B$ be a smooth bundle with fiber F and the structure group $G \subset Diff(F)$. We say that it is *defined on a hypersurface* $H \subset B$ if its restriction to $B - H$ is trivial and there is a map $a : H \to G$ such that the map $A : H \times F \to H \times F : (x, v) \mapsto (x, a(x)v)$ is smooth and the bundle is obtained by gluing the product pieces along H with A. The definition applies also in the case when $B - H$ is connected. If B is the circle, then as the hypersurface one can take a single point.

Given an exact symplectic manifold $(M, \omega = d\beta)$, denote by $Ex(M, \beta)$ the group of exact symplectomorphisms and by $Ex(M, \partial M, \beta)$ the group of exact symplectomorphisms equal to the identity near the boundary.

Proposition 2.11. *If $\pi : E \to B$ is a bundle with compact contact base (B, μ), compact exact symplectic fiber $(F, \omega = d\beta)$, the structure group contained in the group of exact symplectomorphisms $Ex(F, \beta)$ and defined on a hypersurface $H \subset Int\, B$, then E admits a contact form. If the structure group is contained in $Ex(F, \partial F, \beta)$, then the contact form can be chosen equal to the product form $R\mu + \beta$ on a collar of $B \times \partial F$, where R is a large enough constant.*

Proof. Let $A : H \times F \to H \times F$ be the gluing diffeomorphism. By assumptions, for any $x \in H$ we have $(A|\{x\} \times F)^*\beta = \beta + d\varphi_x$, where $\varphi_x \in C^\infty(F)$. Actually, there exists a smooth function $\tilde{\varphi}$ on $H \times F$ such that this equality holds with $\varphi_x = \widetilde{\varphi}(x, \cdot)$. Consider a tubular neighborhood $U \cong H \times [-1, 1]$ of H. For any positive constant R the form $R\mu + \beta$ is contact on $\pi^{-1}(B - U) \cong (B - U) \times F$. On U consider the form

$$\eta = R\mu + \beta + ud\widetilde{\varphi}.$$

where $u : [-1, 0] \to [0, 1]$ is given by formula 2.7. If dimension of F is $2m$ and dimension of B is $2n$, then

$$\eta d\eta^{n+m-1} = C_1 R^m \mu d\mu^{m-1} d\beta^n + C_2 R^{m-1} u' \mu d\mu^{m-2} d\beta^n dt d\widetilde{\varphi} +$$
$$C_3 R^{m-2} u' d\mu^{m-1} \beta d\beta^{n-1} dt d\widetilde{\varphi} + C_4 R^{m-1} d\mu^{m-1} d\beta^n d\widetilde{\varphi},$$

where $C_i, i = 1, 2, 3, 4$ are constants depending only on m, n. For R large enough the first term dominates the whole sum and consequently η is contact. By construction, the forms on $\pi^{-1}U$ and on $\pi^{-1}(B - U)$ agree near $H \times \{\pm 1\} \times F$, hence we obtain smooth contact form on E. □

Consider now two compact manifolds X, Y with non-empty boundaries, of dimensions $2n, 2m$ respectively. Assume that they are endowed with exact symplectic forms $\omega_X = d\beta_X$, $\omega_Y = d\beta_Y$, both with convex type boundaries. Let $\beta_X = e^s \mu_{\partial X}$, $\beta_Y = e^{-s} \mu_{\partial Y}$ in collars $\partial X \times (-1, 0], \partial Y \times [0, 1)$ of boundaries, where $\mu_{\partial X}, \mu_{\partial Y}$ are some contact forms. In this notation $s \in [-1, 1]$ and both boundaries correspond to $s = 0$. Let E be the total space of a bundle over ∂Y with fiber X defined on a hypersurface $H_{\partial Y} \subset \partial Y$ and having the structure group $Ex(X, \partial X, \beta_X)$ of exact symplectomorphisms of X equal to the identity near the boundary. Similarly, assume that the bundle $F \to \partial X$ with fiber Y

is defined on a hypersurface $H_{\partial X} \subset \partial X$, and its structure group is $Ex(Y, \partial Y, \beta_Y)$. The assumptions on structure groups imply that $\partial E = \partial X \times \partial Y = \partial F$.

Proposition 2.12. *Under the above assumptions, $E \cup_{\partial X \times \partial Y} F$ is contact.*

Proof. Consider $\tilde{X} = X \cup \partial X \times [0, \log R_X]$ obtained from X by adding a long collar, with $\beta_X = e^s \mu_{\partial X}$ for $s \in [-1, \log R_X]$. In this way the contact form on the boundary is multiplied by the constant R_X. Analogously, Y is enlarged to $\tilde{Y} = Y \cup \partial Y \times [-\log R_Y, 0]$ with $\beta_Y = e^{-s}\mu_{\partial Y}$ for $s \in [-\log R_Y, 1]$ (we assume $R_X, R_Y \geq 1$). Let $\tilde{E}$ denote the obvious extension of E to a bundle with fiber $\tilde{X}$, and similarly $\tilde{F}$ the extension of F. Applying Proposition 2.11 gives a contact form on $\tilde{E}$ equal to $R_Y\mu_{\partial Y} + \beta_X$ near $\partial\tilde{E} = \partial Y \times \partial X$. The choice of R_Y which yields contactness is determined by the behavior of the forms in the tubular neighborhood of $H_{\partial Y}$. We claim that the choice depends only on X (not on $\tilde{X}$) regardless of R_X. To see this, let us calculate $\eta_E d\eta_E^{n+m-1}$ for $\eta_E = R_Y\mu_{\partial Y} + \beta_X + ud\varphi$ in $\partial X \times [-1, \log R_X] \times H \times [-1, 1]$. Since $\beta_X = e^s\mu_{\partial X}$ for $s \in [-1, \log R_X]$,

$$\eta_E d\eta_E^{n+m-1} = e^{sn}\mu_{\partial X} d\mu_{\partial X}^{n-1}\big(D_1 R_Y^m \mu_{\partial Y} d\mu_{\partial Y}^{m-1} ds + D_2 R^{m-1} u' \mu_{\partial Y} d\mu_{\partial Y}^{m-2} ds dt d\widetilde{\varphi} + D_3 R^{m-2} u' d\mu_{\partial Y}^{m-1} dt d\widetilde{\varphi} + D_4 R^{m-1} d\mu_{\partial Y}^{m-1} ds d\widetilde{\varphi}\big),$$

where $D_i, i = 1, 2, 3, 4$ are again constants depending only on m, n.

It follows from this formula that the choice of R_Y is independent of the extension by the long collar and our claim follows. Thus we can choose $R = R_X = R_Y$ such that there are contact forms on $\tilde{E}$ and $\tilde{F}$ that restrict to $R(\mu_{\partial X} + \mu_{\partial Y})$ on $\partial E = \partial F = \partial X \times \partial Y$.

For simplicity we replace now $\log R$ with K. After the change the parameter in $[-K, 1]$ replacing s with $s+2K$, the form η_F on $\partial X \times \partial Y \times [K, 1+2K]$ becomes $R\mu_{\partial X} + e^{-s+K}\mu_{\partial Y}$. If $\psi : [K, 1+2K] \to \mathbb{R}$ is a positive smooth function such that $\psi = e^{s-K}$ near $s = K$ and $\psi = 1$ near $1 + 2K$, then it can be regarded as a function on F (we simply extend it from the collar $\partial F \times [0, 1]$ to whole F). It yields that $\psi\eta_F$ is contact and it smoothly agrees with η_E along $\partial E = \partial F = \partial X \times \partial Y$. Thus we get a smooth contact form on $E \cup F$. □

Remark 2.13. In some proofs in the sequel we use the following well-known fact: if η_1, η_2 are contact and homotopic on X, then there is a topologically trivial symplectic cobordism $M = X \times [0, 1]$ between (X, η_1) and (X, η_2). In particular, we can always deform the symplectic form on a compact symplectic manifold with boundary of contact type to have the Giroux form on the boundary. Later on we will often assume this property tacitly, especially if we write $\hat{\lambda}$ for a given contact form λ.

One of our tools is the heat flow deformation of a confoliation [AW]. On a closed manifold Y^{2m+1} consider a confoliation, i.e. a 1-form α satisfying the inequality $\alpha \wedge (d\alpha)^m \geq 0$. The points $x \in Y$ where $\alpha \wedge (d\alpha)^m > 0$ are called contact (regular), the other (non-contact) points are called singular and the set of singular points will be denoted by Σ. Altschuler and Wu show that under some assumptions, the heat flow can deform the confoliation to a contact form. To describe those assumptions we choose a Riemannian metric g on Y and consider the form $\tau = \star(\alpha \wedge (d\alpha)^{m-1})$, where $\star$ denotes the Hodge star. Then at every point $x \in Y$ we denote by $\mathcal{D} \subset TY_x$ the orthogonal complement of

$Null(\tau)_p = \{V \in T_pY : \iota_V\tau = 0\}$. At a contact point the subspace $\mathcal{D}$ has dimension $2m$ and it is transversal to the Reeb vector of τ. At a point where rank of $d\alpha$ on $ker\,\alpha$ is $2m-2$, the dimension of $\mathcal{D}$ is 2, and $\mathcal{D}$ is zero at points where rank of $d\alpha|ker\,\alpha$ is less than $2m-2$. A point x is called *accessible* if there is a smooth curve $\sigma : [0,1] \to Y$ such that $z'(t) \in \mathcal{D}$ and is non-zero for all $t \in [0,1]$, $z(0) = x$ and $z(1)$ is a contact point. Thus we see that in the case when the rank of $d\alpha|ker\,\alpha$ is less than $2m-2$ no singular point is accessible. Since we have to reduce the general case to that of corank at most 3, this is one of the main difficulties of our construction.

In the sequel we will use the following theorem.

Theorem 2.14. [AW] *Suppose that Y is a closed manifold with a confoliation α. If every non-contact point of Y is accessible, then Y supports a contact form C^∞-close to α.*

3. Main theorem

Our main theorem is the following.

Theorem 3.1. *If (X^{2m+1}, α) is a closed contact manifold and M^{2n} admits a strong symplectic fold of convex type, then $X \times M$ is contact.*

Proof. Consider the decomposition $M = W_1 \cup (N \times [-1;1]) \cup W_2$ and the forms $\omega_+, \omega_-, \lambda$ given by the strong symplectic fold on M. Here N is the common boundary of W_+, W_-, $N \times [-1,1]$ is a tubular neighborhood of N with $N \times [-1,0] \subset W_-$, $N \times [0,1]) \subset W_+$, $W_1 = W_- - N \times (-1,0], W_2 = W_+ - N \times [0,1)$, $\omega_\pm = d\gamma_\pm$.

We can assume that the contact form α is given by the Giroux construction with the page P, the binding B using $\overline{\beta}, h_1, h_2, \varphi$ and u as described in Section 2. Recall that $d\overline{\beta}$ is symplectic on P and $\overline{\beta}$ restricts to a contact form $\overline{\beta}$ on ∂P.

We define a 1-form $\tilde{\eta}$ on $X \times M$ by separate formulae on $X \times (W_1 \cup W_2)$; $(X - B \times D^2) \times N \times [-1,1]$; $B \times D^2 \times N \times [-1,1]$.

On $X \times W_\pm$ we take $\tilde{\eta} = \alpha + \gamma_\pm$. By the discussion in Section 2, for every $l \in \mathbb{R}$ the form $\hat{\beta} + l\ d\phi$ is well-defined on $X - B \times D^2$. Therefore on $(X - B \times D^2) \times N \times [-1;1]$ we can put $\tilde{\eta} = \overline{\beta} + u(\phi)d\varphi + g(t)d\phi + f(t)\lambda$ with f, g given in Theorem 2.2 .

Finally, let

$$\tilde{\eta} = h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi \tag{3.2}$$

on $B \times D^2 \times N \times [-1,1]$.

Lemma 3.3. *The form $\tilde{\eta}$ is smooth on $X \times M$ and contact in the complement of $(B \times \{0\}) \times (N \times \{0\}) \subset X \times M$.*

Proof. A direct inspection of the definition shows that the form is smooth and in $X \times (W_1 \cup W_2)$ is contact.

In $(X - B \times D^2) \times N \times [-1;1]$ we have

$$\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} = (m+n-1)f^n(f'g - g'f)d\phi(d\overline{\beta})^m \wedge \lambda \wedge (d\lambda)^{n-1} \wedge dt +$$
$$+(m+n)f^{n-1}(d\overline{\beta})^m d\varphi(d\lambda)^{n-1}(g'(t)dtd\phi + f'(t)dt\lambda)u(\phi) + u'(\phi)\kappa$$

for some $2m + 2n + 1$-form κ. As $d\overline{\beta}^m d\varphi$ is a $(2m+1)$-form on P, hence the middle term vanishes. Furthermore, for R big enough $|u'(\phi)\kappa|$ can be made arbitrarily small,

because κ does not depend on R. It follows that $\tilde{\eta}(d\tilde{\eta})^{m+n} > 0$, hence our formula defines a contact form on this part.
It remains to examine $\tilde{\eta}$ on $B \times D^2 \times N \times [-1;1]$. Direct computations give

$$\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} = c_1(f'g(h_1h_2' - h_1'h_2) + fg'h_1'h_2)\beta \wedge d\beta^{m-1} \wedge dt \wedge \lambda \wedge d\lambda^{n-1} \wedge dr \wedge d\phi,$$

where c_1 is a positive constant. Since $h_1h_2' - h_2h_1' > 0$, $f'g \geq 0$, $fg'h_1'h_2 \geq 0$, we see that $\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} \geq 0$ and it vanishes if and only if $f'g = 0$ and $fg'h_1'h_2 = 0$. The equality $f'g = 0$ implies $t = 0$. Furthermore, for $t = 0$ we have $fg' > 0$. To complete the proof notice that our assumptions on h_1, h_2 yield $h_1'h_2 = 0 \Leftrightarrow r = 0$. □

We want to apply Theorem 2.14, so we need the accessibility condition to be satisfied. First we need to establish that $rank\ d\widetilde{\eta} \mid \ker\widetilde{\eta} = 2(m+n-1)$ on $\Sigma = B \times \{0\} \times N \times \{0\}$. Unfortunately, $rank\ d\widetilde{\eta} \mid \ker\widetilde{\eta} < 2(m+n) - 2$ since $d\widetilde{\eta}|T(X \times M)|_\Sigma = 2d\beta + d\lambda$ and $\widetilde{\eta}|T(X \times M)|_\Sigma = 2\beta + \lambda$. In order to remedy this we change the confoliation form making it asymmetric with respect to the decomposition $W_1 \cup (N \times [-1;1]) \cup W_2$. Roughly speaking, we impose in this way some more transversality along the singular set. Define the form η on $X \times M$ by the formula

$$\eta = \begin{cases} e^{-1}(\hat{\beta} + d\phi + \gamma_-) & \text{on } B \times D^2 \times W_1 \\ k(t)(h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi) & \text{on } B \times D^2 \times N \times [-1;1] \\ e(\hat{\beta} - d\phi + \gamma_+) & \text{on } B \times D^2 \times W_2. \end{cases} \tag{3.4}$$

Here (r, ϕ) are polar coordinates on the disk D^2 of radius 2, f, g are functions defined in Theorem 2.2 and $k : M \to [e^{-1}; e]$ is a positive, non-decreasing function satisfying

$$k(t) = \begin{cases} e^{-1} & \text{on } W_1 \\ e^t & \text{on } N \times [-1+\varepsilon; 1-\varepsilon],\ t \in [-1+\varepsilon; 1-\varepsilon] \\ e & \text{on } W_2 \end{cases}$$

with ε small enough.
This formula extends to $X \times M$ as in Lemma 3.3. We get again a confoliation with the critical set $\Sigma = B \times \{0\} \times N \times \{0\}$.
To apply the result of [AW] we need a Riemannian metric $\langle \cdot, \cdot \rangle$ on $X \times M$. Choose $\langle \cdot, \cdot \rangle$ so that near Σ submanifolds N, I, B, D^2 are pairwise orthogonal. We will check that η satisfies the assumption of Theorem 2.14. In fact, we show that for every point $(b, v) \in B \times \{0\} \times N \times \{0\}$, the radial path $z(r) = (b, (r, \phi), 0, v) \subset B \times D^2 \times N \times I$ (with $z'(r) = \frac{\partial}{\partial r} \in TD^2$ for $r \in [0;2]$ and any fixed $\phi \in [0, 2\pi)$) satisfies $z'(t) \in \mathcal{D}$, hence every $x \in \Sigma$ is accessible from a contact point. The proof is divided into two parts. First we show that on Σ we have $\mathcal{D} = TD^2$ and then that $z'(r) \in \mathcal{D}$ for $r \in (0;2]$.

Lemma 3.5. *Under the assumptions above, $\mathcal{D} = TD^2$ on Σ.*

Proof. By Definition 3.4, $\eta = e^t\widetilde{\eta}$, hence

$$d\eta = e^t dt\widetilde{\eta} + e^t d\widetilde{\eta} = \tag{3.6}$$

$$= e^t dt(h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi) +$$

$$+ e^t(h_1'(r)dr\beta + h_1(r)d\beta + f'(t)dt\lambda + f(t)d\lambda + h_2'(r)g(t)drd\phi + h_2(r)g'(t)dtd\phi).$$

Substituting $t = r = 0$ in Formula 3.6 gives that $\eta|T(X \times M)|_\Sigma = 2\beta + \lambda$ and $d\eta|T(X\times M)|_\Sigma = 2d\beta + d\lambda + dt(2\beta+\lambda)$ with $\Sigma = B\times\{0\}\times N\times\{0\}$. As $d\beta^m = 0, d\lambda^n = 0$, we easily calculate:

$$\eta \wedge (d\eta)^{m+n-1} = \eta \wedge (m+n-1)(2d\beta + d\lambda)^{m+n-2} \wedge dt \wedge (2\beta+\lambda) =$$

$$= \eta \wedge (m+n-1)2^{m-1}(d\beta)^{m-1} \wedge (d\lambda)^{n-1} \wedge dt \wedge (2\beta+\lambda) =$$

$$= C\beta \wedge (d\beta)^{m-1} \wedge \lambda \wedge (d\lambda)^{n-1} \wedge dt = Cdvol_B \wedge dvol_N \wedge dt$$

for some positive constant C. Thus $\star(\eta \wedge (d\eta)^{m+n-1}) = \pm C dvol_{D^2}$ and $\mathcal{D} = TD^2$. $\square$

It is not clear yet if we can extend our path $z(r)$ beyond Σ since we do not know the behavior of $\mathcal{D}$ on the complement of Σ, hence in the second part of the proof we deal with the case $r > 0$. The proof is an elementary but long computation, hence we skip some parts of it.

As η is contact on $X \times M - \Sigma$, we have that $\mathcal{D}$ is $2(m+n)$-dimensional and the Reeb field of τ is equal to $\mathcal{D}^\perp$. We will show that with respect to the previously chosen metric $\langle\cdot,\cdot\rangle$ the Reeb field R_τ on $B \times (D^2 - \{0\}) \times N \times \{0\}$ is perpendicular to $\frac{\partial}{\partial r}$. Therefore once we show that for $t = 0$ the Reeb field R_τ is tangent to $T = B \times S^1_r \times N \times I$ (with $S^1_r = \{p \in D^2 : \ |p| = r\}$), or equivalently, that τ is non-degenerate on T, the proof is completed.

From now on we omit the wedge sign from the computations to make them more compact.

As in Lemma 3.5, substituting $t = 0$ in Formula 3.6 gives $\tilde{\eta}|T(X\times M)|_S = h_1(r)\beta + 2\lambda$ and $d\tilde{\eta}|T(X \times M)|_S = h_1' dr\beta + h_1 d\beta + d\lambda - h_2 dtd\phi$ on $S = B \times D^2 \times N \times \{0\}$. We obviously have $(d\eta)^{m+n-1} = (dt\tilde{\eta} + d\tilde{\eta})^{m+n-1} = (d\tilde{\eta})^{m+n-1} + (m+n-1)(d\tilde{\eta})^{m+n-2}dt\tilde{\eta}$ on S. Further, as $d\beta^m = 0, d\lambda^n = 0$ we get

$$(d\tilde{\eta})^{m+n-1} = \binom{m+n-1}{n-1}(d\lambda)^{n-1}(h_1' dr\beta + h_1 d\beta - h_2 dtd\phi)^m +$$

$$+\binom{m+n-1}{n-2}(d\lambda)^{n-2}(h_1' dr\beta + h_1 d\beta - h_2 dtd\phi)^{m+1} =$$

$$= (d\lambda)^{n-1}((d\beta)^{m-1}(D_1 dr\beta + D_2 dtd\phi) + D_3(d\beta)^{m-2}dr\beta dtd\phi) + D_4(d\lambda)^{n-2}(d\beta)^{m-2}dr\beta dtd\phi$$

for some functions D_i $(i \in \{1,2,3,4\})$ of variable r. In a similar manner we calculate $dt\tilde{\eta}(d\tilde{\eta})^{m+n-2}$:

$$dt\tilde{\eta}(d\tilde{\eta})^{m+n-2} = dt\tilde{\eta}(h_1' dr\beta + h_1 d\beta + 2d\lambda - h_2 dtd\phi)^{m+n-2} = dt\tilde{\eta}(h_1' dr\beta + h_1 d\beta + 2d\lambda)^{m+n-2}$$

$$= dt\tilde{\eta}\left(\binom{m+n-2}{n-1}(h_1 d\beta)^{m-1}(2d\lambda)^{n-1} + (m+n-2)(h_1 d\beta + 2d\lambda)^{m+n-3}h_1' dr\beta\right).$$

After arduous, but elementary computation we get that

$$\eta \wedge (d\eta)^{m+n-1} = C_1\beta(d\beta)^{m-1}dr\lambda(d\lambda)^{n-1} + C_2\beta(d\beta)^{m-1}d\phi(d\lambda)^{n-1}dt +$$

$$+C_3(d\beta)^{m-1}d\phi\lambda(d\lambda)^{n-1}dt + C_4\beta(d\beta)^{m-2}drd\phi\lambda(d\lambda)^{n-1}dt$$

$$+C_5\beta(d\beta)^{m-1}drd\phi\lambda(d\lambda)^{n-2}dt + C_6\beta(d\beta)^{m-1}\lambda(d\lambda)^{n-1}dt$$

for some functions $C_i,\ i = 1, \dots, 6$ of variable r. Furthermore, $\hat{\beta} = \star(\beta(d\beta)^{m-2})$ in B and $\hat{\lambda} = \star(\lambda(d\lambda)^{n-2})$ in N both have maximal ranks equal to respectively $2m-2$ and $2n-2$. If we additionally set $\beta_1 = \star((d\beta)^{m-2})$ in B and $\lambda_1 = \star((d\lambda)^{m-2})$ in N, then

$$\tau = \star(\eta \wedge (d\eta)^{m+n-1}) = E_1 dt d\phi + E_2 \lambda_1 dr + E_3 dr \beta_1 + E_4 \hat{\beta} + E_5 \hat{\lambda} + E_6 dr d\phi$$

again for some functions $E_i,\ i = 1, \dots, 6$ of variable r. The pullback of τ to T via the inclusion $T \hookrightarrow M \times M$ to $B \times S^1_r \times N \times \{0\}$ yields

$$\star(\eta \wedge (d\eta)^{m+n-1}) = E_1 dt d\phi + E_4 \hat{\beta} + E_5 \hat{\lambda}.$$

The rank of this form is equal to $2(m-1) + 2(n-1) + 2 < 2(m+n)$, which completes the proof.

□

Theorem 3.1 has a generalization to the case of contact bundles over a strong symplectic fold.

Theorem 3.7. *Let (W, ω) be a compact exact symplectic manifold, $\pi : E \to W$ a bundle over W with a compact contact fiber (X, η_0). If the structure group of the bundle is contained in the group $Cont(X, \eta_0)$ of diffeomorphisms preserving the contact form η_0 (strict contactomorphisms), then E admits a contact form.*

Proof. We will use the symplectization of the fiber and the well-known Thurston construction of symplectic forms on bundles. By assumptions, $\omega = d\alpha$. Let $\{U_s\}_{s\in S}$ be an open cover of W with local trivializations $\Psi_s : \pi^{-1}(U_s) \cong U_s \times X$. If $\{f_s\}_{s\in S}$ is the partition of unity subordinated to $\{U_s\}_{s\in S}$, then we define a symplectic form $\omega = d(K\pi^*\alpha + e^t(\sum_{s\in S} f_s \Psi_s^* \eta_0))$ on $E \times [-\varepsilon, \varepsilon]$ for some K big enough and $\varepsilon > 0$. Let R be the Reeb vector field of η_0, so that the interior products with $\eta_0, d\eta_0$ are $\iota_R \eta \equiv 1, \iota_R d\eta_0 \equiv 0$. Since η_0 is preserved by the structure group of the bundle, there is a horizontal vector field $\tilde{R}$ on E such that its pushforward by Φ_s is equal to R for any $s \in S$. This implies that $\tilde{R}$ is the Reeb field of $\Phi_s^* \eta_0 | \pi^{-1}(w)$ for any s and $w \in W$. Thus, if $\eta = \sum_{s\in S} f_s \Psi_s^* \eta_0$, then we have $\iota_{\tilde{R}} d\eta \equiv 1, \iota_{\tilde{R}} \eta \equiv 0$. Therefore for the Liouville vector field L of ω we have $\iota_L \omega = K\pi^*\alpha + e^t \eta$. If we additionally apply $\iota_{\tilde{R}}$ to the last equation, we get $-\iota_L \iota_{\tilde{R}} \omega = -\iota_L \iota_{\tilde{R}}(e^t dt \eta + e^t d\eta) = e^t \iota_L dt = e^t$. This implies that L is transversal to E, hence $E \cong E \times \{0\} \subset E \times \mathbb{R}$ is contact. □

Now let (X, α) be a closed contact manifold and let $(W_\pm, N, \lambda_\pm)$ be a strong symplectic fold of convex type on W. Consider bundles $E_\pm \to W_\pm$ with fiber X, trivial over the fold locus $N = \partial W_- \cap \partial W_+$ and such that E_- is contact with respect to (X, α), E_+ is contact with respect to (X, α'). Assume that α, α' are homotopic to Giroux forms α_0, α_0' homotopic to $\hat{\alpha}_0$. Then we get the following extension of Theorem 3.1.

Theorem 3.8. *Any contact bundle over a strong symplectic fold of convex type admits a contact form.*

In fact, over the collar $N \times [-1, 1]$ the bundles are product, thus the arguments used in the product case hold. To be more precise, we start from the contact forms on E_+ given by the contactness of those bundles. Since the bundles are trivial over $N \times [-1, 1]$, we can use the homotopies $\alpha \sim \alpha_0, \alpha' \sim \hat{\alpha}_0'$ to get the form $\beta + \alpha_0$ over $N \times [-\varepsilon, 0]$ and $\beta + \hat{\alpha}_0$ over $N \times [0, \varepsilon]$ for some $\varepsilon > 0$. Having established this, we can apply the same arguments which were used to prove Theorem 3.1.

In the next section we use this theorem in the following special case. Let (X, α) be a contact manifold. Consider a contact bundle $p : E \to W$ with fiber (X, α) which is trivial over W_+. Thus over W_+ we can change the (trivial) bundle so that the fiber $(X, \hat{\alpha}_0)$ satisfies the assumptions of Theorem 3.8. In particular, Theorem 3.8 works for any contact bundle over a sphere.

4. Some applications

Results of the previous section give a constructive way to show that some manifolds are contact. We present now a series of examples.

Proposition 4.1. *The following manifolds admit contact structures:*

(1) $S^{k_1} \times ... \times S^{k_r}$, *if* $k_1 + ... + k_r$ *is odd;*
(2) $S^{k_1} \times ... \times S^{k_r} \times X$ *if* X *is a closed contact manifold and* $k_1 + ... + k_r$ *is even;*
(3) $M \times S^{k_1} \times ... \times S^{k_r}$, *if* M *is a closed manifold with a strong symplectic fold and* $k_1 + ... + k_r$ *is odd;*
(4) $M \times X$, *if* M *is a closed orientable 4-manifold and* X *is contact;*
(5) $\Sigma \times X$, *where* Σ *is a closed oriented surface (see* [Bo]*).*

Proof. Both D^{2k} and $D^{2k+1} \times S^{2l+1}$ with $k \geq l$ are Weinstein manifold, thus taking the doubles we see that S^{2k} and $S^{2k+1} \times S^{2l+1}$ admit strong symplectic folds with any k, l. Therefore the first three cases follow by induction. To get (4) one has to use [B]. In the last statement it is enough to notice that any orientable surface has a strong symplectic fold. This statement was first proved in [Bo] for genus $g > 0$. □

Any Lie group of odd dimension is obviously almost contact. However, no general construction of contact forms on compact Lie group is known. In particular, except for rank 1 there is no G-invariant contact forms on G. The product $S^3 \times S^3 \times S^3$ is an example of simply connected Lie group which admits a contact form but no $G-$invariant contact form. Some examples of contact forms on quotient spaces G/H which do not admit $G-$invariant contact forms can be obtained from Theorem 3.8. For instance, the following is true.

Proposition 4.2. *For any even* n, *the homogenous space* $SO(n+3)/SO(n)$ *is contact, but admits no* $SO(n+3)-$*invariant contact form.*

Proof. The space $SO(n+2)/SO(n)$ has a $SO(n+2)-$invariant contact form given by the circle fibration $SO(n+2)/SO(n) \to SO(n+2)/(SO(n) \times U(1))$ with symplectic base. Moreover, the space $SO(n+3)/SO(n)$ has no $SO(n+3)$-invariant contact form. Both statements follow from Alekseevski's description of contact homogeneous spaces [A], see [HT] for detailed explanations. Consider now the bundle $SO(n+3)/SO(n) \to SO(n+3)/SO(n+2)$ with fiber $SO(n+2)/SO(n)$. If n is even, then on the base $SO(n+3)/SO(n+2) = S^{n+2}$ we have the obvious strong symplectic fold. The structure group of the bundle is $SO(n+2)$, thus it is a contact bundle, trivial over the fold locus $S^{n+1} \subset S^{n+2}$. By Theorem 3.8, there exists a contact form on the total space of the bundle. □

Another example is the space $SO(2k+1)/SU(k)$ of "special unitary twistors" on S^{2k} which is fibered over $SO(2k+1)/SO(2k) = S^{2k}$ with fiber $SO(2k)/SU(k)$.

We will describe now examples of a modification which can be performed on a manifold with a strong symplectic fold. Assume that M^{2m} admits a strong symplectic fold $W_- \cup W_+$ with the fold locus N. We say that a surgery on a sphere $S^{k-1} \subset M$ is *symmetric of index* k, if it is performed using an embedding $\phi : S^{k-1} \times D^{2m-k+1} \to M$ such that $\phi = \phi_0 \times id_{D^1}$, where $\phi_0 : S^{k-1} \times D^{2m-k} \to N$ is an embedding and D^1 corresponds to the transversal disk of a tubular neighborhood of N.

Proposition 4.3. *If M' is obtained from M by a symmetric surgery of index $k \leq m = \frac{1}{2} dim\, M$ such that the stable almost complex structure of M extends to M', then M' has a strong symplectic fold structure.*

Proof. Let $M' = (M - \phi(S^{k-1} \times D^{2m-k+1})) \cup (D^k \times S^{2m-k})$. Decompose S^{2m-k} into the sum of two disks $D_- \cup D_+$ such that the decomposition corresponds to cut of the sphere by N. Since the surgery is symmetric, we get accordingly handles $D^k \times D_-, D^k \times D_+$ of index k attached to respectively W_-, W_+, resulting in a decomposition $W'_- \cup W'_+$. Since $k \leq m$ and almost complex structures on W_-, W_+ extend to these handles, thus given contact forms extend to $W_\pm$. □

Corollary 4.4. *If M^{2m} admits a strong symplectic fold, $k + n = 2m$, then so does the connected sum $M\#(S^k \times S^n)$.*

Proof. The previous proposition can be applied, since the connected sum is obtained by the surgery on a trivially embedded sphere S^{k-1}, and we can assume that $k \leq n$. □

We should admit that we do not know any example of closed stably almost complex manifold which admits no strong symplectic folds. The standard Morse - Smale theory shows that for any closed manifold M^{2m} one can find a decomposition $M = W_+ \cup_N W_-$, where $N = \partial W_+ = \partial W_- = W_+ \cap W_-$ with both W_+, W_- having the homotopy type of complexes of dimension at most m. If M is stably almost complex, then $W_\pm$ are almost complex, thus we have exact symplectic forms on both parts by contact surgery. However, the resulting contact forms λ_-, λ_+ on N do not need to agree. What we do know, it is that they define homotopic almost contact structures on N.

5. Concave folds and some further examples

In Section 3 we considered decompositions of a manifold M into the sum of two exact symplectic cobordisms W_1 and W_2 having the same contact boundary N at their convex ends. Then we constructed a contact form on $M \times X = (W_1 \cup_N W_2) \times X$, provided that X is a contact manifold. Theorem 3.8 gives a slight improvement: we can assume that we have fibrations over each cobordism, trivial at N and with appropriate restrictions on the structure group.

Our purpose is now to extend this construction to the case when M is decomposed into several pieces, each being a symplectic cobordism. In this case we have to allow concave ends to meet at a common component of N.

In case when W_1, W_2 meet in such a way that one of the ends is concave and one is convex (and the contact forms at the boundary are equal), we apply standard gluing of two symplectic cobordisms, which assembles two symplectic cobordisms into one, simplifying the decomposition.

Consider now the case when concave ends of two symplectic cobordisms meet at (N, λ). This means that in a collar neighborhood $N \times [-1, 1]$ of N we have the form $e^{-t}\lambda$ for $t \in [-1; 0]$ and $e^t\hat{\lambda}$ for $t \in [0; 1]$.

We explain now how to use Theorem 3.1 to obtain a contact form on the product of the sum of such two cobordisms by a contact manifold X.

Lemma 5.1. *Suppose that (X, α) and (N, λ) are closed contact manifolds. Then there exists a contact form on $X \times N \times [-1, 1]$ equal to $e^{-t}\lambda + \alpha$ near $N \times \{-1\}$ and to $e^t\hat{\lambda} + \alpha$ near $N \times \{1\}$.*

Proof. We apply Theorem 3.1 after switching the role of X and N.

In fact, define positive functions

$$g_1(t) = \begin{cases} 1 & \text{near } t = -1 \\ e^t & \text{near } t = -\frac{1}{2} \end{cases},$$

on $[-1, -\frac{1}{2}]$ and

$$g_2(t) = \begin{cases} e^{-t} & \text{near } t = \frac{1}{2} \\ 1 & \text{near } t = 1 \end{cases}.$$

on $[\frac{1}{2}, 1]$.

The contact form $g_1(t)(\alpha + e^{-t}\lambda)$ on $X \times N \times [-1, -\frac{1}{2}]$ extends to $e^t\alpha + \lambda$ on $X \times N \times [-\frac{1}{2}, 0]$. Similarly, the contact form $g_2(t)(\alpha + e^t\hat{\lambda})$ on $X \times N \times [\frac{1}{2}, 1]$ extends to $e^t\alpha + \hat{\lambda}$ on $X \times N \times [0, \frac{1}{2}]$. Thus we can apply Theorem 3.1 to construct the required form.

□

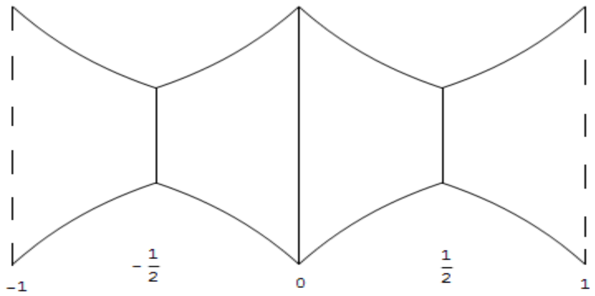
In Lemma 5.1, in order to get a contact structure on the product of this manifold by a contact one, we need a contact form λ on one end of $N \times [-1, 1]$ and $\hat{\lambda}$ on the second, while on the contact factor the form does not depend on $t \in [-1, 1]$. We will show now that the gluing is possible also if the two pairs of forms are equal to (λ, α) and $(\lambda, \hat{\alpha})$ on the two sides. If a symplectic cobordism is just a symplectification of a contact form, then one can consider both λ and $\hat{\lambda}$ as the input of the symplectification. However, if the symplectic cobordism is nontrivial, then such a swap is in general impossible. The same happens if we do not have the product $W_i \times X$, but a bundle. If we have $\hat{\lambda} + \alpha$ over a boundary component, then it can not change the cobordism to get at the boundary the form $\lambda + \hat{\alpha}$.

Lemma 5.2. *Suppose that (X, α) and (N, λ) are closed contact manifolds. Then there exist two contact forms on $X \times N \times [-1, 1]$, both equal to $e^{-t}\lambda + \alpha$ near $N \times \{-1\}$ and one equal to $e^t\hat{\lambda} + \alpha$, the other to $e^t\lambda + \hat{\alpha}$ near $N \times \{1\}$.*

Proof. In case $e^{-t}\lambda + \alpha$ meets $e^{-t}\hat{\lambda} + \alpha$ we apply Lemma 5.1. If it meets $e^{-t}\lambda + \hat{\alpha}$ we use Theorem 3.4 and Lemma 5.1 to $X \times M \times [-1, 1]$ divided into 4 parts:

(1) $X \times N \times [-1, -\frac{1}{2}]$ with the form $e^{-t}\lambda + \alpha$,
(2) $X \times N \times [-\frac{1}{2}, 0]$ with the form $e^t\hat{\lambda} + \alpha$,
(3) $X \times N \times [0, \frac{1}{2}]$ with the form $e^{-t}\hat{\lambda} + \hat{\alpha}$,
(4) $X \times N \times [\frac{1}{2}, 1]$ with the form $e^t\lambda + \hat{\alpha}$.

The forms are defined such that crossing the convex fold at 0 corresponds to passing from (α, λ) to $(\hat{\alpha}, \lambda)$ and crossing concave folds $\pm\frac{1}{2}$ is the swap of (α, λ) to $(\alpha, \hat{\lambda})$ and back. In all cases one of the previously described constructions works. Thus we get a contact form on $X \times N$.



□

All of this enables us to extend the notion of strong symplectic fold to allow concave folds.

Consider a closed hypersurface $N \subset intM$ and denote by $W_i, i = 1, .., k$ the connected components of $M - N$ compactified by adding adjacent components of N. Hence W_i is just a closure of a component of $M - N$. Let $N = \bigcup_s N_s$ denote the decomposition of N into the sum of connected components.

Definition 5.3. *A strong symplectic fold on a compact manifold M is given by:*

1. *a decomposition $\{W_i\}_i$of closed submanifolds, $M = \bigcup_i W_i$, $\partial W_i = \bigcup_s N_{is}$, obtained by cutting M by a hypersurface $N \subset Int\, M$;*
2. *each component N_s of N is endowed with a contact form β_s;*
3. *exact symplectic forms $\omega_i = d\lambda_i$ on W_i such that each ω_i gives a symplectic cobordism on W_i with some convex ends and some concave ends and each pair ω_i, ω_j satisfies one of the following compatibility condition for every connected component N_s of N with $N_s \subset W_i \cap W_j$:*
   - (a) *$\lambda_i = e^t\beta_s$ in $N_s \times [-1, 0] \subset W_i$ and $\lambda_j = e^{-t}\beta_s$ in $N_s \times [0, 1] \subset W_j$ where t is the parameter of $[-1, 1]$ (convex fold: at N_s a convex end of W_i meets a convex end of W_j);*
   - (b) *$\lambda_i = e^{-t}\beta_s$ on $N_s \times [-1, 0] \subset W_i$ and $\lambda_j = e^t\beta_s$ on $N_s \times [0, 1] \subset W_j$, where t is the parameter of $[-1, 1]$ (concave fold: a concave end of W_i meets a concave end of W_j);*
   - (c) *the forms ω_i are either convex or concave along any component of the boundary of M.*

As before, the hypersurface N is called the *fold locus.*

To illustrate usefulness of concave passes we consider a filling question. It is well-known that no overtwisted contact form λ on a compact 3-manifold M is fillable, i.e., there is no compact manifold with boundary of contact type (convex boundary) having overtwisted contact form on the boundary. Constructions based on fibrations, for instance the open book technique, lead to the following question. Is there a similar obstruction to fill up the product of an overtwisted 3-manifold by S^1? In other words, we ask if the form $e^{-t}\lambda + d\theta$ on $M \times [0, \varepsilon) \times S^1$ can be extended to a contact form on a compact manifold

W such that $\partial W = M \times \{0\} \times S^1$. Below we show examples that fillability in this sense is possible. By $d\theta$ we denote the standard form on S^1.

Given two connected contact manifolds $(X, \alpha), (X', \alpha')$ oriented compatibly with contact structures, one can perform 1-surgery such that the resulting manifold is the connected sum $X\#X'$. Then by the contact surgery (Theorem 2.3) we get a contact form on the connected sum. Since we need some choices to perform such operation, the result is not defined uniquely, but its homotopy class is already unique. By slight abuse of language we denote the contact form obtained in this way by $\alpha\#\alpha'$.

Proposition 5.4. *If $n > 0$ and $\lambda)$ is a contact form on S^{2n+1}, then the form $e^{-t}(\lambda\#\hat{\lambda}) + d\theta$ on a collar of the boundary $S^{2n+1}\#S^{2n+1} \times [0, \varepsilon) \times S^1 \subset D^{2n+2} \times S^1$ extends to a contact form on $D^{2n+2} \times S^1$.*

Proof. Consider the symplectizations $e^{-t}\lambda$ on $S^{2n+1} \times [-1, 0]$ and $e^t\hat{\lambda}$ on $S^{2n+1} \times [0, 1]$. Gluing these manifolds along $S^{2n+1} \times \{0\}$ we get a manifold with a concave fold $S^{2n+1} \times \{0\}$ and boundary $S^{2n+1} \cup -S^{2n+1}$. We can perform contact 1-surgery by adding a 1-handle to the boundary which makes the boundary connected, hence diffeomorphic to the sphere. The manifold W obtained by the surgery is diffeomorphic to $S^{2n+1} \times S^1 - D^{2n+2}$ and by Theorem 2.3 the resulting contact form on the boundary sphere is $\lambda\#\hat{\lambda}$ and the boundary is convex (note that we still have the fold in the interior of W). Denote by $S \subset IntW$ the circle given as the sum of intervals $x_0 \times [-1, 1] \subset S^{2n+1} \times [-1, 1]$ and $y_0 \times [-1, 1]$ in the handle, where $y_0 \times \{\pm 1\}$ are attached to $x_0 \times \{\pm 1\}$ by the attaching map of the handle. The (topological) surgery of index 2 on W with the attaching circle S and the standard framing of the normal bundle (determined by the inclusion $W \subset S^{2n+1} \times S^1$) yields the disk D^{2n+2}. Moreover, the standard almost complex structure on W extends to the 2-handle. By Lemma 5.1, on $W \times S^1$ we can find a contact form equal to $\lambda\#\hat{\lambda} + d\theta$ on the boundary. To finish the proof we have to show that this contact form produces a contact form on $D^{2n+2} \times S^1$, obtained from $W \times S^1$ by the 2-surgery multiplied by S^1. The product of a 2-handle attached to W by S^1 corresponds to two surgeries on $W \times S^1$, one of index 2 on $W \times S^1$ and one of index 3 performed on the result of the first one. Since the manifold $W \times S^1$ is of dimension at least 5 and almost contact structure is compatible with the surgeries, we get a contact form on $D^{2n+2} \times S^1$. Finally, the surgeries are done in the interior, hence it preserves the form we have obtained previously in a neighborhood of the boundary sphere.

□

Corollary 5.5. *There exists an overtwisted contact form λ on S^3 such that the form $e^{-t}\lambda + d\theta$ extends from a collar $S^3 \times S^1 \times [0, \varepsilon)$ to a contact form on $D^4 \times S^1$.*

This property can be applied to prove the following.

Proposition 5.6. *If M^5 is closed almost contact and admits an open book decomposition with trivial monodromy, then it is contact.*

Proof. Let P denote the page of the open book. The almost contact structure of M gives a stably almost complex structure on P. But for an open manifold stably almost complex structure determines an almost complex structure. It follows from basic facts of the Morse - Smale theory that there exists a Morse function $f : P \to [0, 4]$ with one minimum ($=0$), constant and maximal ($=4$) on ∂P. This function has critical points only of indices $q=0,1,2,3$ and such that the value of f at a critical point of index q is q. Denote $W_i = f^{-1}[i-\frac{1}{2}, i+\frac{1}{2}],\ i=0,1,2,3,4$. Then W_0 is diffeomorphic to D^4, W_i contains only critical points of indices i and $W_4 = \partial P \times [\frac{7}{2}, 4]$. Let λ be an overtwisted contact form on S^3 such that $e^t\lambda + d\theta$ extends to a contact form on $D^4 \times S^1$. Since P is almost complex, then by the contact surgery we extend the form $e^t\lambda$ to 1-handles of W_1. This makes W_1 a symplectic cobordism with concave end $f^{-1}(\frac{1}{2})$ and convex end $f^{-1}(\frac{3}{2})$. Since the surgeries can be performed far from overtwisted disks, the contact form on the latter can be assumed again overtwisted. Because on an overtwisted 3-manifold one can perform contact surgery on every framing, so the same holds for W_2. In the same manner we make W_3 a symplectic cobordism with concave end $f^{-1}(\frac{7}{2})$ and convex end $f^{-1}(\frac{5}{2})$. In this way we get symplectic structures which agree with the almost complex structure of P. Since the homotopy class of an overtwisted form is determined by the homotopy class of the contact distribution, the contact forms on $f^{-1}(\frac{5}{2})$ obtained from W_2 and W_3 are homotopic, hence by Remark 2.13 can be assumed equal. Finally, on W_4 we put symplectization of the form on $f^{-1}(\frac{5}{2})$ we used for W_3. In this way we get a strong symplectic fold on $W_1 \cup W_2 \cup W_3 \cup W_4$ with fold locus $f^{-1}(\frac{5}{2}) \cup f^{-1}(\frac{7}{2})$, where the fold at $f^{-1}(\frac{5}{2})$ is convex and at $f^{-1}(\frac{1}{2})$ is concave. Therefore, by Theorem 3.1 and Lemma 5.1 we have a contact form on the product with S^1. Since the form on $f^{-1}(\frac{1}{2}) \times S^1$ extends to $D^4 \times S^1$, we get also a contact form on $P \times S^1$. By the construction, this form is the product of a convex form on W_4 by the standard form on S^1 at ∂P. It can be extended to $\partial P \times D^2 \subset M$ exactly as it is done in the case of Giroux' forms. This completes the proof. □

6. Piecewise contact fibered structures

We will compile constructions of previous sections to obtain a notion generalizing both Giroux' structures and constructions of Section 3. This notion is still sufficient to provide a contact form on a manifold endowed with such structure.

Let Y be a compact orientable manifold. Given a hypersurface $H \subset IntY$, let $\{Y_i\}$ denote the collection of connected components of $Y - H$ compactified by adding components of H contained in the closure of Y_i. Our basic assumption is that each Y_i is a fibration of one of the following two types:

(1) a contact fibration with a closed contact fiber (X_i, α_i) over an exact symplectic cobordism $(W_i, d\mu_i)$ trivial in a neighborhood of ∂W_i, or
(2) the fibration over a closed contact manifold (X_i, α_i), defined on a hypersurface in X_i, such that the fiber is an exact symplectic cobordism $(W_i, d\mu_i)$ and the structure group is the group $Ex(W_i, \partial W_i, \mu_i)$ of exact symplectomorphisms equal to the identity in a collar of ∂W_i.

If this is satisfied, then every component of H is the product of X_i by a component of the boundary of the symplectic cobordism W_i. Let us denote by $N_{is}, s = 1,..,l_s$ components of ∂W_i and by λ_{is} the contact form induced on N_{is} by μ_i (which is either convex or concave at N_{is}).

If $N_{is} = N_{jr}$ is a connected component of the intersection $Y_i \cap Y_j \cap H$, then we assume that one of the following conditions is satisfied:

(1) N_{is} is a convex end of W_j, N_{jr} is a convex end of W_j and $X_i = X_j$;
(2) N_{is} is a concave end of W_i, N_{jr} is a concave end of W_j and $X_i = X_j$;
(3) N_{is} is a convex end of W_i, N_{jr} is a convex end of W_j, $N_{is} = X_j,\ N_{jr} = X_i$;
(4) N_{is} is a concave end of W_i, N_{jr} is a concave end of W_j, $N_{is} = X_j,\ N_{jr} = X_i$;
(5) N_{is} is a concave end of W_i, N_{jr} is a convex end of W_j and $X_i = X_j$;
(6) N_{is} is a convex end of W_i, N_{jr} is a concave end of W_j and $X_i = X_j$.

Finally, we assume compatibility of the forms on the adjacent ends of $Y_i's$. In all the cases above we require one the following conditions, according to the list above:

(1) $\lambda_{is} = \lambda_{jr}$ and $\alpha_i = \hat{\alpha}_j$ or $\lambda_{is} = \hat{\lambda}_{jr}$ and $\alpha_i = \alpha_j$;
(2) $\lambda_{is} = \lambda_{jr}$ and $\alpha_i = \hat{\alpha}_j$ or $\lambda_{is} = \hat{\lambda}_{jr}$ and $\alpha_i = \alpha_j$;
(3) $\lambda_{is} = \alpha_j$ and $\alpha_i = \lambda_{jr}$;
(4) $\lambda_{is} = \alpha_j$ and $\alpha_i = \lambda_{jr}$;
(5) $\lambda_{is} = \lambda_{jr}$ and $\alpha_i = \alpha_j$;
(6) $\lambda_{is} = \lambda_{jr}$ and $\alpha_i = \alpha_j$.

If ∂Y_i contains a connected component of ∂Y, then in a collar of that component we have the product of X_i and an end of W_i (either convex or concave).

Remark 6.1. We allow a component of H to be the boundary of two different ends of one Y_i (when $i = j$ in the list above). In particular, it is possible that $Y - H$ is connected.

One can explain our assumptions by saying that the fold locus H divides the manifold M into a number of fibrations carrying contact fibered structure with both fibrations and forms product near any component of H. Under our compatibility conditions we can apply either Theorem 3.8 or Lemma 5.2.

Definition 6.2. *A decomposition of M satisfying the assumptions above is called a piecewise contact fibered structure on M.*

Theorem 6.3. *If M admits a piecewise contact fibered structure, then M is contact.*

Proof. Consider a component Y_i of the decomposition. As we explained in Sections 2 and 3, it admits a contact form equal to $\lambda_{ij}^{\varepsilon} + p^*\alpha_i$, or to $p^*\lambda_{ij} + \alpha_i$, in a collar of the j-th component of ∂Y_i, depending on the type of the fibration on Y_i. Furthermore, $\varepsilon = \pm 1$ depending on convex/concave type of the fold. Under the compatibility conditions we use Theorem 3.8 or Lemma 5.2 to extend those forms through H and we get a global contact form on M.

Remark 6.4. One can allow that instead of equalities in the compatibility conditions one assumes equality up to homotopy, for instance up to the multiplication by a constant. This can be always reduced to the equality case by extending the adjacent end (which is $e^{\pm t}\lambda_j,\ t \in [0,1]$) from $[0,1]$ to $[0,R]$ for R appropriately chosen and applying the trick of Lemma 5.2.

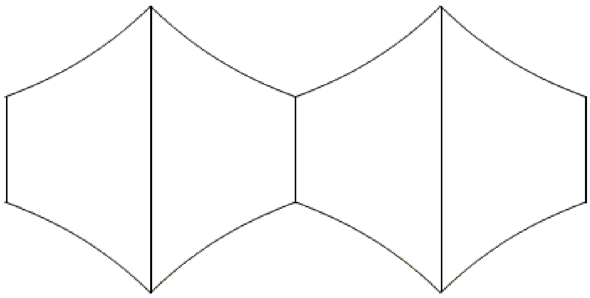
□

Let us illustrate Theorem 6.3 by the following examples.

Example 6.5. Any Giroux' structure is decomposable in the described way. One part is $D^2 \times B$, the other is the fibration over S^1 with fiber P. In this case the fold locus is of type 3.

Example 6.6. If M is a S^1-bundle over $X \times N$, where $(X, \lambda), (N, \lambda')$ are contact, then M is contact (in particular, $X \times S^1 \times S^1$ is). Begin with the trivial bundle. Write $N \times S^1 = N \times [0, \frac{1}{4}] \cup N \times [\frac{1}{4}, \frac{1}{2}] \cup N \times [\frac{1}{2}, \frac{3}{4}] \cup N \times [\frac{3}{4}, 1]$ with $N \times \{0\}$ and $N \times \{1\}$ glued. On these four parts put $e^t\lambda', e^{-t+\frac{1}{2}}\lambda', e^{t-\frac{1}{2}}\hat{\lambda}', e^{-t+1}\hat{\lambda}'$ respectively. This gives a strong symplectic fold structure on $N \times S^1$. Now take products with (X, λ) for $N \times [0, \frac{1}{4}], N \times [\frac{3}{4}, 1]$ and with $(X, \hat{\lambda})$ for $N \times [\frac{1}{4}, \frac{1}{2}], N \times [\frac{1}{2}, \frac{3}{4}]$. So we have the following sequence of forms:

$$e^t\lambda' + \lambda, \quad e^{-t+\frac{1}{2}}\lambda' + \hat{\lambda}, \quad e^{t-\frac{1}{2}}\hat{\lambda}' + \hat{\lambda}, \quad e^{-t+1}\hat{\lambda}' + \lambda.$$



This makes a piecewise contact fibered structure (with all fibrations trivial) on $X \times N \times S^1$, hence there is a contact form on this manifold. By [G3], this extends to any circle bundle over $X \times N$.

Example 6.7. Consider a closed contact manifold M of dimension 5 and a homotopically trivial circle S embedded in M. Then the manifold M' obtained from M by the blow-up along S is contact.

Proof. We can deform given contact form on M to one given by the Giroux structure with page P and the fibration $E \to S^1$. Then S can be deformed to a section of the fibration and, if we deform it to a collar U of $\partial P \times S^1$, we get a product of $U \subset P$ with the circle. On $\overline{\mathbb{C}P^2}$ there is a strong symplectic fold $W_- \cup W_+$ of convex type by [B]. Cutting two small (Darboux) disks in $Int\, P$ and W_- and identifying obtained spheres we get the connected sum $P \# \overline{\mathbb{C}P}^2$. By construction, we have a strong symplectic fold on it and a piecewise fibered structure on M. The fibered pieces are: $B \times D^2$, the product neighborhood of the binding, $(U - D^4) \times S^1$, $(W_- - D^4) \times S^1, W_+ \times S^1$ and the fibration over S^1 with fiber $P - U$ given by the open book structure. Thus it yields a contact form on M.

□

Remark 6.8. Note that $\mathbb{C}P^2 - Int\, D^4$ does not admit any exact symplectic form with contact type boundary, contrary to an incorrect statement in the previous version of the paper. It was explained to us by András Stipsicz that this follows from the fact that any spherical homology 2-class in a closed 4-manifold with self-intersection number -1 is represented by a symplectic submanifold. The same argument, combined with a result of McDuff [McD] shows that there is no strong symplectic fold of convex type on $\mathbb{C}P^2 - Int\, D^4$. That is why to get Example 6.7 we need the general type of strong symplectic folds.

7. Open questions

The main purpose of the paper was to develop constructive techniques in contact topology. Our constructions are based on the strong symplectic fold structures, which exist only on stably almost complex manifolds. The natural question is

Question 7.1. *Does any stably almost complex manifold of even dimension admits a strong symplectic fold?*

In particular, it is anything but obvious if a symplectic manifold has a strong symplectic fold. For instance, is it true for complex projective spaces?

Perhaps more interesting questions concern locally fibered manifolds. For instance, is it possible to construct locally fibered structure on an almost contact manifold starting from open book decompositions with the structure obtained by the Donaldson construction (cf. [MMP])?

8. Appendix: Computations

We present here some of the calculations which led us to the proof of Theorem 3.1. The result was first checked using Mathematica's package "Differential forms" (Frank Zizza and Ulrich Jentschura [FZ]) in low dimensions. Namely, for $t1 = 0$ (for technical reasons we slightly change notation to adapt it for our purposes) and around a point $(b, d, n, 0) \in B \times D^2 \times N \times I$ we take coordinate system in which $\beta = d[z1] + x1d[y1], \lambda = d[z2] + x2d[y2]$. Further, on disk D^2 we take coordinate system (x, y). In this system we set $h_1 = 2 - (x^2 + y^2)^2$ and $h_2 = x^2 + y^2$ (hence in the formula below h_1 is equal to $2 - r^4$ near $r = 0$ so that it is of class C^3). Then the following expressions are equal respectively to η and $d\eta$:

$$\mathbf{eta1{:}{=}(2 - (\boldsymbol{x}\hat{\ }2 + \boldsymbol{y}\hat{\ }2)\hat{\ }2)(\boldsymbol{d}[z1] + x1\boldsymbol{d}[y1]) + (\boldsymbol{d}[z2] + x2\boldsymbol{d}[y2])}$$

$$\begin{aligned}&\mathbf{deta1{:}{=}}\,(-4x^3 - 4xy^2)\, d[x] \wedge d[\mathrm{z1}] + (-4x^3\mathrm{x1} - 4x\mathrm{x1}y^2)\, d[x] \wedge d[\mathrm{y1}]+\\&(-4x^2y - 4y^3)\, d[y] \wedge d[\mathrm{z1}] + (-4x^2\mathrm{x1}y - 4\mathrm{x1}y^3)\, \mathrm{x1}d[y] \wedge d[\mathrm{y1}]+\\&(2 - x^4 - 2x^2y^2 - y^4)\, d[\mathrm{t1}] \wedge d[\mathrm{z1}] + (2 - x^4 - 2x^2y^2 - y^4)\, d[\mathrm{x1}] \wedge d[\mathrm{y1}]+\\&(2\mathrm{x1} - x^4\mathrm{x1} - 2x^2\mathrm{x1}y^2 - \mathrm{x1}y^4)\, d[\mathrm{t1}] \wedge d[\mathrm{y1}] + d[\mathrm{x2}] \wedge d[\mathrm{y2}]+\\&d[\mathrm{t1}] \wedge d[\mathrm{z2}] + \mathrm{x2}\ d[\mathrm{t1}] \wedge d[\mathrm{y2}] - xd[\mathrm{t1}] \wedge d[y] + yd[\mathrm{t1}] \wedge d[x]\end{aligned}$$

Now $\tau = \star(\eta \wedge (d\eta)^3)$ can be computed in two steps: first we calculate

ExteriorProduct[eta1, deta1, deta1, deta1]

and later

HodgeStar[%, *t*[x1, x1] + *t*[y1, y1] + *t*[z1, z1] + *t*[x2, x2]+ *t*[y2, y2] + *t*[z2, z2] + *t*[*x*, *x*] + *t*[*y*, *y*] + *t*[t1, t1]]

where the percent sign refers to $\eta \wedge (d\eta)^3$.

Then $\tau = \star(\eta \wedge (d\eta)^3)$ is given by

$$\begin{aligned}&(24\,(x^2 + y^2)^2\,)\ d\mathrm{x1}\ \hat{}\ d\mathrm{y1} + (-96x(-1 + \mathrm{x1})\mathrm{x1}y\,(x^2 + y^2)^2\,)\ d\mathrm{t1}\ \hat{}\ d\mathrm{x1}+\\&(-24\mathrm{x1}\,(x^2 + y^2)\,(x^2 + \mathrm{x1}y^2)\,)\ d\mathrm{x1}\ \hat{}\ d\mathrm{z1} + (6x\,(-2 + x^4 + 2x^2y^2 + y^4)\,)\ dx\ \hat{}\ d\mathrm{z1}\\&+(6y\,(-2 + x^4 + 2x^2y^2 + y^4)\,)\ dy\ \hat{}\ d\mathrm{z1}+(-24\,(x^2 + y^2)^2\,(-2 + x^4 + 2x^2y^2 + y^4)\,)\ d\mathrm{x2}\ \hat{}\ d\mathrm{y2}+\\&(24\mathrm{x2}\,(x^2 + y^2)^2\,(-2 + x^4 + 2x^2y^2 + y^4)\,)\ d\mathrm{x2}\ \hat{}\ d\mathrm{z2}+(6x\,(-2 + x^4 + 2x^2y^2 + y^4)^2\,)\ dx\ \hat{}\ d\mathrm{z2}+\end{aligned}$$

$(6y\left(-2+x^4+2x^2y^2+y^4\right)^2)\ dy \wedge dz2 + (24x\left(x^2+y^2\right)\left(-2+x^4+2x^2y^2+y^4\right))\ dt1 \wedge dy +$
$(-24y\left(x^2+y^2\right)\left(-2+x^4+2x^2y^2+y^4\right))\ dt1 \wedge dx$
$+(-24(-1+\mathrm{x1})\mathrm{x1}y^2\left(x^2+y^2\right)\left(-2+x^4+2x^2y^2+y^4\right))\ d\mathrm{x1} \wedge dz2$

and τ^4 is equal to

$(-1990656\left(x^2+y^2\right)^6\left(-2+x^4+2x^2y^2+y^4\right)^3)\ dt1 \wedge dx \wedge d\mathrm{x1} \wedge d\mathrm{x2} \wedge dy \wedge d\mathrm{y1} \wedge dy2 \wedge dz1 +$
$(-1990656\mathrm{x2}\left(x^2+y^2\right)^6\left(-2+x^4+2x^2y^2+y^4\right)^3)\ dt1 \wedge dx \wedge d\mathrm{x1} \wedge d\mathrm{x2} \wedge dy \wedge d\mathrm{y1} \wedge dz1 \wedge dz2 +$
$(-1990656\mathrm{x1}\left(x^2+y^2\right)^6\left(-2+x^4+2x^2y^2+y^4\right)^4)\ dt1 \wedge dx \wedge d\mathrm{x1} \wedge d\mathrm{x2} \wedge dy \wedge dy2 \wedge dz1 \wedge dz2 +$
$(-1990656\left(x^2+y^2\right)^6\left(-2+x^4+2x^2y^2+y^4\right)^4)\ dt1 \wedge dx \wedge d\mathrm{x1} \wedge d\mathrm{x2} \wedge dy \wedge d\mathrm{y1} \wedge dy2 \wedge dz2,$

As $\iota_R dvol_{\mathbb{R}^9} = \tau^4$ for some $R \in lin\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$, hence the Reeb field R_τ of τ is equal to R because $\iota_R\tau^4 = \iota_R\iota_R dvol_{\mathbb{R}^9} = 0$. The field R_τ is obviously perpendicular to $\frac{\partial}{\partial r}$ (away from the degenerate set Σ).

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