Presymplectic manifolds

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Abstract

A presymplectic structure on an odd-dimensional manifold is given by a closed 2-form which is non-degenerate, i.e., of maximal rank. We investigate geometry of presymplectic manifolds. Some basic theorems analogous to those in symplectic and contact topology are given and applied to study constructions of presymplectic manifolds. In particular, we describe how to glue presymplectic manifolds along isomorphic presymplectic submanifolds, including surgery on presymplectic circles.
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1 Introduction

Investigation of symplectic and contact structures is an important and developing part of geometry of manifolds. In this paper we study presymplectic manifolds which mediate between those two types of structures.

A *presymplectic form* is a closed differential 2-form of maximal rank on an odd-dimensional manifold. Thus if the manifold M is of dimension $2n+1$, ω is a presymplectic form on M, then the rank of the form is $2n$. This means that at each point $x \in M$ there exists a (non-unique!) codimension one subspace of T_xM such the form is a symplectic linear form on it. This gives a linear symplectic, hence admitting a complex structure, subbundle and a unique subbundle $\mathcal{R} = \{V \in T_xM | \iota_V \omega = 0\}$ of dimension one, which we call *Reeb bundle.* Throughout the paper we assume that M is an oriented manifold. Then there is an orientation of $\mathcal{R}$ given by the orientation of M and the orientation of the symplectic subbundle provided by ω. In particular, TM has a reduction of the structure group to $U(n) \times \mathbf{1}$. Furthermore, the

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Reeb bundle $\mathcal{R}$ defines a foliation of dimension one and, for any choice of non-zero section R of $\mathcal{R}$, a flow without fixed points. We shall call the foliation the *Reeb foliation* of the presymplectic form ω, and any such R a *Reeb vector field.*

There is a natural source of presymplectic structures.

Lemma 1.1 *If $i : Q \subset M$ is a codimension one submanifold of a symplectic manifold (M, Ω), then $i^*\Omega$ is a presymplectic form on Q.*

As we noticed above, if an oriented manifold M admits a presymplectic form, then the tangent bundle of M has a reduction to $U(n) \times \mathbf{1} \subset O(2n+1)$ defined by Reeb subbundle and a choice of a complex structure in a complementary symplectic subbundle. As in the case of almost complex structures defined by a symplectic form, there is a large family of possible choices. Any such choice is provided by a bundle map $J : TM \to TM$ such that $J(\mathcal{R}) = 0$ and $J^2 = -Id$ on the subbundle $Im\, J$. One can see that existence of such a reduction is equivalent to existence of a 2-form of maximal rank (not necessarily closed), cf. [B]. This is analogous to existence of an almost complex structure for a symplectic form.

However, existence of a 2-form of maximal rank is also sufficient for the existence of presymplectic form. It reduces the problem of existence to homotopy theoretical questions due to the following fundamental theorem.

Theorem 1.2 *If $\mathcal{S}_{non-deg}(M)$ is the space of all 2–forms of maximal rank on a closed manifold M^{2n+1} and $\mathbb{S}_{presymp}(M; a)$ is the space of all presymplectic forms in a given cohomology class a, then*

$$\mathbb{S}_{presymp}(M, a) \hookrightarrow \mathcal{S}_{non-deg}(M)$$

is a homotopy equivalence.

Main technical novelty of the present note is that the assumption on the cohomology class can be removed and a relative case also holds, cf. Section 3. This is later applied to various constructions.

The first proof that there is a bijective correspondence of connected components, with a relative version, was given by Dusa McDuff in [MD].

Theorem 1.3 *Any 2-form ω of maximal rank can be deformed in the space of forms of maximal rank, to a presymplectic form ω'. If ω is presymplectic in a neighborhood of a compact set K, then there exists a deformation which is constant in a neighborhood of K.*

With some effort this can be extended to prove the homotopy equivalence. Later Eliashberg and Mishachev [EM] gave a proof based on Gromov's existence theorem of symplectic forms on open manifolds. The idea can be easily explained: ω gives a form $\omega + \rho \wedge dt,\ \rho(R) = 1$, of maximal rank on $M \times \mathbb{R}$ (not closed in general), by Gromov's theorem it can be deformed to a symplectic form. Its restriction to $M \times \{0\}$ yields a presymplectic form by Lemma 1.1.

As an immediate corollary we get that any parallelizable manifold admits a presymplectic form, e.g. any oriented 3-manifold is presymplectic.

The fact that it is relatively simple to construct a presymplectic forms can be useful for contact and symplectic topology. For example, one can ask the following.

Question 1.4 *Does any closed presymplectic manifold admit a contact form?*

A rather natural idea is to consider a presymplectic form of zero cohomology class, $\omega = d\alpha$, and look for a deformation of α to a contact form. By [G], this can be done in dimension 5 for some for any closed simply connected presymplectic manifold. However, except for these results, the classical case of dimension 3 and a number of special cases (e.g., see [GS] for the case of $\mathbb{C}P^2 \times S^1$), no answers to this question are known. In section 5 we show that vanishing of basic cohomology class with respect to its Reeb foliation is an obstruction for a presymplectic form to be of *contact type* (i.e. equal to differential of a contact form).

Consider two other problems, which proved to be very difficult and both are open in dimensions greater than four. First, let M be an odd-dimensional closed manifold.

Question 1.5 *Is it true that if $M \times S^1$ is symplectic, then M fibres over S^1?*

For M of dimension three this question was posed by Taubes and answered positively, after a series of partial results of many authors, by Friedl and Vidussi [FV]. Their proof uses Seiberg -Witten invariants and it does not extend to higher dimensions.

A closed manifold X of dimension $2n$ we shall call *homotopically symplectic* if X is almost complex and there exists $x \in H^2(X, \mathbb{R})$ such that $x^n \neq 0$. These two conditions are the only known necessary conditions for existence of a symplectic structure for a general closed manifolds of dimension greater than four.

Note that the term *c-symplectic* or *cohomologically symplectic* is used for manifolds which satisfy the cohomological part of the condition.

Question 1.6 *Does any closed, homotopically symplectic manifold admit a symplectic form?*

Questions 1.5, 1.6 cannot simultaneously have positive answers. Namely, there are presymplectic manifolds which do not fibre over the circle, but their products with the circle satisfy assumption of 1.6, e.g. connected sum of two copies of tori $T^{2k+1} \# T^{2k+1}$.

The principal aim of the present paper is to give some basic theorems on presymplectic manifolds and to provide means to constructions of presymplectic manifolds. It includes Moser type theorems, tubular neighborhoods of presymplectic submanifolds and constructions of presymplectic manifolds. For symplectic and contact manifolds such theorems open a way to analyze their topology. In the presymplectic case the Reeb foliation plays an important role. For example, on a closed symplectic manifold we have Moser's theorem saying that a path ω_t of symplectic forms with constant cohomology class is given by an isotopy, i.e., there exists an isotopy f_t such that $\omega_t = f_t^* \omega_0$, can not be extended to presymplectic manifolds without an additional assumption on the Reeb foliation. This assumption is, roughly speaking, that the basic cohomology class of the presymplectic form is constant. A similar problems appear if one analyzes behavior of a presymplectic form in a neighborhood of a presymplectic submanifold. For tubular neighborhoods there is no isotopy of different tubular neighborhoods of a given submanifolds, which allows various constructions in symplectic and contact topology. Here we have only a weaker statement of a presymplectic isomorphisms of such neighborhoods.

Despite of the fact that existence of presymplectic form boils down to homotopy theoretical, hence "soft" question, some effective construction methods can be useful. We describe first a Thurston type construction of presymplectic forms on bundles in Section 7.

Another construction which may have interesting application gives a presymplectic structure on any presymplectic open book decomposition. This technique is well known in contact theory. By results of Giroux and Mohsen [GM], any closed contact manifold admits a compatible open book decomposition and in dimension three this was successfully used to some classification problems. For presymplectic manifolds the construction of open book, based on Donaldson method of quasi-holomorphic sections, can be repeated, cf. [MMP]. We work in reverse direction, namely for a given open book decomposition we give sufficient condition to construct a presymplectic form with Reeb bundle transversal to pages.

In fact, we do this in more general case of *star-like structures.* This is a straightforward generalization of the notion of open book decomposition to the case when the complement of a submanifold of codimension $2k$ admits a symplectic fibration over S^{2k-1}.

Finally, we describe how to glue two presymplectic manifolds along tubular neighborhoods of isomorphic presymplectic submanifolds. A particular case of 1-dimensional submanifold, hence a closed leaf of the Reeb foliation, is in fact the classical surgery on a circle. We show that in dimension greater than 3 any presymplectic form is homotopic to one with a closed orbit in each element of a generating set of the fundamental group, thus one can always kill fundamental group using such modifications.

In our terminology we follow McDuff [MD]. The term presymplectic is used in a wider sense in papers on quantization, e.g. [CKT, KT], where a manifold is called presymplectic if it is endowed with a closed, not necessarily non-degenerate, 2-form. What we call presymplectic form was called odd-symplectic form in [Gi] and closed form of maximal rank in [EM]. In hamiltonian mechanics context an exact presymplectic form is called hamiltonian.

In the sequel we will consider only *closed* smooth manifolds and smooth mappings.

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2 Homotopically presymplectic forms

Let M be a closed manifold of dimension $2n+1$. We say that M is *homotopically presymplectic* if the tangent bundle TM of M admits a reduction to $U(n) \simeq U(n) \times \mathbf{1} \subset SO(2n+1)$. The name almost contact manifold has also been used for such structure (see [G1, MMP]). Our terminology attempts to unify presymplectic and symplectic cases. If M is closed, then we call it *strongly homotopically presymplectic* if it is homotopically presymplectic and there exists a class $x \in H^2(M)$ such that x^n is non-zero.

Proposition 2.1 *$M \times S^1$ is homotopically symplectic if and only if M is strongly homotopically presymplectic.*

Proof. Assume that $M \times S^1$ is homotopically symplectic. If $u \in H^2(M \times S^1)$ satisfies $u^{n+1} \neq 0$, then its restriction to $H^2(M \times *)$ satisfies, by the Künneth formula, $x^n \neq 0$. Moreover, any codimension one subbundle of a complex bundle has a reduction to $U(n)$, thus M is strongly homotopically presymplectic.

If $x \in H^2(M)$ satisfies $x^n \neq 0$, then there exists a class $\beta \in H^1(M)$ such that $x^n \cup \beta$ is non-zero, hence $(x + \beta \cup [dt])^{n+1} \neq 0$. Furthermore, if TM admits a reduction to $U(n)$, then we can easily get an almost complex structure J associated with decomposition $TM = \xi \oplus \varepsilon^1$ by setting

$$J\left(\frac{\partial}{\partial t}\right) = X, JX = -\frac{\partial}{\partial t},$$

where X is a non-zero vector field on M corresponding to the trivial subbundle ϵ^1 and J is a complex structure on ξ. $\square$

Consider an odd-dimensional manifold M which is strongly homotopically presymplectic, but do not fibers over the circle, e.g. $T^{2n-1} \sharp T^{2n-1}$. Then $M \times S^1$ is strongly homotopically symplectic. If such manifold is symplectic, then it is a counter-example to Question 1.5.

Any symplectic form on $M \times S^1$ gives a loop of presymplectic forms. By Lemma 1.1, there is a continuous mapping from the space of all symplectic forms on $M \times S^1$ into the space of (free) loops of presymplectic forms on M. Question 1.5 leads to the following: does this mapping induce a surjective map on π_0? Equivalently, one can ask if any loop of presymplectic forms can be deformed to a loop obtained from a symplectic form on $M \times S^1$? We show a counter-example on the 3-torus. In particular, the loop we give is non-contractible.

It is not difficult to construct a loop of non-degenerate 2-forms on T^3 such that the associated almost complex structure J on T^4 has non-vanishing first Chern class. It is obtained from a non-degenerate 2-form on T^3 with non-trivial c_1 by restriction. By Theorem 1.3, we deform this loop to a loop of presymplectic forms. This loop is not obtained from a symplectic form, since for any almost complex structure J tamed by a symplectic structure on T^4 we have $c_1(J) = 0$, as proved by Donaldson.

3 Homotopy of presymplectic forms

In the proof of Theorem 1.2 due to Mishachev and Eliashberg, the first step is to pass from a form ω of maximal rank on M^{2n+1} to a form of maximal rank on $M \times \mathbb{R}$. This is very simple. If R is a Reeb vector field of ω, then there exists a 1-form η such that $\eta(R) \equiv 1$. Let $p : M \times \mathbb{R} \to M$ be the projection and θ be the standard parameter of $\mathbb{R}$, so that $d\theta$ is the standard (translation invariant) 1-form on $\mathbb{R}$. Then $p^*\omega + d\theta \wedge \eta$ is a form of maximal rank (non-degenerate) extending ω to $M \times \mathbb{R}$. However, it can be non-closed even if ω is. The following observation yields a useful improvement.

Lemma 3.1 *Let $C \in \mathbb{R}$. For δ small enough the form $\phi = p^*\omega + \delta(d\theta \wedge \eta - \theta\, d\eta)$ on $M \times [-C, C]$ is non-degenerate. If ω is closed in an open set $U \subset M$, then ϕ is symplectic in $U \times [-C, C]$ and $\phi | M \times \{0\} = \omega$.*

□

Now, after applying the relative version of Gromov's theorem, one comes to a relative version of Theorem 1.2.

Theorem 3.2 *Let $\mathcal{S}_{non-deg}(M, K; a_0)$ be the space of all 2–forms of maximal rank on a closed manifold M^{2n+1} which restricts, in a neighborhood of a compact subset $K \subset M$, to a presymplectic form in the cohomology class a_0. Let furthermore $\mathbb{S}_{presymp}(M, K; a)$ be the space of all presymplectic forms in a given cohomology class a which is equal to a_0 under restriction to a neighborhood of K. Then*

$$\mathbb{S}_{presymp}(M, K; a) \hookrightarrow \mathcal{S}_{non-deg}(M, K; a_0)$$

is a homotopy equivalence.

$\square$

Corollary 3.3 *If $\eta_t,\ t \in [0,1]$ is a path of non-degenerate forms on M connecting some presymplectic forms η_0, η_1, such that $[\eta_0] = [\eta_1]$ and η_t is presymplectic in a neighborhood of K, then η_t can be deformed to a path $\tilde{\eta}_t$ of presymplectic forms which satisfies:*

1. $\tilde{\eta}_0 = \eta_0, \quad \tilde{\eta}_1 = \eta_1$,

2. *$\tilde{\eta}_t$ is equal to η_t in a neighborhood of K for any $t \in [0,1]$.*

In particular, we see that if M^{2n+1} is presymplectic, then for any two cohomologous presymplectic forms ω_1, ω_2 which can be connected by a path of non-degenerate forms ω_t, there is a path ω'_t of presymplectic forms joining ω_1 and ω_2, and ω'_t can be prescribed in a neighborhood of K.

We will show now that the assumption that the forms are cohomologous can be removed. We give a relative version as well.

Theorem 3.4 *Assume that ω, ω' are two presymplectic forms on a closed manifold M^{2n+1} and that they are joined by a path $\{\omega_t\}_{t\in[0;1]}$ of non-degenerate 2-forms. Then ω, ω' are homotopic through presymplectic forms. Furthermore, the following relative version of this theorem also holds: if $i : K \subset M$ is a compact subset, $\{\omega_t\}$ is presymplectic in an open neighborhood of K, $[i^*\omega_t]$ is constant, then there exists a presymplectic homotopy equal to ω_t in a neighborhood of K, up to a presymplectic deformation with fixed ends.*

Proof. Let us define a set $\Lambda_{\omega,K}$ as the set of elements $u \in H^2(M;\mathbb{R})$ which satisfy the following condition:

(*) if $[\omega'] = u$ and there is a path of non-degenerate forms joining ω and ω', satisfying assumptions of the relative part of Theorem 3.4 rel K, then there is a path of presymplectic forms joining ω and ω' equal to the given path in a neighborhood of K.

We will show first that if the condition (*) is satisfied by a form, then it is satisfied by all other forms in the same cohomology class. In fact, assume that we have two presymplectic forms ω_1, ω_2 such that $u = [\omega_1] = [\omega_2]$ and there are two paths of non-degenerate forms ω^1_t, ω^2_t joining ω with, respectively, ω_1, ω_2. By Corollary 3.3, the path $\omega^1_{-t} \star \omega^2_t$ can be homotoped (rel a neighborhood U_K of K) to a path of presymplectic forms in the class u joining ω_1 and ω_2. By assumptions, there is a (rel U_K) path of presymplectic forms joining ω and ω_1. This path in a neighborhood of K is equal to $\omega^1_t \star \omega^1_{-t} \star \omega^2_t$, which is deformable to ω^2_t.

Our assumption that the class of $i^*\omega_t$ is constant implies that $\Lambda_{\omega,K} \subset [\omega] + j^*H^2(M,K,\mathbb{R})$, where $j : (M,\emptyset) \hookrightarrow (M,K)$ is the inclusion.

We are going to prove that $\Lambda_{\omega,K}$ is both open and closed, thus equal to $[\omega]+j^*H^2(M,K,\mathbb{R})$. If $u\in\Lambda_{\omega,K}$ and $[\omega_1]$ is close enough to it, then there exists a presymplectic $\omega_2\in u$ such that $\omega_1=\omega_2+\eta$, where η is small (say, in a norm defined by a Riemannian metric) and vanishes near K. Then $\omega_2+t\eta$ is a presymplectic path connecting ω_2 with ω_1, constant in a neighborhood of K. This path can be used to pass from a path connecting ω with ω_1 to a path connecting ω with ω_2 and backwards. Thus if u' is close enough to an element of $\Lambda_{\omega,K}$, then it also belongs to $\Lambda_{\omega,K}$.

Essentially the same argument shows that $\Lambda_{\omega,K}$ is closed, since if we have an element $\omega_2\in\Lambda_{\omega,K}$ close enough to ω_1, then again we can use the linear segment to draw the required path from ω_2 to ω_1. $\square$

Theorem 3.4 has the following corollary.

Corollary 3.5 *Let (M,ω) be a presymplectic manifold. Then in every class $a\in H^2(M,\mathbb{R})$ there is a presymplectic form homotopic to ω through presymplectic forms.*

More difficult questions of that type are those with Reeb foliation involved. For example, one might ask whether for every non-zero vector field X on M^{2n-1} there is a presymplectic form for which X is the Reeb field. The answer is negative even in dimension three and the following yields a counter-example on T^3.

On $T^3=T^2\times S^1$ consider coordinates $(x,y,t),\ t\in[0,2]$. Take $X=T^2\times[0,1]\subset T^2\times S^1\equiv T^3$ with a coordinate system $((x,y),t)$ and define a vector field R on T^3 by the formula $R=\cos(\pi t)\frac{\partial}{\partial t}+\sin(\pi t)\frac{\partial}{\partial x}$. Note that R points inwards along both boundary tori $T_0=T^2\times\{0\}$ and $T_1=T^2\times\{1\}$ of X. Assume that R is a Reeb field of a presymplectic form ω . Counting orientations of T_0,T_1 given by ω and R, we come to contradiction with equality $\int_{T_0}\omega=\int_{T_1}\omega$.

4 Moser type theorems and contact structures

Among classical results of symplectic topology a special role plays the theorem of Moser. It says that a path $\{\omega_t\}_{t\in[0;1]}$ of cohomologous symplectic forms on a closed manifold is an *isotopy*, i.e. there exists an isotopy ϕ_t starting from $\phi_0=id$ such that $\omega_t=\phi_t^*\omega_0$. Let us denote an obvious obstruction to the isotopy in presymplectic case: the Reeb foliations should be conjugated by diffeomorphisms (see example after Theorem 4.1 below). The crux of the proof is that when we write the equation

$$\sigma_t+\iota(X_t)\omega_t=0, \tag{4.1}$$

where

$$\frac{d}{dt}\omega_t = d\sigma_t \tag{4.2}$$

and X_t generates the isotopy $\{\phi_t\}_{t\in[0;1]}$ we are looking for, then the equation has always a solution. This is due to non-degeneration of ω_t. This obviously does not extend to the case of presymplectic forms. However, it does extend if we know that presymplectic form σ_t in 4.2 satisfies $\sigma_t(R_t) = 0$. This leads to the notion of *basic cohomology* (see for example [Mo]).

For any foliation $\mathfrak{F}$ consider the space $T\mathfrak{F}$ of vectors tangent to the foliation $\mathfrak{F}$ and define the space of *basic n-forms* as

$$\Omega_b^n = \{\alpha \in \Omega^n \mid \forall X \in T\mathfrak{F} \; \iota_X\alpha = \iota_X d\alpha = 0\}.$$

The usual exterior derivative d defines a mapping $d_b^n : \Omega_b^n \to \Omega_b^{n+1}$. Homology groups of the resulting chain complex are called *basic cohomology groups* of $\mathfrak{F}$ and denoted $H_b^*(M;\mathfrak{F})$ (we suppress $\mathfrak{F}$ from the notation if it is clear what foliation is considered). Now, with a stronger assumption that the basic cohomology class is constant, we can repeat Moser's argument to get the following.

Theorem 4.1 *Assume that $\{\omega_t\}_{t\in[0,1]}$ is a path of presymplectic forms in a fixed cohomology class on M. If the Reeb foliation is independent of t and the basic cohomology class $[\omega_t]_b \in H_b^2(M)$ with respect to the Reeb foliation is constant, then $\{\omega_t\}$ is an isotopy.*

□

The assumption on basic cohomology class cannot be relaxed, since the foliation can vary along with the parameter. For example, on T^3 define $\omega = dx \wedge dy$, $\alpha = sin(y)dz$. Then for t small enough $\omega_t = \omega + td\alpha$ is presymplectic. But for many values of t the Reeb foliation of ω_t is irrational (orbits are dense), whereas for ω_0 it is rational (all orbits are closed circles).

However, it is rather easy to strengthen slightly the last theorem as follows. If two foliations are conjugated by a diffeomorphism, then we can identify basic cohomology groups and using this identification we can compare basic cohomology classes.

Theorem 4.2 *If ω_t is a path of presymplectic forms on M, ϕ_t is an isotopy such that $\mathfrak{F}_t$ is conjugated to $\mathfrak{F}_0$ by ϕ_t and the basic cohomology class of ω_t is constant (when we identify $H_b^2(M,\mathfrak{F}_t)$ with $H_b^2(M,\mathfrak{F}_0)$ using ϕ_t), then $\{\omega_t\}$ is an isotopy.*

□

5 Presymplectic forms of contact type

A challenging problem concerning contact forms is to give necessary and sufficient condition to assure that a non-degenerate 2-form on a closed manifold can be deformed to a form of contact type. Recall that we call a 2-form ω a form of contact type if $\omega = d\lambda$, where λ contact. In particular, any results solve the question of existence of contact structures. The problem was addressed by Eliashberg in dimension 3, and for simply connected 5-manifolds by Geiges [G2]. They both proved that there are no obstructions.

Such problems seem to be very difficult in general. We give here a necessary condition for presymplectic form to be of contact type.

Theorem 5.1 *Assume that λ is a contact form on a closed manifold M and $\mathfrak{R}$ its Reeb foliation. Then $[d\lambda]$ is non-zero in $H^2_b(M, \mathfrak{R})$.*

Proof. Let R denote the Reeb vector field of λ, so that $\lambda(R) \equiv 1$. If $[d\lambda]_b = 0$, then there exists α such that $\alpha(R) = 0$ and $d\alpha = d\lambda$. Thus $\phi_0 = \alpha - \lambda$ is closed and equal to 1 on R. It yields existence of a closed form ϕ which is C^1-close to ϕ_0 and such that $\ker \phi$ is integrable with compact leaves, cf. [Ti]. Since $d\lambda$ restricts to a symplectic form on any leaf, its cohomology class would be non-zero, which is of course false. $\square$

One can notice that for open manifolds such existence questions as above are solved positively by Gromov's h-principle (see [EM]). In particular, one can always deform a presymplectic form to a form which is contact outside any given non-empty open set.

6 Some examples

Let ω be a presymplectic form, $\omega = d\alpha$, and R be a positively oriented Reeb vector field for ω. Define $f_\alpha = \alpha(R)$. Any other choice of R corresponds to multiplication of f_α by a positive function. Then we have the following.

1. α is contact if and only if $f_\alpha > 0$ for any choice of R (and then we can choose R such that f_α is constant and positive);

2. α is $R-$invariant if and only if f_α is constant.

Example 6.1 *T^2–bundle over the circle.*

Consider the following example constructed by Carrière in [YC]. The manifold T^3_A he considers is the T^2-bundle over the circle whose monodromy is given by matrix $A \in SL(2, \mathbb{Z})$ such that $trA > 2$,

$$T^3_A = T^2 \times \mathbb{R}/(x, t) \sim (Ax, t + 1).$$

Then both eigenvalues $\lambda, \frac{1}{\lambda},\ \lambda > 1 > \frac{1}{\lambda}$ of A are real and irrational. The eigenvectors define two vector fields μ_1, μ_2 on T^3_A. Using them we can define two 1–forms v_1, v_2 on T^2 by the formulas $v_1(\mu_1) = 1,\ v_1(\mu_2) = 0$ and $v_2(\mu_1) = 0,\ v_2(\mu_2) = 1$. This definition extends to $T^2 \times \mathbb{R}$ by setting

$$\alpha_1 = \lambda^t v_1$$

$$\alpha_2 = \frac{1}{\lambda^t} v_2$$

and then gives two well defined forms α_1, α_2 on T^3_A. By direct calculation, $d\alpha_1 = \ln(\lambda) dt \wedge \alpha_1$ and $d\alpha_2 = -\ln(\lambda) dt \wedge \alpha_2$. Thus $d\alpha_1$ is a presymplectic form on T^3_A with associated Reeb field $R = \mu_2$. In this case, $f_\alpha = 0$. Furthermore,

$$\phi_\epsilon = \alpha_1 + \epsilon\alpha_2$$

is a contact form for ϵ small enough since

$$\phi_\epsilon \wedge d\phi_\epsilon = 2\epsilon \ln(\lambda)\alpha_1\alpha_2 dt > 0.$$

This contact form is C^1-close to the presymplectic form α_1.

Carrière proved that the second basic cohomology group of the Reeb bundle of $d\alpha_1$ vanishes, so by Theorem 5.1 there is no contact form with the Reeb field equal to μ_2. This follows also from Taubes theorem about existence of closed orbits of a contact Reeb field (the Weinstein conjecture).

Example 6.2 $S^4 \times S^1$

By Eliashberg's beautiful result [E], $S^1 \times S^4$ has a contact form. Namely, if a compact almost complex manifold with boundary of even dimension $2n,\ n > 2$, has a Morse function (constant and maximal on the boundary) without critical points of index greater than n, then the boundary is contact. The fact that M is presymplectic is much simpler, since this is enough to notice that M is parallelizable.

We shall build directly a presymplectic form $\omega = d\alpha$ on $M = S^4 \times S^1$ such that the function f_α has both positive and negative values, hence ω cannot be C^1-approximated by a contact form. Let $x_1, ..., x_6$ be standard coordinates in $\mathbb{R}^6$. Consider S^4 as the hypersurface $(x_2 - 2)^2 + ... + x_6^2 = 1$ in $\mathbb{R}^5 = \{x_1 = 0\}$ and rotate it around $\{x_1 = x_2 = 0\}$.

If we set $\alpha = \frac{1}{2}(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 + x_5 dx_6 - x_6 dx_5)$, then $\omega = d\alpha = dx_1 dx_2 + dx_3 dx_4 + dx_5 dx_6$ is the standard symplectic form on $\mathbb{R}^6$. Since ω restricted to any 5-dimensional subspace of $\mathbb{R}^6$ is presymplectic, hence $d\alpha = \omega$ restricts to a presymplectic form on M. If X is orthogonal to M, then $J_{st}X$, where J_{st} is the standard almost complex structure on $\mathbb{R}^6$, is a Reeb field R of $\omega|M$. Having established that, it is easy to calculate that f_α takes both positive and negative values.

7 Thurston's construction

Let $p : M \to B$ be a fibration with base B and a symplectic fibre (F, ω_0). We say that the fibration is *symplectic* if its structure group is $Symp(F, \omega_0)$, and *compact symplectic* if it is symplectic and both B and F are compact. For a discussion of symplectic fibrations see [MS], Ch.6. On each fibre $F_b = p^{-1}(b)$ of a symplectic fibration there is a well defined symplectic form ω_b given by ω_0 and local trivializations. Denote by $i_b : F_b \subset M$ the inclusion of the fibre. The following theorem is given by Thurston's construction, originally applied to symplectic fibrations over a symplectic base to get symplectic structures on total spaces [Th].

Theorem 7.1 *Let $p : M \to B$ be a compact symplectic fibration with fibre (F, ω_0) and a presymplectic base (B, ω_B). For any class $u \in H^2(M, \mathbb{R})$ such that $i_b^* u = [\omega_b]$ there exists $K > 0$ and a presymplectic form ω on M such that $i_b^* \omega = \omega_b$ and $[\omega] = u + Kp^*[\omega_B]$.*

Proof. Let $\{U_\alpha, f_\alpha : p^{-1}U_\alpha \to U_\alpha \times F\}$ be charts of an atlas of local trivializations and χ_α a smooth partition of unity subordinated to $\{U_\alpha\}$. On each set $p^{-1}U_\alpha$ we have a form ω_α which is pull-back of ω by projection to the fibre F. If τ represents u, then $\omega_\alpha - \tau = d\phi_\alpha$ for some 1-forms ϕ_α. The formula $\tau_1 = \tau + d\sum(\chi_\alpha \circ p)\phi_\alpha$ defines a closed form on M which restricts to ω_b on a fibre F_b and represents u. Then for K large enough, $\tau_1 + Kp^*\omega_B$ is a presymplectic form on M representing $u + Kp^*\omega_B$. Compare [Th, MS]. $\square$

We will prove a relative version of Thurston's construction. Assume additionally that $j_0 : N_0 \subset F$ is a smooth compact codimension zero submanifold of F such that N_0 is preserved pointwise by the structure group of the given fibration. In other words, the structure group is the group $Symp(F, N_0, \omega_0)$ of symplectomorphisms which are equal to the identity on N_0. Then N_0 defines submanifold N_b in each F_b and a submanifold $N \subset M$ such that $N \cap F_b = N_b,\ N = N_0 \times B$. Via pull-back of ω_0 under projection on F one has on N a form ω_N equal to ω_0 on fibers and zero in B-direction.

Theorem 7.2 *Let $N_0 \subset F$ be a compact codimension zero submanifold and let $p : M \to B$ be a compact symplectic fibration with fibre (F, ω_0) and structure group $Symp(F, N_0, \omega_0)$.*

For any presymplectic form ω_B on B and any class $u \in H^2(M, \mathbb{R})$ satisfying

1. $i_b^* u = [\omega_b]$,
2. $u|N = [\omega_N]$

there exist a constant $K > 0$ and a presymplectic form τ on M such that $[\tau] = u + Kp^*[\omega_B]$ and $\tau = \omega_N + Kp^*\omega_B$ on N.

Proof. By assumption 2, there exist a form τ representing u such that on N we have $u = \omega_N$. Let ϕ_α, τ_1 be forms defined in the proof of Theorem 7.1. Then $d\phi_\alpha = 0$ on N. Thus the form $\eta = d\sum_\alpha(\chi_\alpha \circ p)\phi_\alpha$ on N is horizontal, since both η and $d\eta(=0)$ vanish on vertical vectors. Thus it is equal to $p^*\eta'$ for a closed form η' on B. Then $\tau_1 - p^*\eta + Kp^*\omega_B$ has the required properties. $\square$

8 Submanifolds

Let (M^{2n+1}, ω) be a presymplectic manifold and let Q^{2k+1} be a smooth closed submanifold. One is tempted to define a presymplectic submanifold as those Q that the restriction of ω is presymplectic. However, the preceding discussion shows that the Reeb foliation plays an important role, thus it is better to impose a condition to have it under control.

Definition 8.1 *Let (M, ω) be a presymplectic manifold. A submanifold $i : Q \hookrightarrow M$ is a presymplectic submanifold if $\omega_Q = i^*\omega$ is presymplectic and the Reeb bundle of ω_Q is equal to the restriction of the Reeb bundle of ω to Q (equivalently, the Reeb bundle of ω is tangent to Q).*

Example 8.2 *If a presymplectic form is invariant under an action of S^1, then any component of the fixed point set is a presymplectic submanifold.*

For a symplectic submanifold Q of a symplectic manifold M and in many other cases, the structure of a tubular neighborhood is determined up to isotopy by Q and the symplectic normal bundle. This is not the case for presymplectic forms, since the Reeb field might not be invariant under contractions of the neighborhood to Q. For contact submanifolds it is possible to obtain such invariance when we allow the contact form to be multiplied by a function. Note that this operation changes, in general, the Reeb bundle.

Let (M, ω) be a presymplectic manifold and $\mathcal{R}$ be the Reeb bundle. Then the associated reduction of TM to a complex bundle is given by a choice of a bundle endomorphism $J : TM \to TM$ such that:

1. $ker\, J = \mathcal{R}$;
2. $J^2 = -Id$ on $Im\, J$;
3. $\omega(JU, JV) = \omega(U, V)$ for all $U \in Im\, J$ and arbitrary V;
4. $\omega(U, JU) > 0$ for all non-zero $U \in Im\, J$.

Denote $\mathcal{S} = Im\, J$. We have $TM = S \oplus \mathcal{R}$ and the formula $g(U, U') = \omega(U, JU')$ defines a Riemannian metric on $\mathcal{S}$. When a Reeb vector field R is chosen, then we get a Riemannian metric on M defined on $\mathcal{R}$ by $g(R,R)=1$. Note that 3 implies that $\mathcal{S}\perp\mathcal{R}$. The choice of J consists in a choice of a linear complement $\mathcal{S}$ of $\mathcal{R}$ and a choice of complex structure on it compatible with ω (note that ω is a linear symplectic form on $\mathcal{S}$). This gives J with required properties if we extend by zero on $\mathcal{R}$. For the resulting Riemannian metric we have $g(JU,JV)=g(U,V)$ for all $U\in\mathcal{S}$ and every V.

If Q is a presymplectic submanifold of (M,ω), then we can assume that J preserves TQ. Consider a linear complement $\mathcal{N}$ of TQ in TM such that $\mathcal{N}\subset\{U\in TM|Q : \forall V\in TQ\ \omega(U,V)=0\}$. Since ω is non-degenerate on $\mathcal{N}$, we can choose J in $\mathcal{N}$ compatible with ω. Then, with respect to the Riemannian metric g (on the whole M) constructed as above, $\mathcal{N}$ is an orthogonal complement to TQ and g defines a horizontal distribution $\mathcal{H}\subset T(\mathcal{N})$. We get now a presymplectic form on the total space of $\mathcal{N}$, linear in every fibre, by the formula

$$\omega^{\mathcal{N}}_W(V_1+H_1,V_2+H_2)=\omega_{p(W)}(p_*H_1,p_*H_2)+\omega_{p(W)}(V_1,V_2),$$

where $W\in\mathcal{N},\ V_1,V_2$ are vertical, H_1,H_2 horizontal vectors in $T_W(\mathcal{N})$ and $p:\mathcal{N}\to Q$ is the bundle projection of TM restricted to $\mathcal{N}$.

In this way we get complex structures on the normal bundle $\nu Q=(TM|Q)/TQ$ and compatible with the presymplectic forms on the total space of νQ via isomorphisms $\mathcal{N}\subset TM|Q\to\nu Q$. Any two such complex structures obtained in this way are isomorphic.

Consider now the exponential map $exp:\nu Q\to M$ given by the metric g. It is a diffeomorphism near Q, thus $\omega^{\mathcal{N}}$ defines a non-degenerate form linearized in the normal direction which we also denote $\omega^{\mathcal{N}}$, in a neighborhood of Q. The form restricted to $TM|Q$ is equal to ω, thus these forms in a small neighborhood of Q are close enough to have a linear segment contained in the space of non-degenerate forms which connects them. Hence, for some open tubular neighborhoods $Q\subset U\subset U_1$ such that $\overline{U}_1-U\cong\partial\overline{U}\times[0,1]$, there is a smooth non-degenerate form ω' such that:

1. $\omega'=\omega$ on $M-U_1$;
2. $\omega'=\omega^{\mathcal{N}}$ on U;
3. $\omega'|\overline{U}_1-U$ is a smooth linear combination $\lambda(t)\omega|\partial\overline{U}_1+(1-\lambda(t))\omega^{\mathcal{N}}|\partial\overline{U}$, where $t\in[0,1]$ and λ is an appropriate smooth function changing from 0 to 1.

With our choices, $t\omega+(1-t)\omega'$ is a path of non-degenerate forms. It connects ω with ω' and is equal to ω outside U_1.

Let us assume that we have two presymplectic forms ω_0,ω_1 which coincide on Q and define isomorphic structures on the normal bundle νQ.

Then for any choice of normal subbundles $\mathcal{N}_0, \mathcal{N}_1$ and horizontal distributions $\mathcal{H}_0, \mathcal{H}_1$ there is a path $(\mathcal{N}_t, \mathcal{H}_t, J_t)$; $t \in [0,1]$ connecting $(\mathcal{N}_0, \mathcal{H}_0, J_0)$ with $(\mathcal{N}_1, \mathcal{H}_1, J_0)$. Thus the forms ω'_0, ω'_1 are homotopic through a path ω'_t of non-degenerate forms.

This in turn gives a path of non-degenerate forms connecting ω_0 with a presymplectic form equal to ω_0 outside U_1 and to ω_1 in a neighborhood of Q. It is defined as follows. Take tubular neighborhoods $Q \subset U' \subset U_2 \subset U$ such that $U_2 - U'$ is a product by an interval. Construct, using ω'_t, a path of non-degenerate forms equal to ω_0 outside U_1, with the end form η'_1 equal to ω'_1 in U'. Next deform η'_1 to a form η_1 equal to ω_1 in U', with the help of the path connecting ω'_1 with ω_1. Finally, there is a presymplectic homotopy connecting ω_0 with a presymplectic form equal to ω_0 on $M - U'_1$ and to ω_1 on U'', where U'_1, U'' are tubular neighborhoods of Q such that $U'' \subset U'$, $U_1 \subset U'_1$. This can be done in two steps. First deform η_1 to a presymplectic form with the required properties by Theorem 1.3, then from obtained path of non-degenerate forms pass to a presymplectic homotopy using Theorem 3.4, relative to $K = M - U'_1$.

This proves the following tubular neighborhood theorem.

Theorem 8.3 *Assume that ω_0, ω_1 are presymplectic forms such that Q is a presymplectic submanifold with respect to both of them. If the presymplectic forms coincide on Q and the complex normal vector bundles to Q induced by ω_1 and ω_2 are isomorphic, then there exist tubular neighborhoods U_0, U_1 of Q, $U_0 \subset U_1$, and a presymplectic homotopy connecting ω_0 to a presymplectic form equal to ω_0 on $M - U_1$ and to ω_1 on U_0.*

□

9 Starlike structures

An open book decomposition of M is defined by a quadruple $\{B, P, \pi, \phi\}$ where B is a codimension 2 submanifold of M, $\pi : M - B \to S^1$ is a smooth fibration with fibre $Int\, P$ and ϕ is the gluing diffeomorphism of the fibration. It is assumed that $\partial P = B$ and ϕ is equal to the identity on $U - B$, where U is a neighborhood of ∂P, so that the normal bundle of B is trivial. If U is a collar, then a neighborhood of B is diffeomorphic to $B \times D^2$. P is called *page* of the decomposition and B it's *binding.*

Consider an open book decomposition of M satisfying the following additional assumptions.

1. P is endowed with a symplectic form ω_0,

2. ϕ preserves ω_0.

Then the fibration $M - B \to S^1$ is symplectic and one can apply Thurston type construction described in Theorem 7.2 to prove the following.

Theorem 9.1 *Consider a closed smooth manifold M^{2n+1} and an open book decomposition of M with symplectic page (P, ω_0) and the gluing diffeomorphism $\phi \in Symp(P, U, \omega_0)$. For any cohomology class $u \in H^2(M, \mathbb{R})$ such that $i_t^* u = [\omega_t]$ there exists a presymplectic form ω on M such that:*

1. $[\omega] = u$;
2. *outside of a neighborhood of the binding B the Reeb vector field is transversal to pages;*
3. *the binding B is a presymplectic submanifold of M.*

We shall prove this for a more general structure of *starlike decomposition* of M. By that we mean a quadruple $\{C, S, \pi, \phi\}$, where C is a codimension $2k$ submanifold of M, $\pi : M - C \to S^{2k-1}$ is a smooth fibration with fibre $Int\, S$ and the gluing map $\phi : S^{2k-2} \to Diff(S, U)$ of the fibration takes values in the group of diffeomorphisms equal to the identity on an open neighborhood U of ∂S. It is assumed also that $\partial S = C$ and thus the normal bundle of C is trivial. We will call S *spine* of the decomposition and C it's *core*.

Assume now that S is endowed with a symplectic form ω_0 and the gluing map has values in the symplectomorphism group $Symp(S, U, \omega_0)$, so that a neighborhood of C is diffeomorphic to $C \times Int\, D^{2k}$. In the sequel we assume that the neighborhood U in the definition is chosen in that way. As above, $S_t = \pi^{-1}(t)$, $i_t : S_t \subset M$ is the inclusion, ω_t is the symplectic form induced by ω_0 on $S_t, t \in S^{2k-1}$.

Theorem 9.2 *Consider a closed smooth manifold M^{2n+1} and a starlike decomposition of M with a symplectic spine (S, ω_0) and a gluing map $\phi : M - C \to Symp(S, U, \omega)$. For any cohomology class $u \in H^2(M, \mathbb{R})$ such that $i_t^* u = [\omega_t]$ and an open neighborhood V of C, there exists a presymplectic form ω on M such that:*

1. $[\omega] = u$;
2. *outside of a tubular neighborhood $W \subset V$ of the core C the Reeb vector field is transversal to spines;*
3. *the core C is a presymplectic submanifold of M.*

Proof. Let us start with the following observation. Consider a manifold X with boundary. If η is a symplectic or non-degenerate form on X, then one can deform η near the boundary to get a non-degenerate (but perhaps not closed) form which is of the form $p^*\eta_0 + dt \wedge p^*\phi$ on a collar $W = \partial X \times [0,1)$ of the boundary. Here $p : W \to \partial X$ is the projection and η, ϕ are some forms on ∂X. We will use this to construct a non-degenerate form on M and to define a presymplectic form by Theorem 1.3.

Let a tubular neighborhood $W_1 \subset U$ of the core diffeomorphic to $C \times Int\, D^{2k}$ be given. Applying the remark above to the standard symplectic form on $\mathbb{D}^{2k}$, we get on D^{2k} a non-degenerate 2-form α equal to the standard symplectic form in a neighborhood of 0 and product near the boundary, i.e. equal to $dt \wedge \lambda + d\lambda$ on $(1-\epsilon, 1+\epsilon) \times S^{2k-1}$, where λ is a standard contact structure on S^{2k-1}. Here $\{1+\epsilon\} \times S^{2k-1}$ is the boundary, $t \in (1-\epsilon, 1+\epsilon]$. Thus on $C \times D^{2k}$ we have a presymplectic form $\Omega_1 = \omega_0|C \times \alpha$ which has required properties near C.

Let $W_2 \subset W_1$ be a subcollar corresponding to the interval $[0,1) \subset [0, 1+\epsilon)$. The manifold $M' = M - W_2$ is diffeomorphic to the fibre bundle over S^{2k-1} with fibre $S' = S - (S \cap W_2)$ provided by the starlike decomposition. Let Ω_3 be a presymplectic form constructed in Theorem 7.2 on $M - W_2$ such that $i_t^*\Omega_3 = \omega_t$ and equal to the product $\omega_0 \times d\lambda$ in $W_1 - W_2$. As above, by deforming ω_0 near $C = \partial S$ we can assume that ω_0 is of the form $\omega_0|C + dt \wedge \eta$, for a form η on C. Since the resulting form is non-degenerate, η is non-zero on the Reeb subbundle in TC. So we have now a non-degenerate form which is presymplectic outside a neighborhood of the boundary and product near the boundary.

Now let $A \subset W_1$ be a subcollar corresponding to the interval $[0, 1-\epsilon') \subset [0, 1+\epsilon)$, where $\epsilon' < \epsilon$, and B a subcollar corresponding to $[0, 1+\epsilon')$. Consider Ω_1 on A and Ω_3 on $M - B$. We connect these two forms by a form on $B - \overline{A} = C \times (a,b) \times S^{2k-1}$ as follows. Define $\Omega_2 = \omega_0|C \times dt \wedge (h(t)\lambda + (1-h(t))\eta) \times d\lambda$, for appropriately chosen smooth function $h : (a,b) \to \mathbb{R}$ vanishing near a and equal to 1 near b. Notice that for Reeb fields $R_1 = ker\, \omega_0|C$, $R_3 = ker\, d\lambda$ we have $\lambda(R_1) = \eta(R_3) = 0$, so that $ker \Omega_2$ is of dimension 1 spanned by $(h-1)R_3 + hR_1$. This implies that Ω_2 is non-degenerate and we get a smooth non-degenerate form on M. Deforming it according to (the relative version of) Theorem 1.3 we get required presymplectic structure. $\square$

10 Presymplectic surgery

The classical surgery on smooth manifolds is performed by cutting a product neighborhood $S^k \times D^{n-k}$ of an embedded sphere S^k and gluing $D^{k+1} \times S^{n-k-1}$ along the boundary. This operation can be described as deleting $S^k \times D^{n-k}$ from M^n and $S^n = S^k \times D^{n-k} \cup D^{k+1} \times S^{n-k-1}$ and gluing the manifolds obtained in this way along the obvious diffeomorphism of boundaries. In the present context, prominent examples of constructions of that type are given by Eliashberg for contact, and by Gompf for symplectic manifolds. Gompf assumes that two symplectic submanifolds V, V' of codimension two embedded in respectively M, M', are symplectomorphic and that the normal bundles ν, ν' of these submanifolds satisfy $c_1(\nu) + c_1(\nu') = 0$. Then one can perform surgery along tubular neighborhoods of these submanifolds resulting in a new symplectic manifold. We want to settle a presymplectic analog of Gompf's construction.

Consider two presymplectic submanifolds $V \subset (M, \omega)$, $V' \subset (M', \omega')$. We assume that there exists a presymplectic diffeomorphism $g : V \to V'$ and an orientation reversing linear mapping $G : \nu V \to \nu V'$ of normal bundles covering g. Using some auxiliary Riemannian metrics on M, M' we identify total spaces of the normal unit disk bundles with neighborhoods N, N' of V and V'. We can assume that G is an isometry of normal bundles. Let N_δ, N'_δ correspond to $\delta-$disk bundles. Fix $0 < \epsilon < \frac{1}{2}$. G induces a diffeomorphism of $N_{1-\varepsilon}$ to $N'_{1-\varepsilon}$ which we also denote by G. Glue $M - N_\varepsilon$ with $M' - N_\varepsilon$ along boundaries using G. We claim that the resulting manifold, denoted by $M \cup_g M'$ admits a presymplectic structure.

Theorem 10.1 *Under the above assumptions, there exists a presymplectic form on $M \cup_g M'$, equal to the given presymplectic forms in $M - N \cup M' - N' \subset M \cup_g M'$.*

Proof. As in Section 3, take a 1-form η on M such that $\eta(R) = 1$, where R is a Reeb field of ω, and define a symplectic form $\phi = p^*\omega + \delta(d\theta \wedge \eta - \theta\, d\eta)$ on $M \times \mathbb{R}$. When we restrict ϕ to $M \times \{1\}$, we get a presymplectic form $\omega + d\eta$ homotopic to ω, if δ is small enough. By Theorem 8.3, there is a deformation of ω', supported in $N'_{1-\varepsilon}$, to a presymplectic form ω'' equal to $(G^{-1})^*(\omega + d\eta)$ in N'_ε.

Consider the disjoint sum

$$Z = M \sqcup (U \times \mathbb{R}) \sqcup M',$$

and identify

$$M' \supset N'_{1-\varepsilon} \ni x \to (G^{-1}(x), 1) \in U \times \{1\} \subset U \times \mathbb{R}.$$

In $N_{1-\varepsilon} \times \mathbb{R}$ consider a hypersurface $(M - N_\varepsilon) \cup (\partial N_{1-\varepsilon} \times [0, 1]) \cup (M' - N'_\varepsilon)$, where on the level $\theta = 1$ we use the above identification.

This is a standard exercise that one can smooth out the corners of the hypersurface we defined, so we get a smooth hypersurface in a symplectic manifold, hence a presymplectic form on it. Now a scrutiny of forms and identifications shows that we get a presymplectic form satisfying the requirements of the theorem. $\square$

Classical surgery can be used to simplify manifolds, for instance to construct a simply connected manifold cobordant to any given oriented one. As an application of Theorem 10.1 we will show that one can get a 1-connected manifold out of a given presymplectic manifold by presymplectic surgeries on a number of circles.

Proposition 10.2 *If each generator of the fundamental group of a presymplectic manifold M can be represented by a closed orbit of the Reeb foliation, then presymplectic surgeries on these circles transform M into a simply connected presymplectic manifold.*

Proof. Take $S^{2n+1} = S^1 \times D^{2n} \cup_{\mathrm{id}_\partial} D^2 \times S^{2n-1}$. It admits a presymplectic form such that $S^1 \times \{0\}$ is a closed orbit of the Reeb foliation. Since presymplectic manifolds are orientable by definition, thus normal bundles of any embedded circle is trivial.Therefore for each closed orbit of the Reeb field one can perform the surgery of M with $(M', V') = (S^{2n+1}, S^1 \times \{0\})$. Notice that we perform in fact the classical surgery on a 1-sphere so that the homotopy class of the circle is killed. The proposition follows. $\square$

In order to be able to apply the last proposition we need a simple lemma. Denote the space of linear forms of maximal rank on $\mathbb{R}^k$ by $\Omega(\mathbb{R}^k)$. This means that $\Omega(\mathbb{R}^{2n})$ is the space of symplectic linear forms and $\Omega(\mathbb{R}^{2n+1})$ is the space of presymplectic linear forms. Moreover, let $\Omega^+(\mathbb{R}^{2n})$ denote the component of symplectic forms compatible with the orientation of $\mathbb{R}^{2n}$. Then the following holds.

Lemma 10.3 *The space $\Omega(\mathbb{R}^{2n+1})$ is simply connected.*

Proof. There exists a fibration $\Omega^+(\mathbb{R}^{2n}) \xrightarrow{i} \Omega(\mathbb{R}^{2n+1}) \xrightarrow{\pi} S^{2n}$, defined by $\pi(\omega) = R$, where R is the unit Reeb vector of ω compatible with the orientation of $\mathbb{R}^{2n+1}$ and the orientation provided by ω on the orthogonal complement of the Reeb line. Since $\Omega^+(R^{2n})$ is simply connected ([MS], Ch. 2), hence so is $\Omega(\mathbb{R}^{2n+1})$. $\square$

If $S^1 \hookrightarrow M$ is any embedded circle and $S^1 \times D^{2n}$ is a tubular neighborhood of this circle, then by Lemma 10.3 we can assume that on a smaller tubular neighborhood $S^1 \times D^{2n}_1$ our form ω is the pullback $\pi^*\omega_{st}$ (where ω_{st} denotes the standard symplectic form on $D^{2n}_1 \subset \mathbb{R}^{2n}$). Finally, Theorem 1.3 shows that one can always find a presymplectic form which enables to apply presymplectic surgery to kill the fundamental group.

Proposition 10.4 *If M is a presymplectic manifold, then for every cohomology class $a \in H^2(M, \mathbb{R})$, any given homotopy class of non-degenerate forms and arbitrary elements $g_1, \ldots, g_n \in \pi_1(M)$, there exists a presymplectic form ω in the given homotopy class such that $[\omega] = a$ and there exists embedded circles representing $g_1, \ldots, g_n$ which are closed orbits of the Reeb foliation.* $\square$

Remark 10.5 By Theorem 3.4, we have also the following result. If we are given a presymplectic form on M (under assumptions of Proposition 10.4) in a prescribed homotopy class, we can deform it, through presymplectic forms, to a form satisfying all the requirements of Proposition 10.4.

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