Titles and abstracts

Maciej Błaszak (Adam Mickiewicz University, Poznań)

From autonomous Stäckel systems to non-autonomous Painlevé systems

In cooperation with Krzysztof Marciniak and Ziemowit Domański.

Among all second order nonlinear integrable ordinary differential equations (ODE's) there are two distinguished classes, playing important roles in modern mathematics and physics. The first class is represented by nonlinear equations of *Stäckel-type*, with an autonomous Hamiltonian representation on a symplectic manifold

$$\frac{d\xi}{dt_r} = X_r(\xi) \equiv \pi dh_r(\xi), \quad r = 1, \dots, n, \tag{1}$$

which are Frobenius integrable (also known as Liouville integrable in this case)

$$[X_r, X_s] = 0, \quad r, s = 1, \dots, n.$$

Moreover, the autonomous equations (1) are represented by the isospectral Lax equations

$$\frac{d}{dt_k} L(\lambda; \xi) = [U_k(\lambda; \xi), L(x; \xi)], \quad k = 1, \dots n. \tag{2}$$

The second class is represented by nonlinear ordinary differential equations of *Painlevé-type*, with a non-autonomous Hamiltonian representation

$$\frac{d\xi}{dt_r} = Y_r(\xi, t) = \pi dH_r(\xi, t), \quad r = 1, \dots, n, \tag{3}$$

where $t = (t_1, \dots, t_n)$. The set of n equations (3) constitutes a non-autonomous Painlevé-type system if is Frobenius integrable

$$\frac{\partial X_s}{\partial t_r} - \frac{\partial X_r}{\partial t_s} + \{X_r, X_s\} = 0, \quad r, s = 1, \dots, n \tag{4}$$

and the system is represented by the isomonodromic Lax representation

$$\frac{d}{dt_k} L(x; \xi, t) = [U_k(x; \xi, t), L(x; \xi, t)] + \frac{\partial U_k(x; \xi, t)}{\partial x}, \quad k = 1, \dots, n. \tag{5}$$

In this lecture we present a systematic deformation of autonomous Stäckel-type systems to non-autonomous Painlevé-type hierarchies. In particular we construct the infinite hierarchies of Painlevé I (P_I), Painlevé II (P_{II}), Painlevé III (P_{III}) and Painlevé IV (P_{IV}).

[1] Błaszak M., Marciniak K., Sergyeyev A., *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep Math Phys **87** (2021) 249-263

[2] Błaszak M., Marciniak K., Domański Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability*, Stud. Appl. Math. **148** (2022) 1208-1250

[3] Błaszak M., Marciniak K., Domański Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation*, Stud. Appl. Math. **149** (2022) 364-415

[4] Błaszak M., *Multi-component Painlevé ODEs and related non-autonomous KdV stationary hierarchies*, Stud. Appl. Math. **151** (2023) 5-34

Debora Choińska (University of Warsaw)

Singularity curves of two component Yang-Baxter maps

Jan Cieśliński (University of Białystok)

Integrable discretization of Nambu mechanics using discrete gradients. Applications to rigid body dynamics

Ziemowit Domański (Poznań University of Technology)

Time evolution in quantum mechanics with minimal time scale

Many theories of quantum gravity suggest that there should exists a minimal scale with which it is possible to measure distances and time. We present a formalism of quantum mechanics exchibiting a minimal time scale. The work is based on the Page-Wootters formalism which gives a possible solution to the problem of time in quantum mechanics. In this approach an operator is associated with a clock which is used to measure passage of time. The clock is entangled with the rest of the system which in turn results in a method to measure time in the system. In our work we modify the commutation relations between the time and energy operators leading to a minimal uncertainty of time measurement. We further investigate the effects of such modification and explore its physical consequences.

Maciej Dunajski (University of Cambridge)

Gravitational Instantons Old and New

Toric Ricci–flat metrics in dimension four correspond to certain holomorphic vector bundles over a twistor space. We construct these bundles explicitly, by exhibiting and characterising their patching matrices, for the five–parameter family of Riemannian ALF metrics constructed by Chen and Teo. This characterises the Chen–Teo instantons using the integrable systems techniques (Riemann Hilbert problem, and anti–self–dual Yang–Mills equations).

Piotr P. Goldstein (National Centre for Nuclear Research, Otwock)

An interpretation of complex resonant indices in the Painlevé test

In the Painlevé test, resonant indices having nonzero imaginary part do not provide first integrals. This means that the number of the first integrals is insufficient to construct the general solution. Hence, occurrence of such indices means that the general solution of the considered system of differential equations does not have the Painlevé property. However, special solutions often exist which may be expanded into the Laurent series about a movable singularity. In these cases the complex resonant indices can characterise unstable oscillatory behaviour of the special solutions in the neighbourhood of the singularity. This way, the complex indices can provide information on the transition to chaos.

To see this behaviour, we proceed beyond the standard Painlevé test to its perturbative version [1], obtaining an interesting picture of the perturbation dynamics.

[1] R. Conte, A.P. Fordy and A. Pickering, *A perturbative Painlevé approach to nonlinear differential equations*, 1993, *Physica* **D69**, 33.

Alfred Michel Grundland (Université de Montréal, Université du Québec à Trois-Rivières)

Applications of quasi-rectifiable Lie algebras to hydrodynamic-type systems

In this talk, we define and employ the concept of families of quasi-rectifiable vector fields in order to obtain multiple Riemann wave solutions of a quasilinear first-order hyperbolic system of differential equations. We show how to construct quasi-rectifiable Lie algebras of vector fields which allow us to study several hydrodynamic-type systems admitting k-wave solutions. We demonstrate the usefulness of this approach on several examples.

Pavlos Kassotakis (University of Warsaw)

On quadrirational pentagon maps

The *pentagon equation* serves as one of a handful of very important equations in mathematical physics. It appears in two equally significant versions, the operator and the set-theoretical one. In this talk we will focus on the set-theoretic version of the pentagon equation. Firstly, we will introduce a specific class of matrices which participate in factorization problems that turn to be equivalent to pentagon maps, expressed in totally non-commutative variables. Secondly, we will propose a classification scheme for rational solutions of a specific type of the pentagon equation. That is, we will give a full list of representatives of *quadrirational maps* that satisfy the pentagon equation, modulo an equivalence relation that is defined on birational functions on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Finally, we will show how from a pentagon map that admits a *partial inverse*, we can obtain set theoretical solutions of the so-called *entwining pentagon equation.*

Andrzej Maciejewski (University of Zielona Góra)

Non-integrability of the charged three body problem

Joint work with Maria Przybylska.

Omid Makhmali (UiT The Arctic University of Norway, Tromsø)

Zero-curvature subconformal structures and dispersionless integrability in dimension five

It is well-known that the notion of dispersionless integrability is related to Einstein-Weyl structures and self-dual conformal structures in dimension three and four, respectively. We extend this relation to dimension five using certain subconformal contact manifolds with vanishing curvature, which are examples of parabolic geometries. We discuss twistorial constructions and symmetry reductions in this case, as well as a classification of parabolic geometries that are related to dispersionless integrability. This is a joint work with Boris Kruglikov.

Krzysztof Marciniak (Linköping University)

Algebraic curves, separable multi-Hamiltonian systems and Miura maps

In this talk I will describe the idea of constructing various types of finite-dimensional integrable and separable Hamiltonian systems from *parameter-dependent* planar algebraic curves

$$\varphi(\lambda, \mu, a) = 0, \quad a \in \mathbf{R}^k \tag{1}$$

where (λ, μ) are coordinates on a plane and where a is a multiparameter. I will focus on two particular constructions. First, I will consider separable systems in $\mathbf{R}^{2n}$generated by algebraic curves (1) depending on a set of $k = n+n$ parameters. Each such curve leads to two distinct integrable Hamiltonian systems. I will demonstrate that these systems are related by a Stäckel transform [3, 4, 6, 2] and also how solutions of these two systems are related by reciprocal (multi-time) transformations. I also specify these results to the case of Stäckel systems. Then, I will consider algebraic curves (1) depending on $n+N$, $N > 0$, parameters and having a certain block-type structure. These curves leads to families of integrable and separable Hamiltonian systems that can be related with each other by a finite-dimensional analogue of Miura maps, which yields in turn their multi-Hamiltonian formulation. These results generalize the particular results obtained earlier in [1] and in [5].

This talk is an effect of a joint work with prof. Maciej Blaszak, Poznan, Poland.

[1] Błaszak M., *Bi-Hamiltonian representation of Stäckel systems,* Phys. Rev. E **97** (2009) 056607

[2] Błaszak M., Marciniak K., *On Reciprocal Equivalence of Stäckel Systems*, Stud. Appl. Math. 129, issue 1 (2012), 26-50

[3] Boyer C.P., Kalnins E.G. and Miller W. Jr., *Stäckel-equivalent integrable Hamiltonian systems*, SIAM J. Math. Anal., **17** (1986), 778–797

[4] Hietarinta J., B. Grammaticos B., Dorizzi B. and Ramani A., *Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems*, Phys. Rev. Lett. **53** (1984), 1707–1710

[5] Marciniak K., Błaszak M., *Miura maps for Stäckel systems,* J. Math. Phys. **64**, 122903 (2023)

[6] Sergyeyev, A., Błaszak, M., *Generalized Stäckel transform and reciprocal transformations for finite-dimensional integrable systems*, J. Phys. A **41** (2008), no. 10, 105205, 20 pp

Maciej Nieszporski (University of Warsaw)

On a unfication of discrete analytic functions theories

There are two theories of discrete analytic functions. The linear theory by Duffin can be viewed as a limit of the nonlinear theory which is usually attributed to Thurston. In contrast we will show that both theories can be viewed as special cases of a system of difference integrable equations.

Andriy Panasyuk (Cardinal Stefan Wyszyński University, Warsaw)

Hirota dispersionless system and hidden symmetries of heavenly equation

Joint work with Adam Szereszewski.

In 2021 Konopelchenko, Schief and Szereszewski observed that that solutions of 4D dispersionless Hirota system also solve the general heavenly equation describing self-dual vacuum Einstein metrics in neutral signature. They also noticed that the symmetry $f \mapsto \phi(f)$ of the Hirota system essentially change the properties of the corresponding metric.

In this talk we shall restate these observations in the context of II Plebański heavenly equation (IIPHE). Namely, we first extend the hierarchy for this equation found by Dunajski and Mason in 2000 to odd dimensions. We then consider the corresponding 5D system from this hierarchy with a special type of symmetry generalizing the tri-holomorphic symmetry of IIPHE. The reduction with respect to this symmetry (which in a sense imitate the reduction of self-dual vacuum Einstein metrics with respect to a tri-holomorphic symmetry ending in special Einstein–Weyl structures) gives an analogue of the dispersionless Hirota system for IIPHE. Such a point of view allows to reinterpret the symmetry $f \mapsto \phi(f)$ mentioned and obtain explicit formulas for the metric depending on ϕ and for its Weyl spinor. We present some examples showing that this last changes essentially along with ϕ.

Miłosz Panfil (University of Warsaw)

Navier-Stokes equations for nearly integrable quantum gases

Analtolij Prykarpatski (Lviv Polytechnic National University, Cracow University of Technology)

Super-conformal vector fields on the supercircle and related heavenly and KP-types integrable systems

Joint work with Oksana Hentosh.

Maria Przybylska (University of Zielona Góra)

Top on a smooth plane

A rigid body, apart from the material point, is the most important model in classical mechanics that represents various physical systems. The analysis of the dynamics of a rigid body with one fixed point in the gravitational field, called the heavy top, has been a research problem for hundreds of years and a testing ground for which various methods have been used and gave impetus to the creation of new methods. This paper provides a complete integrability analysis of an almost forgotten top model with an ideal sharp tip that slides on a perfectly smooth horizontal plane in a constant gravity field.

Stefan Rauch-Wojciechowski (University of Linköping)

Understanding reversals of a Rattleback

The rattleback is a rigid body having a boat like shape (modelled as a bottom half of an 3-axial ellipsoid) having asymetric (chiral) distribution of mass. When the rattleback is spun on its bottom in the “wrong” direction then it starts to rattle, it slows down and acquires rotation in the opposite, preferred sense of spinning. This behaviour defies our intuition about conservation of angular momentum as the force and the torque responsible for changing the angular momentum (and the direction of spinning) is not easily discernible. Overwhelming majority of papers on the rattleback motion study the dependence of stability for spinning solutions: on the sense of rotation, on the shape of rattleback’s surface and on the distribution of mass. There has been no simple explanation of the rattleback behaviour in terms of physical forces and torques.

This question has been the subject of our paper with M. Przybylska that have appeared in Regular and Chaotic Dynamics, a journal of Russian Academy of Sciences. In this paper we study the motion of a toy rattleback by using frictionless Newton equations of motion for a rigid body rolling without sliding in a plane. In these equations it is the reaction force of the supporting surface that is the source of the torque turning the rattleback in the preferred sense of rotation. The picture is, however, more subtle as it appears that the direction of the torque depends on the initial conditions and a frictionless, low energy rattleback admits reversals in both directions (!). I will present a simple, intuitive understanding of how the rattleback’s motion depends on initial conditions and will discuss how it is consistent with numerical simulations of rattlebacks equations for tapping and for spinning initial conditions. Simulations show also that long time behaviour of such rattleback is, for low energy initial conditions, quasi-periodic and there are inifinitly many reversals in both directions.

Artur Sergyeyev (Silesian University in Opava)

Two-component integrable extension of general heavenly equation

In this talk, based on the joint work with W. Kryński (arXiv:2402.10317), we introduce an integrable two-component extension of the general heavenly equation. It turns out that the solutions of this extension are in one-to-one correspondence with 4-dimensional hyper-para-Hermitian metrics, and if the metrics in question are hyper-para-Kähler, then our system reduces to the general heavenly equation. We present a Lax pair, an infinite hierarchy of nonlocal symmetries and a recursion operator for the system under study.

Błażej Szablikowski (Adam Mickiewicz University, Poznań)

Non-autonomous deformations of integrable soliton hierarchies

I will present results of joint work with *Maciej Błaszak* (Poznań, Poland) and *Krzysztof Marciniak* (Linköping, Sweden).

Research in the area of integrable non-autonomous hierarchies of soliton type is quite rare compared to research on integrable autonomous hierarchies. In our work we want to partially fill this gap. Similar research program in the case of finite-dimensional integrable systems, followed by series of papers, was initiated in [1].

Starting with an autonomous integrable hierarchy of commuting systems

$$u_{t_n} = K_n(u, u_x, u_{xx}, \ldots) \qquad n \in \mathbb{N}, \qquad [K_i, K_j] = 0, \tag{1}$$

we are interested in deforming it to the time-dependent hierarchy

$$u_{t_n} = \mathbb{K}_n(t_1, \ldots, t_n; u, u_x, \ldots) \qquad n \in \mathbb{N}, \tag{2}$$

where each vector field $\mathbb{K}_n$ explicitly depends on its own time t_n and other times, $t_1, \ldots, t_{n-1}$, associated with lower order members of the hierarchy. To preserve integrability, existence of common multi-time solutions, and commutativity ($u_{t_i,t_j} = u_{t_j,t_i}$), we need to require Frobenius conditions, which in the case of the hierarchy (2) reduce to the system in the triangular form

$$\frac{\partial \mathbb{K}_j}{\partial t_i} + [\mathbb{K}_i, \mathbb{K}_j] = 0 \qquad i < j, \tag{3}$$

as $\frac{\partial \mathbb{K}_i}{\partial t_j} = 0$ for $i < j$.

We attack the problem of constructing the non-autonomous deformations (2) by reformulating the system of Frobenius conditions (3) as an initial value problem on an arbitrary Lie algebra and by providing formal solution to this problem. However, applying this solution to the trivial Lie algebra given by (1) leads to the trivial uninteresting result. To overcome this obstacle, we extend this trivial algebra over additional master symmetries σ_i (idea proposed in [2]) and thus we work with the so-called hereditary Lie algebra

$$[K_n, K_m] = 0, \qquad [\sigma_n, K_m] = (\alpha m + \rho - 1) K_{n+m}, \qquad [\sigma_n, \sigma_m] = \alpha(m-n)\sigma_{n+m}.$$

As result, we provide, in this setting, nontrivial and (general) solutions for the constructing of the integrability preserving non-autonomous deformations (2) of the integrable autonomous hierarchies (1). We illustrate our theory with examples of well-known hierarchies, starting with the Korteweg-de Vries hierarchy. Additionally, we present construction of (isomondromic) Lax representations for derived non-autonomous hierarchies (2).

[1] Błaszak M., Marciniak K., Sergyeyev A., *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep Math Phys **87** (2021) 249–263

[2] Błaszak M., *Multicomponent Painlevé ODEs and related nonautonomous KdV stationary hierarchies*, Stud. Appl. Math. **151** (2023) 5–34

Marzena Szajewska (University of Białystok)

Geometrical aspects of Coxeter groups

The polyhedra with exact reflection symmetry group G in the real 3D space is considered. Modifications of the shell of polyhedra that preserve the symmetry are described. The recursive rules for finding orbits with smaller radii, which provide the structures of nested polytopes, are demonstrated.

[1] Myronova, M., Patera, J., Szajewska, M., *Nested polyhedra and indices of orbits of Coxeter groups of non-crystallographic type* Symmetry 12 (2020), no. 10, 1–18.

[2] Grabowiecka Z., Patera, J., Szajewska, M., *Reduction of orbits of finite Coxeter groups of non-crystallographic type* J. Math. Phys. 59 (2018), no. 101705, 1–18.

[3] Szajewska M., *Geometrical structures of nested polyhedra* J. Phys.: Conf. Ser. 2667 (2023), 1–7.

Wojciech Szumiński (University of Zielona Góra)

Dynamics and integrability of variable-length double pendulums

We are studying the dynamics and integrability of different types of pendulum systems - mainly focusing on the double-spring pendulum. Since these are Hamiltonian systems with three degrees of freedom, their analysis is quite challenging. To gain insight into the systems' dynamics, we employ various numerical methods, including Lyapunov exponent spectra, phase-parametric diagrams, and Poincaré cross-sections. The novelty of our work lies in the integration of these three numerical methods into one powerful tool. We aim to identify parameter values or initial conditions that result in hyperchaotic, chaotic, quasi-periodic, and periodic motion. As far as I know, this is the first attempt to use Lyapunov exponents to systematically search for the first integrals of the system. We demonstrate how to effectively apply Lyapunov exponents as an indicator of integrable dynamics and provide the explicit forms of integrable and superintegrable systems. In addition to the numerical analysis, we are also providing analytical proofs of the systems' non-integrability. These proofs are based on analysing the properties of the differential Galois group of variational equations along specific solutions.

Ivan Tsyfra (AGH University of Science and Technology, Kraków)

On reducing and finding solutions of nonlinear partial differential equations via nonclassical and generalized symmetry methods

We study symmetry reductions of nonlinear partial differential equations via nonclassical point symmetry and generalized symmetry methods. We find ansatse reducing these equtions to ordinary differential equations. Ansatse are constructed using nonclassical (condtional) symmetry and generalized symmetry of ordinary differential equations. The method applied gives the posibility to find exact solutions which cannot be obtained by virtue of classical Lie method. We also prove the integrability by quadratures of ordinary differential equations with coefficients that are solutions of stationary Calogero-Bogoyavlenskii-Shiff equation.