

DIFFEOMORPHISMS AND ALMOST COMPLEX STRUCTURES ON TORI

BOGUSŁAW HAJDUK¹ AND ALEKSY TRALLE¹

ABSTRACT. We prove that there exist diffeomorphisms of tori, supported in a disc, which are not isotopic to symplectomorphisms with respect to the standard symplectic structure. This yields a partial negative answer to a question of Benson and Gordon about the existence of symplectic structures on tori with exotic differential structure.

1. INTRODUCTION

The initial motivation for this work was the following remark in the paper of Benson and Gordon [2]: *it is not known, if there exist Kähler structures on exotic tori.* In fact, it is *even not known if there are symplectic structures on a torus with an exotic differential structure.* Motivated by this, we study the possibility of building symplectic structures on exotic tori using the classical construction of Thurston. This construction gives a symplectic form on a compact manifold M fibred over a symplectic manifold with symplectic fiber provided the fibration is symplectic and the following condition is satisfied: there exists a class in $H^2(M, \mathbb{R})$ which restricts to the cohomology class of the symplectic form of the fibre (see [10], 6.3).

Some exotic tori have the structure of a fiber bundle with the base and the fiber being even dimensional tori with standard differential structures. This is obtained as follows. In the sequel we denote by \mathbb{T}^k the standard k -torus and we write \mathcal{T}^k if we consider a torus with an exotic structure. Let $f : \mathbb{T}^{2n} \rightarrow \mathbb{T}^{2n}$ be a diffeomorphism supported in a disc, i.e., equal to the identity outside an embedded disc D^{2n} . This diffeomorphism corresponds to a diffeomorphism \hat{f} of the $2n$ -sphere S^{2n} and hence gives a $(2n+1)$ -dimensional homotopy sphere $\Sigma_f = D^{2n+1} \cup_{\hat{f}} D^{2n+1}$. Simply consider the disc where f is supported as embedded in the sphere S^{2n} , say, as the upper hemisphere, and extend f from the disc to the whole sphere by identity. It is rather straightforward to check that, once orientations are fixed, the isotopy class of \hat{f} does not depend on the choices made. It is known that if Σ_f is an exotic sphere, then the connected sum $\mathcal{T}_f^{2n+1} = \mathbb{T}^{2n+1} \# \Sigma_f$ is an exotic torus, i.e. it is topologically, but *not smoothly* homeomorphic to \mathbb{T}^{2n+1} . Moreover, it fibers over S^1 with the gluing map f , see [7], 4.3. This implies that $\mathcal{T}^{2n+2} = (\mathbb{T}^{2n+1} \# \Sigma_f) \times S^1$ is an exotic torus (cf. Sec. 4) which fibers over \mathbb{T}^2 with fiber \mathbb{T}^{2n} .

1991 *Mathematics Subject Classification.* 53C15, 53D35.

¹Both authors were partially supported by grant 2P03A 036 24 of the Polish Committee for the Scientific Research (KBN)

Note that if f were isotopic to a symplectomorphism, this fibration would be symplectic. Clearly, the cohomology ring $H^*(\mathcal{T}^{2n+2})$ is isomorphic to $H^*(\mathbb{T}^{2n}) \otimes H^*(\mathbb{T}^2)$. Hence, the cohomological condition of Thurston's construction were also satisfied and we would get a symplectic structure on the exotic torus \mathcal{T}_f^{2n+2} . These observations motivate the following questions.

Problem 1. *Is there a symplectic structure on a torus with an exotic differential structure?*

Problem 2. *Is the fibration $\mathbb{T}^{2n} \rightarrow \mathcal{T}^{2n+2} \rightarrow \mathbb{T}^2$ symplectic?*

Equivalently, one can ask

Problem 3. *Given a diffeomorphism $f : \mathbb{T}^{2n} \rightarrow \mathbb{T}^{2n}$ supported in an embedded disc but non-isotopic to the identity, is there a symplectomorphism in the isotopy class of f ?*

Our goal is to give negative examples to Problem 3 in the particular case of the standard symplectic structure on \mathbb{T}^{2n} . Note that this problem is related to the question posed by McDuff and Salamon [10], p. 328, whether every symplectomorphism of a torus which acts trivially on homology is isotopic to the identity.

Let $\pi_0(\text{Diff}_+(M))$ denote the group of isotopy classes of orientation preserving diffeomorphisms of a smooth oriented manifold M . Assume now that M is $2n$ -dimensional and admits almost complex structures, and let $\mathbb{J}M$ denote the set of homotopy classes of such structures, compatible with the given orientation. Any diffeomorphism f acts on the set of all almost complex structures by the rule

$$f_*J = dfJdf^{-1},$$

where $df : TM \rightarrow TM$ denotes the differential of f . This action clearly descends to the action of $\pi_0(\text{Diff}_+(M))$ on $\mathbb{J}M$.

Let now $\mathfrak{G}(M)$ denote the subgroup of $\pi_0(\text{Diff}(M))$ generated by diffeomorphisms with supports in discs.

We have seen above that the positive answer to Problem 3 would imply the positive answer to Problems 1 and 2. In the sequel we will show that the answer to Problem 3 is *negative* in case $2n \equiv 0 \pmod{8}$, and the standard symplectic structure ω_0 : there exist diffeomorphisms $f : \mathbb{T}^{8k} \rightarrow \mathbb{T}^{8k}$ supported in a disc, whose isotopy classes $[f] \in \mathfrak{G}(M)$ do not preserve the homotopy class $[J_0] \in \mathbb{J}M$ of the standard complex structure. Therefore, they cannot be isotopic to symplectomorphisms with respect to the standard symplectic structure ω_0 . Indeed, given a symplectic form ω , any diffeomorphism preserving ω preserves also the homotopy class of any almost complex structure compatible with ω (since the space of all such almost complex structures is contractible, hence connected).

Now, let us describe in more detail the results of the paper. We will obtain first a necessary homotopic condition on a diffeomorphism to be isotopic to a symplectomorphism.

Theorem 1. *Let $f \in \text{Diff}(\mathbb{T}^{4n})$ be supported in a disc $D^{4n} \subset \mathbb{T}^{4n}$. If f is isotopic to a symplectomorphism with respect to the standard symplectic structure, then df restricted to its support disc D^{4n} gives in $\pi_{4n}SO(4n)$ the trivial homotopy class.*

Remark. The same proof is valid for any symplectic form compatible with a parallelizable (as *complex* vector bundle structure on the tangent bundle) almost

complex structure. It is not known whether \mathbb{T}^{2n} admits a non-parallelizable symplectic structure. For example, for any almost complex structure on \mathbb{T}^4 compatible with a symplectic structure its first Chern class vanishes and no examples of symplectic forms on \mathbb{T}^4 non-homotopic to the standard form are known.

For $\mathcal{T}_f^{8k+1} = \mathbb{T}^{8k+1} \# \Sigma_{\hat{f}}$ consider the Atiyah - Milnor - Singer invariant, alias the \hat{a} -genus \hat{a} [1,7] (see Section 4). Since the \hat{a} -genus is additive with respect to the operation of connected sum and it is nontrivial for some homotopy spheres in dimension $8k+1$, one easily concludes that there exist isotopy classes of diffeomorphisms f with support in a disc such that $\hat{a}(\mathcal{T}_f^{8k+1}) \neq 0$. On the other hand, we prove the following result.

Theorem 2. *In the notation of Theorem 1, if $[df] = 0$, then $\hat{a}(\mathcal{T}_f) = 0$.*

Comparing Theorem 1 and Theorem 2 we come to the main conclusion of the paper.

Theorem 3. *For any $k > 0$ there exist diffeomorphisms $f : \mathbb{T}^{8k} \rightarrow \mathbb{T}^{8k}$ with support in a disc which are not isotopic to a symplectomorphism of $(\mathbb{T}^{8k}, \omega_0)$.*

Of course, Theorem 3 implies all observations we have mentioned, and, in particular, a partial negative answer to problem 2 in case of the standard symplectic torus \mathbb{T}^{2n} .

The paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorem 1, Section 4 describes the technique of the \hat{a} -genus, and, finally, Section 5 contains the proofs of Theorems 2 and 3.

Acknowledgement. We are grateful to Dusa McDuff for valuable advice and to Yuli Rudyak for reading the manuscript and several useful remarks. We are also indebted to the referee for pointing out a gap in our previous version of Lemma 3.

2. PREPARATORY WORK FOR THE PROOF OF THEOREM 1

To prove Theorem 1, we need some preparatory work.

2.1 Interpretation of $[df]$. Consider diffeomorphisms $f : D^m \rightarrow D^m$ which are equal to the identity near the boundary. The group of such diffeomorphisms we denote by $\text{Diff}_\epsilon(D^m, \partial D^m)$. The differential df of such a diffeomorphism yields a map $(D^m, \partial D^m) \rightarrow (GL(m, \mathbb{R}), \{id\})$ when we use a trivialization of the tangent bundle of the disc. Since such a trivialization is unique up to homotopy, we get a well defined homotopy class $[df] \in \pi_m GL(m, \mathbb{R}) \cong \pi_m SO(m)$.

Given an embedding of D_m in a manifold, we can always extend f to the whole manifold, since $f = id$ near the boundary of the disc. In particular, f extended by the identity map id to S^m gives the diffeomorphism

$$\hat{f} : S^m = D^m \cup_{\partial D^m} D^m \rightarrow S^m = D^m \cup_{\partial D^m} D^m.$$

If M is an arbitrary manifold, and $f \in \mathfrak{G}(M)$ is a diffeomorphism with support in a disc, the restriction of f to this disc yields as above a homotopy class in $\pi_m SO(m)$. For simplicity, we will denote this class by the same symbol $[df] \in \pi_m SO(m)$.

2.2 Homomorphism $\Gamma_m \rightarrow \pi_{m-1} SO(m-1)$. Let us denote $\Gamma_m = \pi_0 \text{Diff}_+ S^{m-1}$, where $\text{Diff}_+ S^k$ denotes the group of orientation preserving diffeomorphisms of the k -sphere. It is well known that

$$\Gamma_m = \pi_0 \text{Diff}_+ S^{m-1} \cong \pi_0 \text{Diff}_\epsilon(D^{m-1}, \partial D^{m-1}).$$

It follows that there is a well defined homomorphism

$$\Gamma_m \xrightarrow{\cong} \pi_0 \text{Diff}_\epsilon(D^{m-1}, \partial D^{m-1}) \xrightarrow{\mu} \pi_{m-1}SO(m-1),$$

where

$$\mu([f]) = [df]. \quad (1)$$

2.3 Stabilization maps. Consider the natural inclusions

$$SO(m-1) \hookrightarrow SO(m) \hookrightarrow SO(m+1)$$

and the induced maps of homotopy groups

$$\pi_{m-1}SO(m-1) \xrightarrow{j_m} \pi_{m-1}SO(m) \xrightarrow{J_m} \pi_{m-1}SO(m+1)$$

which we will call the *stabilization maps*. From the work of Kervaire [8] one can obtain all the homotopy groups $\pi_{m-1}SO(m-1), \pi_{m-1}SO(m), \pi_{m-1}SO(m+1)$ for all m , as well as the kernels of the stabilization maps J_m . In the sequel we will need this information. Because of that, we give the table of the homotopy groups together with $\text{Ker } J_m$ up to $m = 15$. It is well known that for $m > 7$ the homotopy groups we are interested in are 8-periodic.

Table 1

m	$\pi_{m-1}SO(m-1)$	$\text{Ker } J_m$	$\pi_{m-1}SO(m)$	$\pi_{m-1}SO(m+1)$
3	0	0	0	0
4	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}_2	\mathbb{Z}_2	0
6	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}	0
7	0	0	0	0
8	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}
9	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}_2
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}_2
11	\mathbb{Z}_4	\mathbb{Z}_2	\mathbb{Z}_2	0
12	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}
13	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}_2	\mathbb{Z}_2	0
14	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}	0
15	\mathbb{Z}_4	\mathbb{Z}_2	\mathbb{Z}_2	0

2.4 Image $j_m([df])$. We will need the following result.

Lemma 1. *The following equality holds for any m :*

$$j_m([df]) = 0.$$

Proof. The proof is a consequence of the relations between homotopy groups from the Table. The necessary calculation falls into the cases:

- (i) $m \equiv 4 \pmod{8}$,
- (ii) m is odd, $m \neq 3, 7$,
- (iii) $m \equiv 2 \pmod{8}$, or $m \equiv 6 \pmod{8}$,
- (iv) $m = 3, 7$,
- (v) $m \leq 4$.

Let us start with the following observation. Consider the homotopy sphere $\Sigma_f^m = D^m \cup_f D^m$ and the tangent bundles TS^m and $T\Sigma_f^m$. Since, up to homotopy, m -dimensional vector bundles over homotopy spheres are classified by elements of $\pi_{m-1}SO(m)$ one can check that, if $\tau \in \pi_{m-1}SO(m)$ represents TS^m , then the element $\tau + j_m([df])$ represents $T\Sigma_f^m$. Furthermore, the stable tangent bundle of Σ_f^m is trivial. For $m > 4$ it is proved in [9] and for $m \leq 4$ the group Γ_m is trivial by [12], [5]. In particular, $j_m([df]) \in \text{Ker } J_m$.

Again by Kervaire and Milnor [9], the group Γ_m is finite for $m > 5$, since it is isomorphic to the group of h-cobordism classes of homotopy spheres (cf. Cerf [6]). Now we are ready to exhibit our case by case calculation.

- Case (i) For $m \equiv 4 \pmod{8}$, the group $\pi_{m-1}SO(m-1)$ is torsion free (see Table 1), while Γ_m is always finite. Thus $[df] = \mu([f])$ must be zero.
- Case (ii) Here the table yields $\text{ker } J_m = \mathbb{Z}_2$. Since the vector bundles TS^m and $T\Sigma_f^m$ are both nontrivial, we see that both τ and $\tau + j_m([df])$ are non-zero elements of \mathbb{Z}_2 . The only possibility for that is $\tau = \tau + j_m([df])$, and again $j_m([df]) = 0$.
- Case (iii) Using Table 1 again, one can notice that $\pi_{m-1}SO(m-1)$ is finite, while $\text{Ker } J_m$ is torsion free. Hence, $j_m([df]) \in \text{Ker } J_m$ can be only zero.
- Case (iv) If $m = 3, 7$, then $\text{Ker } J_m = 0$, and there is nothing to prove.
- Case (v) For $m \leq 4$ the group Γ_m is trivial.

The proof is complete. □

Corollary 1. *For any homotopy sphere Σ of dimension n its tangent bundle is isomorphic to the tangent bundle of S^n .*

This implies for example the equality $\text{span } \Sigma = \text{span } S^n$. Here span denotes the maximal number of linearly independent vector fields. This equality was proved (for any stably parallelizable manifold) in [4], [13] by different arguments.

2.5 $\pi_0(\text{Diff}(M))$ -action on $\mathbb{J}M$. If M^{2n} is a parallelizable smooth manifold then any choice of an almost complex structure J_0 on M with a complex trivialization of (TM, J_0) determines a bijection

$$\mathbb{J}M \xrightarrow{\cong} [M, SO(2n)/U(n)]$$

such that J_0 corresponds to the class of constant map (cf. [11], Prop.2.48). The correspondence can be briefly described as follows. Any two complex structures on \mathbb{R}^{2n} compatible with a given orientation are equivalent up to a linear, orientation preserving isomorphism. For a manifold, any local complex trivialization of (TM, J_0) give locally an identification of complex structures on $T_x M$ with $GL_+(2n, \mathbb{R})/GL(n, \mathbb{C})$. Globally, any almost complex structure corresponds to a

(smooth) section of the bundle with fiber $GL_+(2n, \mathbb{R})/GL(n, \mathbb{C})$, associated to the tangent bundle. If the manifold is parallelizable, then any fixed complex parallelization provides a 1-1 correspondence between the set of such sections and the set of maps $M \rightarrow GL_+(2n, \mathbb{R})/GL(n, \mathbb{C})$. Passing to homotopy classes and replacing the target space by its homotopy equivalent $SO(2n)/U(n)$ we get $[M, SO(2n)/U(n)]$. Note that taking the parallelization we assume J_0 to give the trivial complex structure on the tangent bundle of M^{2n} , but other structures can be arbitrary. The correspondence above measures the difference between any almost complex structure and the given one. In particular J_0 is sent to the class of the constant map, and if J is trivial, then there exists a map $g : M \rightarrow GL_+(2n, \mathbb{R})$ such that $J = gJ_0g^{-1}$. For a nontrivial J such a map exists only locally. However, different local choices differ by a function to $GL(n, \mathbb{C})$, thus give a well-defined map to $GL_+(2n, \mathbb{R})/GL(n, \mathbb{C})$.

Using such correspondence we readily see the action of $Diff_+(M)$ on $\mathbb{J}(M)$ as follows. Given f and J , represent $[df]$ by a map $\delta : M \rightarrow SO(2n)$ and $[J]$ by $\phi : M \rightarrow SO(2n)/U(n)$. Then we have $[f]_*[J] = [\delta \cdot \phi] \in [M, SO(2n)/U(n)]$, where $\delta \cdot \phi$ is given by the natural action of $SO(2n)$ on $SO(2n)/U(n)$.

Consider the bundle

$$U(n) \rightarrow SO(2n) \rightarrow SO(2n)/U(n)$$

and the corresponding part of the long homotopy exact sequence

$$\dots \rightarrow \pi_{2n}U(n) \rightarrow \pi_{2n}SO(2n) \rightarrow \pi_{2n}SO(2n)/U(n) \rightarrow \dots$$

Our first goal is to prove that if $[f] \in \pi_0(\text{Diff}(\mathbb{T}^{8k}))$ is determined by a diffeomorphism with support in a disc, then it acts on $\mathbb{J}\mathbb{T}^{8k}$ with fixed points if and only if $[df] \in \text{Im}(\pi_{8k}U(4k)) \subset \pi_{8k}SO(8k)$. The proof requires two technical facts we will give now.

Lemma 2. *Let $D^{2n} \subset M^{2n}$ be an embedded disc of codimension zero and let θ denote the map $M \rightarrow S^{2n}$ obtained by shrinking the complement $M \setminus \text{Int } D^{2n}$ to a point:*

$$\theta : M \rightarrow S^{2n} = \text{Int } D^{2n} \cup \{pt\}, \theta(M \setminus \text{Int } D^{2n}) = \{pt\}, \theta|_{\text{Int } D^{2n}} = \text{id}.$$

If $M = \mathbb{T}^{2n}$, then the induced map of the sets of homotopy classes of maps

$$[S^{2n}, SO(2n)/U(n)] \xrightarrow{\hat{\theta}} [M, SO(2n)/U(n)]$$

given by $[\psi] \rightarrow [\psi \circ \theta]$, $[\psi] \in [S^{2n}, SO(2n)/U(n)]$, is injective.

Proof. Let M' denote the complement of $\text{Int } D^{2n}$ and let $X = SO(2n)/U(n)$ (in this particular argument the target space X may be arbitrary). Consider the Puppe long exact sequence of homotopy classes ([15], 6.21)

$$\dots \rightarrow [SM, X] \rightarrow [SM', X] \rightarrow [M/M', X] \rightarrow [M, X] \rightarrow [M', X],$$

where S denotes the unreduced suspension. Note that X is simply connected so we do not care about basepoints. It is clear, that $[M/M', X] \rightarrow [M, X]$ can be

identified with the map $\hat{\theta} : [S^n, X] \rightarrow [M, X]$, and, therefore, the injectivity of $\hat{\theta}$ will follow, if one proves the *surjectivity* of the term

$$[SM, X] \rightarrow [SM', X]$$

in the Puppe long exact sequence. The latter is naturally identified with the map $[SM, X] \rightarrow [S(M \setminus \{pt\}), X]$ induced by the inclusion

$$i : S(M \setminus \{pt\}) \hookrightarrow SM.$$

Now, take into consideration the homotopy equivalence

$$S\mathbb{T}^{2n} \simeq \vee S^j. \quad (2)$$

The latter can be explained as follows. Use the well-known formulae

$$S(X \vee Y) = SX \vee SY \text{ and } S(X \wedge Y) = X \wedge Y \wedge S^1 = SX \wedge Y = X \wedge SY$$

and

$$S^1 \wedge X = SX,$$

where $X \wedge Y$ denotes the smash-product of spaces X and Y . Applying these formulae to $\mathbb{T}^{2n} = S^1 \times \dots \times S^1$ one obtains (2). In the latter equivalence, the number of spheres of dimension j equals to the Betti number b_{j-1} of \mathbb{T}^{2n} . We have the following (homotopy) commutative diagram

$$\begin{array}{ccc} S(\mathbb{T}^{2n} \setminus \{pt\}) & \longrightarrow & \vee S^j \setminus S^{2n+1} \\ i \downarrow & & i_S \downarrow \\ S\mathbb{T}^{2n} & \longrightarrow & \vee S^j \end{array}$$

where i_S is an obvious map of the wedges of spheres. Indeed, if one suspends the CW-complex \mathbb{T}^{2n} with a standard cell decomposition, the suspension of every k -cell will give a $(k+1)$ -dimensional sphere, and therefore, cutting the top cell and suspending the rest will give the wedge of spheres with the top one shrunk to a point. It follows that there is a retraction $r : S\mathbb{T}^{2n} \rightarrow S(\mathbb{T}^{2n} \setminus \{pt\})$ corresponding to the right vertical arrow of the previous diagram, and given by shrinking the sphere of the maximal dimension to the point. It follows that the induced map $i_* = (i_S)_* : [S\mathbb{T}^{2n}, X] \rightarrow [S(\mathbb{T}^{2n} \setminus \{pt\}), X]$ has a right inverse r_* . This is the same as the surjectivity of i_* , as required. \square

Lemma 3. *Consider a diffeomorphism f of \mathbb{T}^{2n} with support in a disc and a parallelizable almost complex structure J_0 on \mathbb{T}^{2n} . Then f preserves the homotopy class of J_0 if and only if $[df] \in \text{Im}(\pi_{2n}U(n)) \subset \pi_{2n}SO(2n)$.*

Proof. Let $*$ denotes the base point of $SO(2n)/U(n)$ corresponding to $U(n)$ and D_0 the disc in \mathbb{T}^{2n} containing the support of f . By the constant map we will understand here a map which sends every point to $*$. As we have explained above, the correspondence $\mathbb{J}(\mathbb{T}^{2n}) \cong [\mathbb{T}^{2n}, SO(2n)/U(n)]$ can be chosen such that J_0 corresponds to the class of the constant map, denoted ϕ . Let df be represented by

$\delta : \mathbb{T}^{2n} \rightarrow SO(2n)$, constant outside D_0 , and assume $[f^* J_0] = [J_0]$. This gives a homotopy $H : \delta = \delta \cdot \phi \sim \phi$. We have to prove that there is another homotopy which is constant outside D_0 . The homotopy H gives a map $h : \mathbb{T}^{2n} - D_0 \rightarrow \Omega_* SO(2n)/U(n)$ to the space of loops at $*$. Every subtorus $\mathbb{T}^k \subset \mathbb{T}^{2n}$ is the image of a retraction $r : \mathbb{T}^{2n} \rightarrow \mathbb{T}^k$, thus any map on \mathbb{T}^k extends to \mathbb{T}^{2n} . In particular we have a map $\Phi = hr : \mathbb{T}^{2n} \rightarrow \Omega_* SO(2n)/U(n)$ equal to h on \mathbb{T}^k . Inverting Φ pointwisely and composing the given homotopy H with the resulting homotopy $\Phi^{-1} : \phi \sim \phi$ we get a new homotopy $\delta \sim \phi$ which is homotopy trivial on \mathbb{T}^k . Thus there is also a homotopy $\hat{f} \sim \phi$ which is constant on a neighborhood of T^k .

Let $\mathbb{T}^{2n} = S^1 \times \dots \times S^1$, and denote by N_i a tubular neighborhood of $\mathbb{T}_i = S^1 \times \dots \times \{x_i\} \times \dots \times S^1$, where x_i is a point in i -th S^1 factor. Then $\mathbb{T}^{2n} - \bigcup_{i=1}^{2n} N_i$ is a disc and we can assume that it is equal to D_0 . Applying inductively the above construction to all subtori \mathbb{T}_i we get eventually a homotopy $\delta \sim \phi$ which is constant in the complement of D_0 . Thus the image of $[df]$ in $\pi_{2n} SO(2n)/U(n)$ is zero, and the exactness of the homotopy sequence of the fibration $U(n) \rightarrow SO(2n) \rightarrow SO(2n)/U(n)$ yields our claim. \square

Remark. It is conceivable that Lemma 3 is valid without assumption that J_0 is parallelizable.

3. PROOF OF THEOREM 1

By Lemma 3, if $[f]$ is isotopic to a symplectomorphism, then $[df] \in \text{Im}(\pi_{4n} U(2n))$. We will show, that if $[df]$ is nonzero in $\pi_{4n} SO(4n)$, then $[df] \notin \text{Im}(\pi_{4n} U(2n))$, and this will complete the proof.

The argument goes as follows. Both $U(2n+1)$ and $SO(4n+2)$ act transitively on S^{4n+1} and yield diffeomorphisms

$$S^{4n+1} \cong SO(4n+2)/SO(4n+1) \cong U(2n+1)/U(2n).$$

Consider then the following commutative diagram of fibrations and natural inclusions

$$\begin{array}{ccccc} SO(4n+1) & \longrightarrow & SO(4n+2) & \longrightarrow & S^{4n+1} \\ j \circ i \uparrow & & \uparrow & & \text{id} \uparrow \\ U(2n) & \longrightarrow & U(2n+1) & \longrightarrow & S^{4n+1}, \end{array}$$

where $i : U(4n) \hookrightarrow SO(4n)$ and $j : SO(4n) \hookrightarrow SO(4n+1)$ denote the natural inclusions. This diagram yields the commutative diagram of the group homomorphisms

$$\begin{array}{ccccc} \pi_{4n+1} S^{4n+1} & \xrightarrow{\partial} & \pi_{4n} SO(4n+1) & \xrightarrow{J_{4n+1}} & \pi_{4n} SO(4n+2) \\ \text{id} \uparrow & & j_{4n+1} \circ i_* \uparrow & & \uparrow \\ \pi_{4n+1} S^{4n+1} & \xrightarrow{\partial'} & \pi_{4n} U(2n) & \longrightarrow & \pi_{4n} U(2n+1) \end{array}$$

where the horizontal rows represent parts of the long homotopy sequences of the corresponding fibrations and ∂, ∂' denote the connecting homomorphisms. Let α and β denote, respectively, the generators of cyclic groups (see [3]):

$$\pi_{4n+1} S^{4n+1} \cong \mathbb{Z} \cong \langle \beta \rangle, \text{ and } \pi_{4n} U(2n) \cong \mathbb{Z}_{(2n)!} \cong \langle \alpha \rangle.$$

By [8, Lemma I.1]

$$\alpha = \partial' \beta.$$

Now, the kernel $\text{Ker } J_{4n+1}$ is nontrivial and

$$\text{Ker } J_{4n+1} = \mathbb{Z}_2.$$

It follows that

$$\partial(\pi_{4n+1} S^{4n+1}) = \text{Ker } J_{4n+1} = \mathbb{Z}_2.$$

In particular, $\partial\beta \neq 0$. From the commutativity of the diagram

$$0 \neq \partial\beta = (j_{4n+1} \circ i_*) \partial' \beta = j_{4n+1}(i_* \alpha).$$

It follows that $i_* \alpha \notin \text{Ker } j_{4n+1}$. In fact, it gives more: since $\pi_{4n} SO(4n)$ is a 2-torsion group and $\pi_{4n} U(2n)$ is cyclic, it follows that $\text{Im } i_* \cap \text{Ker } j_{4n+1} = \{0\}$. But from Lemma 1 we know that $[df] \in \text{Ker } j_{4n+1}$, thus $[df] \in \text{Im}(\pi_{4n} U(2n))$ only if it is zero, as required. □

4. \hat{a} -GENUS AND FAMILY INDEX

The tool that yields an obstruction to isotoping the diffeomorphism f (supported in a disc) to a symplectomorphism is a KO -theoretical invariant, namely, the \hat{a} -genus of a closed spin manifold. In this section we summarize the properties of the \hat{a} -genus and explain the argument. A beautiful presentation of techniques used in this section one can find in the monograph of Lawson and Michelson [11].

We consider closed spin manifolds. It is known that stably parallelizable manifolds are spin, hence \mathbb{T}^m and Σ_f are. Also, connected sums of spin manifolds admit spin-structures, which implies that $\mathcal{T}_f = \mathbb{T}^m \# \Sigma_f$ is spin.

The \hat{a} -genus can be defined as the KO -theoretical index of the Dirac operator. It is known that the coefficient groups $KO^{-*}(pt)$ are the following

$$KO^{-m}(pt) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \pmod{8}; \\ \mathbb{Z}_2 & \text{if } m \equiv 1, 2 \pmod{8}; \\ 0 & \text{for any other } m. \end{cases}$$

Let $f : M \rightarrow \{pt\}$ denote the obvious collapsing map. For a spin structure on M we have the Gysin (or direct image) map

$$f_! : KO^0(M) \rightarrow KO^{-m}(pt),$$

where $m = \dim M$.

Definition. The \hat{a} -genus of a closed spin manifold M^m is an element of $KO^{-m}(pt)$ given by the formula

$$\hat{a}(M) = f_!(1).$$

This is the topological index, with values in $KO^{-*}(pt)$, of the Dirac operator on M . In fact we use only the torsion part of \hat{a} called the *Hitchin invariant*.

Now we formulate explicitly the properties of \hat{a} used in the sequel.

Proposition 1. *The \hat{a} -genus has the following properties:*

(i) *for any closed spin manifolds X and Y*

$$\hat{a}(X \# Y) = \hat{a}(X) + \hat{a}(Y), \text{ and } \hat{a}(X \times Y) = \hat{a}(X) \cdot \hat{a}(Y),$$

(ii) *\hat{a} is a spin cobordism invariant,*

(iii) *for any $m > 2$ and any spin structure on the standard torus \mathbb{T}^m we have $\hat{a}(\mathbb{T}^m) = 0$.*

Proof. See [11]. Note that (iii) follows from (i) and (ii) since any spin structure on \mathbb{T}^m is given as product of spin structures on circles. Recall that S^1 has two spin structures, one of them has $\hat{a} \neq 0$, and does not bound. However, the third power of the non-zero element of $KO^{-1}(pt)$ vanishes in the ring $KO^{-*}(pt)$. □

Corollary 2. *The manifold $\mathcal{T}_f = (\mathbb{T}^{8k+1} \#_{\Sigma_f}) \times S^1$ is homeomorphic, but not diffeomorphic to the standard torus \mathbb{T}^{8k+2} , if $\hat{a}(\Sigma_f) \neq 0$. In fact, the \hat{a} -genus of \mathcal{T}_f does depend on the choice of the spin structure.*

Proof. Apply Proposition 1 and use the spin structure on S^1 with non-vanishing \hat{a} . □

Remark. A classification of smooth structures on topological tori is described in the chapter "Fake Tori" in [14].

Let there be given a smooth fiber bundle

$$F \longrightarrow M \xrightarrow{\pi} B \tag{3}$$

with fiber and base being spin manifolds. For any continuous family of elliptic differential operators P on fibers of such bundle there is a well-defined *family index*

$$\text{Ind}_m P \in KO^{-m}(B), \quad m = \dim F.$$

(see Atiyah and Singer [1]). Consider the particular case of the parametric Dirac operator D . We assume that the spin structure on M is the one induced by spin structures on B and F . Then we have the formula [1,7].

$$\text{Ind}_m D = \pi_!(1) \in KO^{-m}(B).$$

From the functoriality of the Gysin map we have

Lemma 4. *If the family index of the Dirac operator on the fiber bundle (3) vanishes, then $\hat{a}(M) = 0$.* □

Now we are going to describe a condition ensuring the vanishing of the family index $\text{Ind}_m D$ of the parametric Dirac operator. This is certainly known to experts, but we haven't found any appropriate reference in the literature.

For a parallelizable manifold, a given trivialization of the tangent bundle induces a spin structure and a Riemannian metric on (the tangent bundle of) the manifold. We consider a fiber bundle with a continuous family of parallelizations of fibers and corresponding spin structures and Dirac operators on fibers.

Lemma 5. *For any closed spin manifold F of dimension $8k$ and any fiber bundle $F \rightarrow M \rightarrow B$ which admits a fiberwise parallelization, the family index $\text{Ind}_{8k} D$ vanishes.*

Proof. The topological index of the family is given as an element of the Real K-theory of B , $KR^{-8k}(B) \cong KO^{-8k}(B)$. Given a parallelization of F , the KR -symbol class of the Dirac operator of F is identified with an element of $KR(F \times \mathbb{R}^k \times \mathbb{R}^k)$. This element is given by a map of product bundles $F \times \mathbb{R}^k \times \mathbb{R}^k \times S_+ \rightarrow F \times \mathbb{R}^k \times \mathbb{R}^k \times S_-$ where S_+, S_- are half-spin representations of the group $Spin(8k)$ and over a point (x, u, v) the map is the multiplication by $u + iv$. In dimension $8k$ parallelizable manifolds have trivial \hat{a} -genera, thus the symbol class of the Dirac operator on F becomes zero after passing to $KR^{-8k}(pt)$. In fact, any parallelization yields a trivialization of the resulting Real bundle. If we consider a fiberwise parallelization, then we obtain the product of the above by B . In particular, $\text{Ind}_{8k} D \in KR^{-8k}(B)$ is zero. □

5. PROOFS OF THEOREMS 2 AND 3

Lemmas 4 and 5 yield the following observation we can use to complete the proofs of Theorems 2 and 3.

Proposition 2. *Assume that M is fibred over a closed spin manifold B with closed spin fiber F of dimension $8k$. If the fibration admits a fiberwise parallelization and the spin structure on M is the one induced from the parallelization and the spin structure of B , then $\hat{a}(M) = 0$.* □

Proof of Theorem 2. Consider $\mathcal{T}_f = \mathbb{T}^{8k+1} \# \Sigma_f$. We have already mentioned that \mathcal{T}_f fibers over S^1 with \mathbb{T}^{8k} as a fiber and the gluing map \hat{f} .

If $[df] = 0$, then the fibration admits fiberwise parallelization. Proposition 2 implies that $\hat{a}(\mathcal{T}_f) = 0$. □

Proof of Theorem 3. If $[f]$ preserves a homotopy class $[J]$ of an almost complex structure, then $[df] = 0$ and, by Theorems 1 and 2, $a(\mathcal{T}_f) = 0$. We have $\hat{a}(\mathcal{T}_f) = \hat{a}(\mathbb{T}^{2n+1} \# \Sigma_f) = \hat{a}(\Sigma_f)$. However, it is well known [7] that in dimensions $8n + 1$ there are homotopy spheres Σ_f with $\hat{a}(\Sigma_f) \neq 0$. This completes the proof. □

Remark. By Section 2, $\hat{a}(\mathcal{T}_f)$ is necessarily a torsion element, so \hat{a} -genus can detect nontriviality of $[df]$ only in dimensions $8n$. However, the group where the class $[df]$ can take values is equal to \mathbb{Z}_2 for any even dimension. We do not know whether in even dimensions $\neq 8n$ there are diffeomorphisms of spheres having non-trivial homotopy class of the differential (while the results of Section 2 show that $[df] = 0$ if the dimension is odd).

REFERENCES

- [1] M.F. Atiyah, I.M. Singer, *The index of elliptic operators IV, V*, Ann. of Math. **93** (1971), 119-138, 139-149.

- [2] C. Benson and C. Gordon, *Kähler and symplectic structures on nilmanifolds*, *Topology* **27** (1988), 513-518.
- [3] R. Bott, *The stable homotopy of the classical groups*, *Proc. Nat. Acad. Sci. USA* **43** (1957), 933-935.
- [4] G.E Bredon, A. Kosinski, *Vector fields on π -manifolds*, *Ann. of Math.* **84** (1966), 85 – 90.
- [5] J. Cerf, *La nullité de $\pi_0(DiffS^3)$* , *Sem. Henri Cartan 1962/63*, Exp. 10-12, 20-21, Paris 1964.
- [6] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudoisotopie*, *Publ. IHES* **39** (1970), 5 –173.
- [7] N. Hitchin, *Harmonic spinors*, *Adv. Math.* **14** (1974), 1-55.
- [8] M. Kervaire, *Some nonstable homotopy groups of Lie groups*, *Illinois J. Math.* **4** (1960), 161-169.
- [9] M. Kervaire, J. Milnor, *Groups of homotopy spheres*, *Ann. of Math.* **77** (1963), 504-537.
- [10] H.B.Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, 1989.
- [11] D. McDuff, D.Salamon, *Introduction to Symplectic Topology*, Oxford Univ. Press, 1998.
- [12] S. Smale, *Diffeomorphisms of the 2-sphere*, *Proc. of AMS* **10** (1959), 621 – 626.
- [13] E. Thomas, *Cross - sections of stably equivalent vector bundles*, *Quart. J. of Math.* **17** (1966), 53 – 57.
- [14] C.T.C. Wall, *Surgery on Compact Manifolds (Second Edition, Edited by A.A. Ranicki)*, American Mathematical Society, 1999.
- [15] G.W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math. 61, Springer, Berlin, 1978.

**Bogusław Hajduk: Mathematical Institute, Wrocław University,
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland**

hajduk@math.uni.wroc.pl

<http://www.math.uni.wroc.pl/~hajduk>

**Aleksy Tralle: Department of Mathematics and Information
Technology, University of Warmia and Mazury,**

Zolnierska 14A, 10-561 Olsztyn, Poland

tralle@matman.uwm.edu.pl

<http://matman.uwm.edu.pl/~tralle>