On vector fields having properties of Reeb fields

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Abstract

We study constructions of vector fields with properties which are characteristic to Reeb vector fields of contact forms. In particular, we prove that all closed oriented odd-dimensional manifold have geodesible vector fields.
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1 Introduction

One of the major open questions in contact topology can be formulated as follows.

Conjecture 1.1 *Does any closed, oriented, odd-dimensional smooth manifold M having a non-degenerate two-form admit a contact form?*

Here non-degeneracy means that the rank of the form is maximal at each point, hence it is $2n$ if $\dim M = 2n+1$. A non-degenerate two form ω exists if and only if the structure group of the tangent bundle TM has a reduction to the group $U(n) \subset SO(2n+1)$. The kernel of such a form is a 1-dimensional subbundle $\mathbf{R}$ of TM, hence it gives a 1-dimensional foliation $\mathfrak{R}$ of M. Properties of Reeb fields are studied intensively in contact topology and some sophisticated analytical techniques are used for this purpose, as for instance symplectic field theory (SFT) or contact homology. In the present note we deal with properties of Reeb foliations which can be detected by topological methods. The general question, which is our motivation, is the following: given a structure on M which arises in the presence of a contact form on M, find constructive methods to build such a structure, at least on some classes of manifolds.

In the paper we consider the following two properties which are satisfied by the Reeb foliation $\mathfrak{R}$ of any contact form:

Property A: $\mathfrak{R}$ admits a connection (equivalently, it is geodesible);
Property B. the basic cohomology class of exterior derivative of a connection on $\mathfrak{R}$ is non-zero.

In the next section we show that in fact these properties are valid for Reeb fields (hence for foliations they define) of of contact forms. Our main result is that on any closed oriented odd-dimensional manifold M a vector field having both properties can be constructed starting from an open book decomposition of M.

There are obstructions for a foliation to have Property A found by Dennis Sullivan [S]. However, Herman Gluck announced [G] that any closed manifold of odd dimension admits a foliation with Property A, but according to our knowledge, the proof was never published. In this paper, we give a simple proof of this hypothesis, based on existence of open book decompositions, cf. Theorem 3.4.

We give also an example showing that a presymplectic form (i.e., nondegenerate and closed) does not need to satisfy neither A nor B.

2 Properties of Reeb vector fields

Throughout this paper we assume all manifolds to be smooth, closed and oriented. Non-zero vector fields are denoted by $V, R, ..$, and the 1-dimensional foliations generated by the fields will be denoted by the corresponding Gothic letter $\mathfrak{V}, \mathfrak{R}$... The contraction of a form η with a vector field V will be denoted by $\iota_V \eta$.

A *contact form* on M^{2n+1} is a one-form α such that $\alpha \wedge (d\alpha)^n > 0$. With every contact form we associate a vector field called its *Reeb field* R. It is uniquely defined by two conditions $\alpha(R) = 1$ and $\iota_R d\alpha = 0$. The condition that defines a contact form implies that $d\alpha$ is non-degenerate (i.e. it has rank $2n$).

A closed 2-form ω such that ω is non-degenerate is called *presymplectic.*. In this case we have the *Reeb foliation* $\mathfrak{R}$ of ω defined by $\iota_{\mathfrak{R}}\omega = 0$. The $2n$-form ω^n defines an orientation on any subbundle transversal to $\mathfrak{R}$, hence $\mathfrak{R}$ has non-zero sections (non-vanishing vector fields tangent to $\mathfrak{R}$). All such fields will be called Reeb fields. By definition, $\iota_R \omega = 0$.

By [MD], any non-degenerate two-form can be deformed to a presymplectic form. Thus Conjecture 1.1 is equivalent to a problem if any presymplectic manifold is contact. So it might be a reasonable strategy to look for a presymplectic form such that its Reeb foliation has properties of contact forms. We consider this note as a step in this direction.

The following theorem gives a property which is shared by all Reeb vector fields of contact forms. Before stating the theorem, let us recall the notion of *basic cohomology.*

If M is a closed manifold equipped with a foliation $\mathfrak{F}$, then consider a cochain complex (C^n, d^n) where C^n is the set of n-forms on M such that

$$\iota_Y \alpha = 0 \text{ and } \iota_Y d\alpha = 0$$

for any vector Y tangent to $\mathfrak{F}$ and d^n is the usual exterior derivative. Then by *n-th basic cohomology group* of $(M, \mathfrak{F})$ we mean the group $H^n_b(M, \mathfrak{F}) = \ker d^n / \operatorname{im} d^{n-1}$.

Theorem 2.1 *Assume that λ is a contact form on a closed manifold M with its Reeb field R. Then the basic cohomology class $[d\lambda] \in H^2_b(M, R)$ is non-zero.*

Proof. By the definition of Reeb field, $\lambda(R) \equiv 1$. Assume that $[d\lambda]_b = 0$, so that there exists α such that $\alpha(R) = 0$ and $d\alpha = d\lambda$. Thus $\phi_0 = \alpha - \lambda$ is closed and equal to 1 on R. It yields existence of a closed form ϕ which is C^1-close to ϕ_0 and such that $\ker \phi$ is integrable with compact leaves by [Ti]. On each leaf $d\lambda$ restricts to a symplectic form, so we get a contradiction, since the cohomology class of a symplectic form on a closed manifold is non-zero. □

Theorem 2.1 is no longer true for presymplectic forms. In fact, one can have $H^2_b(M, \mathfrak{F}) = 0$ for a Reeb foliation of a presymplectic form. Consider the following example studied by Carrière in [C].

Example 2.2 Let T^3_A denotes the T^2-bundle over the circle whose monodromy is given by matrix $A \in SL(2, \mathbb{Z})$ such that $trA > 2$,

$$T^3_A = T^2 \times \mathbb{R}/(x, t) \sim (Ax, t + 1).$$

Then both eigenvalues $\lambda, \frac{1}{\lambda}$ of A are real and irrational. Let μ_1, μ_2 be corresponding eigenvectors. Define 1–forms v_1, v_2 on T^2 by $v_1(\mu_1) = 1,\ v_1(\mu_2) = 0$ and $v_2(\mu_1) = 0,\ v_2(\mu_2) = 1$. Extend v_1, v_2 to $T^2 \times \mathbb{R}$ by setting

$$\alpha_1 = \lambda^t v_1$$
$$\alpha_2 = \frac{1}{\lambda^t} v_2.$$

Now the forms α_1, α_2 are also well defined on T^3_A. By direct calculation, $d\alpha_1 = \ln(\lambda) dt \wedge \alpha_1$ and $d\alpha_2 = -\ln(\lambda) dt \wedge \alpha_2$. Thus $d\alpha_1$ is a presymplectic form on T^3_A with associated Reeb field $R = \mu_2$ (and $d\alpha_2$ is presymplectic with the Reeb field μ_1). Furthermore, $\alpha_1(\mu_2) = 0$, hence by definition $[d\alpha_1] = 0$ in $H^2_b(T^3_A, \mathfrak{R})$.

Carrière also shows that the group $H^2_b(T^3_A, \mathfrak{R})$ vanishes. □

Second property of vector fields we consider is related to the condition $\eta(R) = 1$. Let $\mathfrak{V}$ be a 1-dimensional foliation on M.

Definition 2.3 *A 1-form η is a connection on $\mathfrak{V}$ if $\iota_V d\eta = 0$ and $\eta(V) \equiv 1$ for a vector field V tangent to $\mathfrak{V}$.*

The corresponding notion for vector fields is the following.

Definition 2.4 *Let V be a nowhere vanishing vector field on a manifold M. We say that a 1-form η is a connection form on V if $\iota_V d\eta = 0$ and $\eta(V) \equiv 1$.*

Thus a 1-dimensional foliation has a connection if a vector field tangent to the foliation has one. Note that any connection form on V is V-invariant, since $L_V \eta = \iota_V d\eta + d\iota_V \eta = 0$. If V generates a (locally) free circle action, then what we have defined becomes the standard notion of a connection of a principal bundle.

There are obstructions for a non-vanishing vector field V to have a connection form found by Dennis Sullivan [S]. They are certain currents on M can be arbitrarily well approximated by the boundary of a two-chain tangent to V.

Vector fields described in Example 2.2 have no connection forms. To show this, we will use the following observation.

Lemma 2.5 *If $d\phi$ is a presymplectic form on a manifold M, R is the Reeb field of $d\phi$ such that $\phi(R) \geq 0$ and η is a connection form for R, then $\beta = K\phi + \eta$ is a contact form provided that K is large enough. Its Reeb vector field is $(K\phi(R) + 1)^{-1}R$.*

Proof. To prove that β is contact, it is enough to show that $d(K\phi + \eta) = Kd\phi + d\eta$ is non-degenerated on a subbundle $R^{\perp}$ transverse to R and that $(K\phi + \eta)(R) > 0$. Since the second property is obvious, we only show the first. Non-degeneracy is an open condition,thus $d\phi + \frac{1}{K}d\eta$ is non-degenerated if K is large enough. By assumptions, $\iota_R(Kd\phi + d\eta) = 0$, so R is the Reeb field of $d\beta$. Finally, $\beta(R) \geq \eta(R) > 0$. □

Lemma 2.5 implies that if the vector field R described in Example 2.2 had a connection form, then R would be the Reeb field of a contact form and by Theorem 2.1 the basic cohomology class would be non-zero. This is not possible, as $H^2_b(T^3_A, \mathfrak{R})$ is trivial. This contradicts also the Taubes theorem about existence of closed orbits of a contact Reeb field (the Weinstein conjecture) since trajectories of R are irrational lines in T^2.

Sullivan proved also that a vector field V admits a connection if and only if it is *geodesible,* i.e. there exists a Riemannian metric g that makes the orbits of V geodesics. For the proof that every closed, oriented odd-dimensional manifold has such field we will use open book decompositions.

Definition 2.6 *An open book decomposition of M is given by*

1. *a codimension two submanifold* $B \subset M$,

2. *a fibration* $\pi : M \backslash B \to S^1$ *with fibre* P,

3. *a tubular neighborhood* U *of* B *diffeomorphic to* $B \times D^2$

such that the monodromy of the fibration π *is equal to the identity in* $P \cap U$ *and* $\pi|U$ *identifies with the standard projection* $B \times (D^2/\{0\}) \to S^1$.

The submanifold B is called the *binding* and the closure of P is the *page.*

The existence of an open book decomposition for an odd-dimensional closed and oriented manifold has been proved by Frank Quinn ([Q]). See also the discussion in Chapter 29 and in Appendix of [RW].

3 Existence of vector fields with Properties A,B

By the preceding section, any Reeb field V of a contact form has Properties A and B. In this section we prove that vector fields having both properties exist on every closed oriented odd-dimensional manifold.

Theorem 3.1 *Any closed oriented odd-dimensional manifold* M^{2n+1} *admits a vector field with connection (hence a geodesible vector field).*

The proof is based on the following lemma.

Lemma 3.2 *Consider an open book decomposition of a closed manifold* N. *Then any vector field with connection on the binding extends to a vector field with connection on* N.

Proof. By assumptions, we have a vector field X_B on B tangent to B together with a connection form α for X_B. The mapping $\pi : B \times (D^2/\{0\}) \to S^1$ can be given in polar coordinates by $\pi(b,(r,\varphi)) = \varphi$, hence we set a connection form $\pi^* d\varphi$ for the vector field $\frac{\partial}{\partial\varphi}$ on the boundary of $B \times D^2$. This pair can easily be extended to $N \backslash (B \times D^2)$ by taking any lift of $\frac{\partial}{\partial\varphi}$ that coincides on the boundary $\partial(B \times D^2)$ with $\frac{\partial}{\partial\varphi}$, while the connection form for $\frac{\partial}{\partial\varphi}$ on $N \backslash (B \times D^2)$ remains the same and is equal to $\pi^* d\varphi$. The proof is completed by showing that for suitably chosen functions $f, g : [0;1] \to [0;1]$ ($f, g = 0$ for $r \in [0;\varepsilon)$ and $f, g = 1$ for $r \in (1-\varepsilon;1]$) the form

$$\eta = f(r)d\varphi + (1 - f(r))\alpha$$

is a connection form for

$$V = g(r)\frac{\partial}{\partial\varphi} + (1 - g(r))X_B.$$

We have

$$\eta(X) = f(r)g(r) + (1 - f(r))(1 - g(r)) > 0,$$

and furthermore

$$d\eta = f'(r)drd\varphi - f'(r)dr \wedge \alpha + (1 - f(r))d\alpha,$$

hence

$$\iota_V d\eta = (-f'(r)g(r) + f'(r)(1 - g(r)))dr = f'(r)(1 - 2g(r))dr.$$

This means that if f is not constant, then $g(r) = \frac{1}{2}$. Such functions are easy to construct. □

Proof of Theorem 3.1. The proof is by induction on n. For $n = 1$ this follows from the fact that every closed oriented 3-manifold is contact (c.f. [TW],[Ma]), and it is trivial for $n = 0$. In general, M has an open book decomposition and we apply Lemma 3.2 to complete the proof. □

Now we deal with Property B. The following lemma will be useful for recognizing whether, for a connection form η on a vector field V, the class $[d\eta]_b \in H^2_b(M, \mathfrak{V})$ is non-zero.

Lemma 3.3 *Let η be a connection form for a vector field V. If V has a closed orbit γ which is homologically trivial and $\int_\gamma \alpha \neq 0$, then $[d\eta]_b \neq 0$. The same is true if there are two closed orbits γ, γ' of V field representing the same homology class and $\int_\gamma \eta \neq \int_{\gamma'} \eta$.*

Proof. If $[d\eta]_b = 0$, then we have 1-form ϕ such that $d\phi = d\eta$ and $\phi(\mathcal{R}) = 0$. Consequently, $\eta - \phi$ is closed and $\int_\gamma (\eta - \phi) \neq 0$, contrary to homological triviality of γ. This argument works also for two closed orbits with different values of $\int_\gamma \eta$. □

Corollary 3.4 *Every closed oriented odd-dimensional manifold M^{2n-1} has a geodesible vector field V with such a connection form η that $[d\eta]_b \neq 0$ in $H^2_b(M, V)$.*

Proof. By Theorem 3.1, it is sufficient to prove that for a vector field V constructed in 3.1 we can choose η such that $[d\eta]_b \neq 0$. Observe that near the boundary $\partial(B \times D^2)$ we have $V = \frac{\partial}{\partial t}$, hence it has a contractible closed orbit γ. The connection form η given by Lemma 3.2 in this subset is dt, hence $\int_\gamma \eta \neq 0$. Lemma 3.3 completes the proof. □

Remark 3.5 For presymplectic manifolds M^{2n+1} Theorem 3.4 can be derived directly from [MMP], where the authors proved that M has an open book decomposition with presymplectic binding. As in Theorem 3.4, we proceed with induction on n.

4 Presymplectic confoliation

Assume that a codimension one subbundle ξ on M^{2n+1} is equal to the kernel of a 1-form α. If $\alpha \wedge (d\alpha)^n \geq 0$, then ξ is called a (positive) *confoliation* on M. The form is determined by ξ up to a positive function and is contact if $\alpha \wedge (d\alpha)^n > 0$. A form defining a confoliation will be called confoliation form.

Example 4.1 Let $S^1 \to M^{2n+1} \to B$ be a principal S^1 fibration and η an invariant connection such that its curvature defines a symplectic form on B. Then η is the Boothby-Wang contact form [BW]. If we assume only that $(d\eta)^n \geq 0$, then we get a confoliation form.

Of course, any closed nowhere vanishing 1-form α is confoliation form, but then ξ is integrable and if M is closed, then it fibers over a circle. More exactly, there is a nowhere vanishing 1-form α' which is closed, C^1-close to α and $\xi' = \ker \alpha$ is tangent to fibers of a fibration over S^1 ([Ti]). □

Some methods to deform a confoliation form to a contact form were found, see for example second chapter of [ET] for dimension 3 case and [AW] for general case. Generally, there are two types of obstacles for a confoliation form α to be contact. First of all the rank of $d\alpha$ can be strictly less then $2n$. Even if the rank of $d\alpha$ is maximal (hence $d\alpha$ is presymplectic), then its Reeb field R might lie in $\ker \alpha$ (thus $\alpha \wedge (d\alpha)^n = 0$) at some points.

Definition 4.2 *A 1-form α is a presymplectic confoliation form if α is a (positive) confoliation form and $d\alpha$ is presymplectic.*

Direct calculation shows that conditions defining presymplectic confoliation form are equivalent to the following properties of $\alpha : d\alpha$ is presymplectic and $\alpha(R) \geq 0$. Under the assumption that the Reeb field admits a connection, one can deform a presymplectic confoliation form into a contact form by Lemma 2.5.

Proposition 4.3 *Assume that α is a presymplectic confoliation form and the Reeb field of $d\alpha$ has a connection form η. Then for ε small enough the form $\alpha + \varepsilon\eta$ is contact.*

Example 4.4 The forms α_1, α_2 defined in Example 2.2 are presymplectic confoliation forms, but their Reeb fields have no connections. However, the form $\phi_\varepsilon = \alpha_1 + \varepsilon\alpha_2$ is contact for $\varepsilon > 0$ since $\phi_\varepsilon \wedge d\phi_\varepsilon = 2\varepsilon \ln(\lambda)\alpha_1 \wedge \alpha_2 \wedge dt > 0$. Recall that exactly one of eigenvalues $\lambda, \frac{1}{\lambda}$ of A is greater than 1, thus we must choose suitable α_i to have that $\phi_\varepsilon \wedge d\phi_\varepsilon$ is positive; therefore we take α_1 from Example 2.2 and perturb it linearly in the direction of α_2.

Furthermore, contact forms ϕ_ε is C^1-approximate the presymplectic confoliation form α_1. □

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