ON SIMPLY CONNECTED K-CONTACT NON SASAKIAN MANIFOLDS

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Abstract. We solve the problem posed by Boyer and Galicki about the existence of simply connected K-contact manifolds with no Sasakian structure. We prove that such manifolds do exist using the method of fat bundles developed in the framework of symplectic and contact geometry by Sternberg, Weinstein and Lerman.

Keywords: fat bundle, K-contact manifold, Sasakian manifold, symplectic manifold, contact manifold.

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1. Introduction

In [BG] Boyer and Galicki asked the following question (see Open Problem 7.4 on page 235 in this book).

Problem 1.1. *Do there exist simply connected closed K-contact manifolds with no Sasakian structure ?*

In this work, we answer this question positively.

Theorem 1.2. *There exist simply connected K-contact manifolds which do not carry any Sasakian structure.*

Let (M,η) be a co-oriented contact manifold with a contact form η. We say that (M,η) is ***K-contact*** if there is an endomorphism Φ of TM such that the following conditions are satisfied:

(1) $\Phi^2=-Id+\xi\otimes\eta$, where ξ is the Reeb vector field of η;
(2) the contact form η is compatible with Φ in the sense that
$$d\eta(\Phi X,\Phi Y)=d\eta(X,Y)$$
for all X,Y and $d\eta(\Phi X,X)>0$ for all nonzero $X\in\operatorname{Ker}\eta$;
(3) the Reeb field of η is a Killing vector field with respect to the Riemannian metric defined by the formula
$$g(X,Y)=d\eta(\Phi X,Y)+\eta(X)\eta(Y).$$

In other words, the endomorphism Φ defines a complex structure on $\operatorname{Ker}\eta$ compatible with $d\eta$, hence orthogonal with respect to the metric $g=d\eta\circ(\Phi\otimes Id)$. By definition, the Reeb field ξ is orthogonal to $\operatorname{Ker}\eta$.

For a contact manifold (M, η) define the ***metric cone*** or ***the symplectization*** as

$$\mathcal{C}(M) = (M \times \mathbb{R}^{>0}, t^2\eta + dt^2).$$

Given a K-contact manifold (M, η, Φ, g), the almost complex structure I on $\mathcal{C}(M)$ is defined by:

(1) $I(X) = \Phi(X)$ on $\operatorname{Ker}\eta$;
(2) $I(\xi) = t\frac{\partial}{\partial t}, I(t\frac{\partial}{\partial t}) = -\xi$.

A K-contact manifold is called ***Sasakian***, if the almost complex structure I is integrable, hence defines a dilatation-invariant complex structure on $\mathcal{C}(M)$, endowing $\mathcal{C}(M)$ with a Kähler structure.

Geometry of metric contact manifolds is important because of their applications, for instance in the theory of Einstein metrics [BG], [BGM]. K-contact manifolds have nice topological properties (see [BG], Chapter 7 and [GNT]). For example, they admit Cohen-Macauley torus actions (this is an analogue of the equivariant formality). Most of the known examples of K-contact manifolds are Sasakian, although examples of *non-simply connected* K-contact manifolds with no Sasakian structure are known [BG]. This rises the following general question: *given a contact manifold M, find conditions which ensure that there exists a Sasakian metric compatible with the contact structure.* This was posed by Ornea and Verbitsky in [OV].

Although the main result of the paper belongs to the framework of metric contact geometry, our methods come from symplectic and contact geometry and are based on the notions of symplectic and contact fatness developed by Sternberg and Weinstein in the symplectic setting [W] and by Lerman in the contact case [L1], [L2]. Let $G \to P \to B$ be a principal bundle with a connection. Let θ and Θ be the connection one-form and the curvature form of the connection, respectively. Both forms have values in the Lie algebra $\mathfrak{g}$ of the group G. Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle\, , \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is ***fat***, if the two–form

$$(X, Y) \to \langle \Theta(X, Y), u \rangle$$

is nondegenerate for all *horizontal* vectors X, Y. Note that if a connection admits at least one fat vector then it admits the whole coadjoint orbit of fat vectors. In is important to notice that in this work we consider manifolds with contact forms rather than contact structures, but all the results from contact geometry which we use are valid in this particular situation. Let (M, η) be a contact co-oriented manifold endowed with a contact action of a Lie group G. Define a ***contact moment map*** by the formula

$$\mu_\eta : M \to \mathfrak{g}^*,\ \langle \mu_\eta(x), X \rangle = \eta_x(X_x^*)$$

for any $x \in M$ and any $X \in \mathfrak{g}$. We denote by X^* the fundamental vector field on M generated by $X \in \mathfrak{g}$. Note that the moment map depends on the contact form. The result below is due to Lerman [L2] and yields a construction of fiberwise contact forms on the total space of the bundle associated to a principal bundle with fat connection.

Theorem 1.3. *Let there be given a contact G-manifold (F, η) with the contact moment map ν. Assume that*

$$G \to P \to M$$

is a principal fiber bundle endowed with a connection such that the image $\nu(F) \subset \mathfrak{g}^*$ consists of fat vectors. Then there exists a fiberwise contact structure on the total space of the associated bundle

$$F \to P \times_G F \to M.$$

If the fiber (F, η) is K-contact and G preserves the K-contact structure, then the total space of the associated bundle is also K-contact.

The second part of this theorem yields an explicit construction of a fibered K-contact structure on a fiber bundle and it is our tool to construct examples needed to prove Theorem 1.2. On the other hand, it is known that any closed Sasakian manifold M has Betti numbers $b_p(M)$ even for p odd and not exceeding $\frac{1}{2}(dim\, M + 1)$. We show, using Sullivan models of fibrations [FT], that in our examples b_3 is odd, hence they cannot be Sasakian.

Conceptually, contact manifolds are odd dimensional analogues of symplectic manifolds, while manifolds with Sasakian structures are treated as odd dimensional counterparts of Kähler manifolds. Thus, Problem 1.1 is an odd dimensional analogue of the topic "symplectic vs. Kähler", which was quite important a decade ago ([CFM], [C], [IRTU], [RT]). Interestingly enough, like in the symplectic case, the construction of non-simply connected K-contact non-Sasakian manifold was found before the simply connected one (compare [TO]).

Finally, as a byproduct of our arguments and as a side remark we answer positively a question posed by Alan Weinstein in 1980 [W] about the existence of symplectically fat fiber bundle whose total space is non-Kähler and simply connected. Note that his question was asked in relation with a problem of constructing simply connected non-Kähler symplectic manifold. As we have already noted, the latter was solved by McDuff using the symplectic blow-up construction [McD]. Nevertheless, the question itself remained open.

2. Sullivan models

We use Sullivan models of fibrations as a tool of calculating cohomology in low dimensions. In the sequel our notation follows [FT]. In this Section $\mathbb{K}$ denotes any field of zero characteristic. We consider the category of commutative graded differential algebras (or, in the terminology of [FT], cochain algebras). If (A, d) is a cochain algebra with a grading $A = \oplus_p A^p$, the degree p of $a \in A^p$ is denoted by $|a|$.

Given a graded vector space V, consider the algebra $\Lambda V = S(V^{even}) \otimes \Lambda(V^{odd})$, that is, ΛV denotes a free algebra which is a tensor product of a symmetric algebra over the vector space V^{even} of elements of even degrees, and an exterior algebra over the vector space V^{odd} of elements of odd degrees.

We will use the following notation:

- by $\Lambda V^{\leq p}$ and $\Lambda V^{>p}$ are denoted the subalgebras generated by elements of degree $\leq p$ and of degree $> p$, respectively;
- if $v \in V$ is a generator, Λv denotes the subalgebra generated by $v \in V$,
- $\Lambda^p V = \langle v_1 \cdots v_p \rangle$, $\Lambda^{\geq q} V = \oplus_{i \geq q} \Lambda^i V$, $\Lambda^+ V = \Lambda^{\geq 1} V$.

Definition. A ***Sullivan algebra*** is a commutative graded differential algebra of the form $(\Lambda V, d)$, where

- $V = \oplus_{p \geq 1} V^p$,
- V admits an increasing filtration $V(0) \subset V(1) \subset ... \subset V = \cup_{k=0}^{\infty} V(k)$ with the property $d = 0$ on $V(0)$, $d : V(k) \to \Lambda V(k-1)$, $k \geq 1$.

Definition. A Sullivan algebra $(\Lambda V, d)$ is called ***minimal***, if

$$\mathrm{Im}\ d \subset \Lambda^+ V \cdot \Lambda^+ V.$$

Definition. A ***Sullivan model*** of a commutative graded differential algebra (A, d_A) is a morphism

$$m : (\Lambda V, d) \to (A, d_A)$$

inducing an isomorphism $m^* : H^*(\Lambda V, d) \to H^*(A, d_A)$. If X is a CW-complex, there is a cochain algebra $(A_{PL}(X), d_A)$ of polynomial differential forms. For a smooth manifold X we take a smooth triangulation of X and as the model of X the Sullivan model of $A_{PL}(X)$. If it is minimal, it is called the ***Sullivan minimal model of*** X. We will need the following known fact. If $(\mathfrak{m}_X, d) = (\Lambda V, d)$ is the minimal model of finite simply connected CW complex X, then

$$V \cong Hom_{\mathbb{Z}}(\pi_*(X), \mathbb{K}). \tag{2}$$

Definition. A ***relative Sullivan algebra*** is a graded commutative differential algebra of the form $(B \otimes \Lambda V, d)$ such that

- $(B, d) = (B \otimes 1, d), H^0(B) = \mathbb{K}$,
- $1 \otimes V = V = \oplus_{p \geq 1} V^p$,
- $V = \cup_{k=0}^{\infty} V(k), V(0) \subset V(1) \subset ...$,
- $d : V(0) \to B, d(V(k)) \to B \otimes \Lambda V(k-1)$, $k \geq 1$

Lemma 2.1. *Consider a minimal Sullivan algebra $(\Lambda V, d)$ such that $V^1 = \{0\}$. Let ω be a cocycle of degree 4 such that $[\omega] \in H^4(\Lambda V, d)$ is a non-zero cohomology class. Define the relative Sullivan algebra $(\Lambda V \otimes \Lambda y, d)$ with generator y of degree 3 by $dy = \omega$. Then*

$$H^3(\Lambda V \otimes \Lambda y, d) \cong H^3(\Lambda V, d).$$

Proof. Since $V^1 = \{0\}$ and the degree of y is 3, one can write

$$dy = \sum v_i \cdot v_j + u,\ |v_i| = 2,\ u \in V^4.$$

If $z \in Z^3(\Lambda V \otimes \Lambda y)$, one can write

$$z = x + \alpha y,\ \alpha \in \mathbb{K},\ x \in V^3.$$

Since z is a cocycle, we have

$$0 = dz = dx + \alpha(\sum v_i v_j + u).$$

Since $\Lambda V \otimes \Lambda y$ is a free algebra and u is indecomposable of degree 4, by minimality, either $u = 0$, or $\alpha = 0$. Thus $z \in V^3$. This means that

$$Z^3(\Lambda V \otimes \Lambda y) = Z^3(\Lambda V),$$

and the proof follows.

□

Remark 2.2. One can consult [TO] for a simple description of a method of calculating minimal models of free cochain algebras.

Relative Sullivan algebras are models of fibrations. Let $p : X \to Y$ be a Serre fibration with the homotopy fiber F. Choose Sullivan models

$$m_Y : (\Lambda V_Y, d) \to (A_{PL}(Y), d_Y),\ \bar{m} : (\Lambda V, \bar{d}) \to A_{PL}(F).$$

There is a commutative diagram of cochain algebra morphisms

$$\begin{array}{ccccc} A_{PL}(Y) & \longrightarrow & A_{PL}(X) & \longrightarrow & A_{PL}(F) \\ {\scriptstyle m_Y}\uparrow & & {\scriptstyle m}\uparrow & & {\scriptstyle \bar{m}}\uparrow \\ (\Lambda V_Y, d) & \longrightarrow & (\Lambda V_Y \otimes \Lambda V, d) & \longrightarrow & (\Lambda V, \bar{d}) \end{array}$$

in which $m_Y, m, \bar{m}$ are all Sullivan models (see Proposition 15.5 and 15.6 in [FT]).

3. Betti numbers of Sasakian manifolds

In what follows all cochain algebras and cohomologies are assumed to be defined over $\mathbb{K} = \mathbb{R}$. We will need the following property of Sasakian manifolds.

Theorem 3.1. *If M is a closed Sasakian manifold of dimension $2n+1$, then for any odd $p \leq n+1$ the Betti numbers b_p are even.*

This was proved by several authors, cf. [BG, F, T] and Theorem 7.4.11 in [BG]. For completeness as well as better presentation of our methods we will give a short proof.

Consider a smooth principal S^1-orbibundle $\pi : E \to B$ over a compact Kähler orbifold B with smooth total space E. By this we mean that E is a smooth manifold, the circle acts smoothly on E with all isotropy subgroups finite and $\pi : E \to B$ is the natural projection onto the orbit space of the action. For the definition and relevant properties of orbibundles see [BG], Chapter 4.

Any compact Kähler orbifold B satisfies the hard Lefschetz property. By this we mean that there is a cohomology class $v \in H^2(B)$ such that for $p = 0, 1, ...n-1,\ dim\, B = 2n$, the linear maps

$$L_v^k : H^{n-p}(B) \to H^{n+p}(B) : x \mapsto v^p \cup x$$

are isomorphisms. There is also the Gysin sequence for any sphere orbibundle $p : E \to B$. Both of these properties are the consequences of the fact that the categories of Kähler orbifolds and orbibundles are rationally equivalent to the categories of Kähler manifolds and bundles. Explicit proofs, using calculations of basic cohomology of the foliation of E by fibers, can be found in [EK, NR, KT, WZ].

Lemma 3.2. *If $p \leq n+1$ is odd, then the Betti numbers $b_p(E)$ are even.*

Proof. It follows from the Lefschetz property and the duality that for a compact Kähler orbifold B the Betti numbers $b_p(B)$ are even for p odd. The Gysin sequence for a principal circle orbibundle is the long exact sequence

$$... \to H^{p-2}(B) \overset{L_v^1}{\to} H^p(B) \overset{\pi^*}{\to} H^p(E) \to H^{p-1}(B) \overset{L_v^1}{\to} H^{p+1}(B) \to ... .$$

By assumption $L_v^i = L_v^{i-1}L_v^1$ is an isomorphism, thus for $i \leq n$ the linear map $L_v^1 : H^i(B) \to H^{i+2}(B)$ is a monomorphism. The Gysin sequence gives that π^* is onto and $H^p(E) = H^p(B)/H^{p-2}(B)$ is even dimensional for any $p \leq n+1$.

□

Theorem 3.1 now follows from the fact that any Sasakian manifold is the total space of a circle orbibundle over a Kähler orbifold. This can be found in [OV], where the authors show that if M is Sasakian, than it admits also a quasi-regular Sasakian structure. They essentially adapted the argument of Rukimbira [R], who noticed this for K-contact manifolds. For a quasi regular Sasakian manifold M it had been known (and is rather obvious) that M admits a locally free circle action such that the orbit space is a Kähler orbifold. Note that in the statement of Theorem 1.11 of [OV] the word "fibration" should be understood as "orbibundle".

□

4. Contact fat circle bundles

Let (M, ω) be a symplectic manifold with integral symplectic form. Consider the principal circle bundle $\pi : P \to M$. Note that in the case of circle bundles the curvature has real values, if we identify the Lie algebra of S^1 with the reals. This implies that if a principal S^1-bundle is fat, then the only non-fat vector is the zero vector. Assume that the bundle is determined by the cohomology class $[\omega] \in H^2(M, \mathbb{Z})$. Fibrations of this kind were first considered in [BW] and are called ***the Boothby-Wang fibrations***. By [K], P carries an invariant contact form θ. Namely, there exists an invariant connection on P with the curvature 2-form equal to $\pi^*\omega$. Since the Reeb vector field of such form is the infinitesimal generator of the circle action on P, the moment map is constant and nonzero. Moreover, by [W] principal circle bundles are fat if and only if they are the Boothby-Wang fibrations.

Consider now an S^1-contact manifold and the bundle associated to π. The result below follows from Theorem 1.3.

Theorem 4.1. *Let there be given a Boothby-Wang fibration*

$$S^1 \to P \to M.$$

Assume that (F, η, S^1) is a contact manifold endowed with an S^1-action preserving η and the moment map has only non-zero values. Then the associated bundle

$$F \to P \times_{S^1} F \to M$$

admits a fiberwise contact form. If (F, η) is K-contact, the same is valid for the fiberwise contact structure on $P \times_{S^1} F$.

□

5. A construction of simply connected K-contact non Sasakian manifold

5.1. **Proof of Theorem 1.2.**

Proposition 5.2. *Let (X, ω) be a closed simply connected symplectic manifold and let $L \to X$ be the complex line bundle corresponding to the cohomology class $[\omega] \in H^2(X)$. Consider the Whitney sum $L \oplus L \to X$ and the unit sphere bundle*

$$S^3 \to M \to X.$$

The total space M has a relative Sullivan model of the form

$$(\mathfrak{m}_X \otimes \Lambda y, d), dy = z \neq 0,$$

where $\mathfrak{m}_X$ is a minimal model of X, $z \in \mathfrak{m}_X$, $|z| = 4, |y| = 3$ and z represents $[\omega]^2$.

Proof. The Whitney sum $L \oplus L \to X$ is the rank 2 complex vector bundle. From the multiplication formula for the total Chern class of the Whitney sum of bundles one obtains $c_2(L \oplus L) = c_1(L)^2 = [\omega]^2 \neq 0$. This enables us to apply the argument of Example 4 on page 202 of [FT]. Namely, any spherical fibration $p : Z \to X$ has the model

$$(A_{PL}(Z), \hat{d}) \simeq (A_{PL}(X) \otimes \Lambda y, d), dy = z \in A_{PL}(X).$$

If, moreover, the spherical fibration arises as a unit sphere bundle of a vector bundle $\xi : E \to X$ of even real rank $k + 1$, then the cohomology class $[z]$ is the Euler class of E. Since the given S^3-sphere bundle is the sphere bundle of $L \oplus L$, and the latter has the Euler class equal to the second Chern class $[\omega]^2 \neq 0$, we get the model of M of the form

$$(A_{PL}(X) \otimes \Lambda y, d),\ dy = z \neq 0, [z] = [\omega]^2.$$

By the theory described in Section 2, the above cochain algebra has the required Sullivan model (of M).

□

Proposition 5.3. *Let (X,ω) be any compact simply connected symplectic manifold such that $b_3(X)$ is odd. If*

$$S^1 \to P \to X$$

is a Boothby-Wang fibration with the Euler class equal to $[\omega]$, then the total space of the fiber bundle

$$S^3 \to P \times_{S^1} S^3 \to X$$

associated to $P \to X$ by the Hopf action of S^1 on S^3 admits a K-contact structure, but no Sasakian structure.

Proof. Boothby-Wang fibrations are fat and for the contact form given by the curvature of an invariant connection the moment map has nonzero values. Therefore, if F is the total space of such fibration, then the assumptions on (F,η) in Theorem 4.1 are fulfilled. The sphere S^3 is the Boothby - Wang fibration over S^2, thus it is K-contact and the Hopf S^1-action preserves the K-contact structure. This is the special case of Theorem 4.1, with $F = S^1$. However, it is not difficult to give direct calculations (see below), and one can also use descriptions of K-contact manifolds in [BG], Theorem 6.1.26, or in [BMS]. By Theorem 4.1, the total space $P \times_{S^1} S^3$ of the associated bundle admits a K-contact structure as well. We claim that there exists a fibration of this type which cannot be Sasakian. To prove the latter, let us make the following observations. Denote by $L \to X$ the complex line bundle given by $P \to X$. Consider the Whitney sum $L \oplus L$ of L as in Proposition 5.2. This bundle is associated to the principal bundle $S^1 \to P \to X$ if the circle acts diagonally on $\mathbb{C}\oplus\mathbb{C}$. If we pass to the sphere bundle of $L\oplus L$, then we see that this bundle is associated to $P \to X$ by the Hopf circle action on S^3 and its total space is $M = P\times_{S^1} S^3$. The Euler class of the S^3-bundle is equal to $c_2(L\oplus L) = c_1(L)^2 = [\omega]^2 \neq 0$. This implies that M has a relative Sullivan model given by Proposition 5.2. Since X is simply connected, $(\mathfrak{m}_X, d) = (\Lambda V, d)$ and, by formula (2), we have $V^1 = \{0\}$. Applying Lemma 2.1 we get $b_3(M) = b_3(X)$ odd. This contradicts Theorem 3.1 and the proof is complete.

□

Proposition 5.3 can be extended to get a whole class of examples as follows.

Theorem 5.4. *Let (X,ω) be a closed symplectic manifold and $\xi : P \to X$ be the principal circle bundle with the Euler class $[\omega]$. Consider a unitary representation of S^1 in $\mathbb{C}^{n+1}$ whose all weights are positive, and the associated sphere bundle $E = P \times_{S^1} S^{2n+1} \to X$. Then for all $n \geq 0$ the total space E is K-contact. If, moreover, X is simply connected and $b_3(X)$ is odd, then E admits no Sasakian structure.*

Proof. Consider the standard K-contact form $\eta_{st} = \iota_{\frac{\partial}{\partial r}}\omega_{st}$, on $S^{2n+1} \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Here $\omega_{st} = \sum_{i=1}^{n+1} dx_i dy_i$ is the standard symplectic form and $\frac{\partial}{\partial r}$ is the radial vector field.

The form η_{st} is $U(n+1)$-invariant and its Reeb field has the form

$$R = \sum_{i=1}^{n+1} (y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}).$$

If J is the standard complex structure of $\mathbb{C}^{n+1}$, then $J(\frac{\partial}{\partial r}) = R$ and J preserves Ker η_{st}. In particular, the Riemannian metric $d\eta_{st} \circ (J \otimes Id)$ is the standard round metric on S^{2n+1}, which is the restriction of the Euclidean metric on $\mathbb{R}^{2n+2}$. Moreover, R generates the action of the circle on S^{2n+1} obtained from the diagonal representation in $\mathbb{C}^{n+1}$ (with all weights equal to 1).

Consider now a unitary representation in $\mathbb{C}^{n+1}$ with positive weights $w_1, ..., w_{n+1}$. We claim that the associated bundle $S^{2n+1} \to E \to X$ satisfies the assumptions of Theorem 1.3. Since ξ is fat and η_{st} is $U(n+1)$-invariant, we need only to check that the image of the moment map does not contain zero. As R is orthogonal to Ker η_{st}, this is equivalent to the non-vanishing of the scalar product $< R, V >$, where V generates the S^1-action on S^{2n+1}. The representation is given by the formula

$$\lambda(z_1, .., z_{n+1}) = (\lambda^{w_1} z_1, ..., \lambda^{w_{n+1}} z_{n+1}),\ \lambda \in S^1.$$

Thus in any factor of $\mathbb{C}^n = \mathbb{C} \times ... \times \mathbb{C}$ the i-th component of the generating field is equal to the corresponding component of R multiplied by w_i. From this we get that

$$< V, R >= \sum_{i=1}^{n+1} w_i |z_i|^2 > 0$$

and by Theorem 1.3 we get that E is K-contact. Note that the S^1-action on S^{2n+1} defines a locally free action on E and therefore the latter is an orbibundle over the symplectic orbifold $\xi[S^{2n+1}/S^1]$. Moreover, for $n = 0$ we have the given Boothby - Wang fibration, hence a K-contact structure on its total space.

To prove the second part of the theorem, consider separately the cases $n = 1$ and $n > 1$. In the first case the relevant information is that the Euler class of the bundle $P \times_{S^1} (\mathbb{C} \oplus \mathbb{C})$ associated to P by the representation with weights w_1, w_2 is $w_1 w_2 [\omega]^2$. This follows by calculating the homomorphism induced by the classifying map on H^4. Then we proceed as in the proof of Proposition 5.3.

If $n > 1$, then the argument is much simpler, since M has the homotopy 3-type of X, thus it also has b_3 odd. Again we come to the contradiction with Theorem 3.1 and this completes the proof.

$\square$

Note that in any dimension greater or equal 8 there are examples of closed simply connected symplectic manifolds with odd b_3 (see, e.g. [TO],[IRTU]. In particular, the blow up $\widetilde{\mathbb{C}P}^5$ of $\mathbb{C}P^5$ along the symplectically embedded Kodaira-Thurston manifold has $b_3(\widetilde{\mathbb{C}P}^5) = 3$ [McD]. More involved are examples in dimension 8 given by Gompf [G].

6. A solution of a problem of Weinstein about fatness

Let (M, ω) be a closed symplectic manifold with a Hamiltonian action of a Lie group G and the moment map $\Psi : M \to \mathfrak{g}^*$. Consider the associated Hamiltonian bundle

$$(M, \omega) \to E := P \times_G M \to B.$$

Sternberg [S] constructed a certain closed two–form $\Omega \in \Omega^2(E)$ associated with the connection θ. It is called the ***coupling form*** and pulls back to the symplectic form on each fibre and it is degenerate in general. However, if the image of the moment map consists of fat vectors then the coupling form is nondegenerate, hence symplectic. This was observed by Weinstein in [W, Theorem 3.2] where he used this idea to give a new construction of symplectic manifolds. In the sequel, the bundles with a nondegenerate coupling form will be called ***symplectically fat.*** Let us state the result of Sternberg and Weinstein precisely.

Theorem 6.1 (Sternberg-Weinstein)**.** *Let (M, ω) be a a symplectic manifold with a Hamiltonian action of a Lie group G and the moment map $\nu : M \to \mathfrak{g}^*$. Let $G \to P \to B$ be a principal bundle. If there exists a connection in the principal bundle P such that all vectors in $\nu(M) \subset \mathfrak{g}^*$ are fat, then the coupling form on the total space of the associated bundle*

$$M \to P \times_G M \to B$$

is symplectic.

In [W] A. Weinstein asked the following question: *are there symplectically fat fiber bundles whose total spaces are simply connected, symplectic but non-Kähler?* Note that this problem was called *the Thurston-Weinstein problem for fiber bundles* in [TO]. The answer to this question is positive and is a byproduct of previous sections.

Following Lerman [L3] consider the Kodaira-Thurston manifold N. Recall that the latter is a compact nilmanifold $(N_3/\Gamma) \times (\mathbb{R}/\mathbb{Z})$, where N_3 denotes the 3-dimensional Heisenberg group of all unipotent 3×3 matrices, and Γ is a co-compact lattice in it. It is well known and easy to see that N is symplectic. It is also known that N can be symplectically embedded into $\mathbb{C}P^5$ (see, for example [TO]). Consider $\mathbb{C}P^6$ and define a circle action on it by the formula

$$\lambda \cdot [z_0 : z_1 : \, ... \, : z_6] = [\lambda z_0 : z_1 : \, ... \, : z_6],\ \lambda \in S^1. \tag{3}$$

This action is trivial on the hyperplane

$$\mathfrak{H} = \{[0 : z_1 \ldots : z_6] \, z_i \in \mathbb{C}\} \cong \mathbb{C}P^5.$$

Embed N symplectically into $\mathfrak{H}$. Blow-up $\mathbb{C}P^6$ along N. On the resulting manifold $\widetilde{\mathbb{C}P}^6$ we have the symplectic circle action extending (3), since N is embedded in the fixed point set (see [L3]).

Theorem 6.2. *There exist symplectically fat fiber bundles whose total spaces are simply connected, symplectic but non-Kähler. In particular, the fiber bundle*

$$\widetilde{\mathbb{C}P}^6 \to S^3 \times_{S^1} \widetilde{\mathbb{C}P^6} \to S^2$$

associated with the Hopf bundle $S^1 \to S^3 \to S^2$ admits a fiberwise symplectic structure, but the total space of this bundle does not admit any Kähler structure.

Proof. The general theory of symplectic fat bundles ensures the existence of the non-degenerate coupling form on the total space. Thus, it is sufficient to prove that the total space cannot have Kähler structures. By a theorem of Lalonde and McDuff [LMcD], any hamiltonian fiber bundle over a sphere has the property of c-splitting, that is, the cohomology of the total space of such bundle is the tensor product (as vector spaces) of the cohomology of the fiber and the base. It follows that

$$H^*(S^3 \times_{S^1} \widetilde{\mathbb{C}P}^6) = H^*(S^2) \otimes_{v.s.} H^*(\widetilde{\mathbb{C}P}^6).$$

It follows from the degree reasons that

$$b_3(S^3 \times_{S^1} \widetilde{\mathbb{C}P}^6) = b_3(\widetilde{\mathbb{C}P}^6) = 3.$$

Hence the total space cannot be Kähler.

□

Remark 6.3. Examples of non-simply connected symplectically fat fiber bundles whose total spaces are not Kähler, were given in [KTW]. According to our knowledge, simply connected examples of this type were not presented in the literature, since techniques like symplectic blow-up construction and the Lalonde-McDuff c-splitting theorem were discovered after the question was posed. Since the argument presented here is in the main line of reasoning in this paper, we decided to write it down.

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References

[BLG] D.E. Blair, S.I.Goldberg, *Topology of almost contact manifolds*, J.Different. Geom. 1(1967), 347–354.

[BG] C.P. Boyer, K. Galicki, *Sasakian Geometry*, Oxford Univ. Press, Oxford, 2007.

[BGM] C.P. Boyer, K. Galicki, B. Mann, *The geometry and topology of 3-Sasakian manifolds*, J. reine angew. Math. 455(1994), 183-220

[BMS] F. Belgun, A. Moroianu, U. Semmelmann, *Symmetries of contact metric manifolds*, Geom. Dedic. 101(2003), 203-216

[BW] W. M. Boothby, H.C. Wang, *On contact manifolds*, Annals Math. 68(1958), 721-734

[C] G. Cavalcanti, *The Lefschetz property, formality and blowing-up in symplectic geometry*, Trans. Amer. Math. Soc. 359(2007), 333-348.

[CFM] G. Cavalcanti, M. Fernández, V. Muñoz, *Symplectic resolutions, Lefschetz property and formality*, Advances Math. 218(2008), 576-599.

[F] T. Fujitani, *Complex-valued differential forms on normal contact Riemannian manifolds*, Tôhoku Math.J. 18(1966), 349–361.

[FT] Y. Felix, J.-C. Thomas, S. Halperin, *Rational homotopy theory*, Springer, Berlin, 2002

[GNT] O. Goertsches, H. Nozawa, D. Töben, *Equivariant cohomology of K-contact manifolds*, Math. Ann. 354(2012), 1555-1582.

[G] R. Gompf, A new construction of symplectic manifolds, Annals Math. 142(1995), 527-595

[EK] A. El Kacimi Alaoui, *Operateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. 73(1990), 57-106

[K] S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tôhoku Math. J. 8(1956), 29–45.

[KT] F.Kamber, P.Tondeur, *Duality theorems for foliations*, Asterisque 116 (1984), 458 - 471.

[IRTU] R. Ibáñez, Y. Rudyak, A. Tralle, L. Ugarte, *On certain geometric and homotopy properties of closed symplectic manifolds*, Topology Appl. 127(2003), 33-45

[KTW] J. Kędra, A. Tralle, A. Woike, *On non-degenerate coupling forms*, J. Geom. Phys. 61(2011), 462-475.

[LMcD] F. Lalonde, D. McDuff, *Symplectic structures on fiber bundles*, Topology 42(2003), 309-347

[L1] E. Lerman, *How fat is a fat bundle?* Lett. Math. Phys. 15(1988), 335-339.

[L2] E. Lerman, *Contact fiber bundles*, J. Geom. Phys. 49(2004), 52-66

[L3] E. Lerman, *A compact symmetric symplectic non-Kähler manifold*, Math. Res. Lett. 3(1996), 587-590

[McD] D. McDuff, *Examples of symplectic simply connected manifolds with no Kähler structure*, J. Different. Geom. 20(1984), 267-277

[NR] M. Nicolau, A. Reventos, *Euler class and Gysin sequence for S^p-leaves* , Israel J. Math. 47 (1984), 323–334.

[OV] L. Ornea, M. Verbitsky, *Sasakian structures on CR-manifolds*, Geom. Dedicata 125(2007), 159-173

[RT] Y. Rudyak, A. Tralle, *On Thom spaces, Massey products and non-formal simply connected symplectic manifolds*, Internat. Math. Res. Notices, no. 10(2000), 495-513

[R] P.Rukimbira, *Chern - Hamilton's conjecture and K-contactness*, Houston J. Math. 21(1995), 709–718.

[S] S. Sternberg, *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*, Proc. Natnl. Acad. Sci. USA 74(1977), 5253-5254

[T] S. Tanno, *Harmonic forms and Betti numbers of certain contact Riemannian manifolds*, J. Math. Soc. Japan 19 (1967), 308–316.

[TO] A. Tralle, J. Oprea, *Symplectic manifolds with no Kähler structure*, Springer, Berlin, 1997.

[WZ] Wang Z. Z. and D. Zafran, *A remark on the hard Lefschetz theorem for Kähler orbifolds*, Proc. Amer. Math. Soc. 137(2009), 2497-2501

[W] A. Weinstein, *Fat bundles and symplectic manifolds*, Adv. Math. 37(1980), 239-250

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