Dynamics of multiple pendula

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Let us consider a system of 3 autonomous, ordinary differential equations of first order

\[
\frac{dx_1}{dt} = F_1(x_1, x_2, x_3), \\
\frac{dx_2}{dt} = F_2(x_1, x_2, x_3), \\
\frac{dx_3}{dt} = F_3(x_1, x_2, x_3).
\]

The plane \(S\) is the surface of cross section \(x_3 = \text{constant}\). Each time the trajectory pierces \(S\) in downward direction.
If the phase trajectory intersect the cross-section along a closed loop, then it lies on a two-dimensional invariant surface. This surface is a torus. If $\omega_1/\omega_2$ is

- **rational**
  - closed orbit,
  - the solution laying on that torus is periodic,
  - finite number of intersections.

- **irrational**
  - a single orbit covers the torus densely,
  - the motion is quasi-periodic,
  - the intersection points form continuous loops.

- A chaotic trajectory intersect the plane in scattered points.
THE DOUBLE PENDULUM

\[ x_1 = l_1 \sin \theta_1, \]  
\[ x_2 = x_1 + l_2 \sin \theta_2, \]  
\[ y_1 = l_1 \cos \theta_1, \]  
\[ y_2 = y_1 + l_2 \cos \theta_2. \]  

(1)
A brief introduction to the Poincare mapping

**The double pendulum**

The simple triple pendulum

The triple "flail" pendulum

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**The double pendulum**

The double pendulum consists of two particles, $m_1$ and $m_2$, with lengths $l_1$ and $l_2$, respectively. The motion can be described by the equations:

\[ x_1 = l_1 \sin \theta_1, \quad y_1 = l_1 \cos \theta_1, \]
\[ x_2 = x_1 + l_2 \sin \theta_2, \quad y_2 = y_1 + l_2 \cos \theta_2. \]  

(1)

The Hamiltonian $H$ is given by:

\[ H = \frac{l_2^2 m_2 p_1^2}{2 l_1^2 l_2 m_2} + l_1 (m_1 + m_2) p_2^2 - 2 m_2 l_1 l_2 p_1 p_2 \cos(\theta_1 - \theta_2) \]
\[ - \frac{2 l_1 l_2 m_2}{l_1^2 l_2^2 m_2 [m_1 + \sin^2(\theta_1 - \theta_2) m_2]} \]
\[ - m_2 g l_2 \cos \theta_2 - (m_1 + m_2) g l_1 \cos \theta_1. \]  

(2)
Dynamics described by canonical equations takes place in four-dimensional phase space \((\theta_1, \theta_2, p_1, p_2)\). To visualise the dynamics using the Poincaré sections we need to eliminate one variable and use result we reduce dimension by one. To do this we use the conservation of energy law \(H = E = \text{const.}\) to determine \(p_1\).

\[
p_1 = \frac{1}{2l_2^2 m_2} \left\{ (2l_1 l_2 m_2 p_2 \cos(\theta_1 - \theta_2) \\
+ \sqrt{2} \sqrt{l_1^2 l_2^2 m_2 (2m_1 + m_2 - m_2 \cos[2(\theta_1 - \theta_2)])} \\
\times \sqrt{[2El_2^2 m_2 - p_2^2 + 2gl_2^2 m_2 (l_1 (m_1 + m_2) \cos \theta_1 + l_2 m_2 \cos \theta_2)]} \right\}
\] (3)
We will analyse the dynamics of considered system for the following constant parameters:

\[ m_1 = 3, \ m_2 = 1, \ l_1 = 2, \ l_2 = 1, \ g = 1. \]  \hspace{1cm} (4)

and we choose the cross section plane: \( \theta_1 = 0, \ p_1 > 0. \)
We will analyse the dynamics of considered system for the following constant parameters:

\[ m_1 = 3, \quad m_2 = 1, \quad l_1 = 2, \quad l_2 = 1, \quad g = 1. \] (4)

and we choose the cross section plane: \( \theta_1 = 0, \quad p_1 > 0. \)
Fixing values \((\theta_1, \theta_2, p_1, p_2) = (0, 0, 0, 0)\) into the Hamiltonian (2), we can easy to check that the energy minimum corresponding to a state of rest for the pendulum is equal to \( E_0 = -9. \)
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The triple "flail" pendulum

Analytical results for triple "flail" pendulum

Figure: $E = -8.95$: regular behaviour.
Figure: $E = -7.3$: The invariant tori becomes deformed.
Figure: $E = -7.0$: The first appearance of chaotic region.
Figure: $E = -6.9$: The size of chaotic region increases.
Figure: Poincare section for $E = 0$: the global chaos in the center.
\begin{equation}
E = 0, \ \theta_1 = 0, \ \theta_2 = 3.14, \ p_1 = 0.003, \ p_2 = 3.24. \tag{5}
\end{equation}

**Figure:** (a) Fast oscillations of the first pendulum around the equilibrium point, (b) rotations of the second pendulum around its fixed point.
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THE SIMPLE TRIPLE PENDULUM

The Lagrange function for this system in the absence of gravity looks as follows
The simple triple pendulum

\[ x_1 = l_1 \sin \theta_1, \]
\[ y_1 = l_1 \cos \theta_1, \]
\[ x_2 = x_1 + l_2 \sin \theta_2, \]
\[ y_2 = y_1 + l_2 \cos \theta_2, \]
\[ x_3 = x_1 + x_2 + l_3 \sin \theta_3, \]
\[ y_3 = y_1 + y_2 + l_3 \cos \theta_3. \]

The Lagrange function for this system in the absence of gravity looks as follows
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\[ L = T = \frac{1}{2} \left\{ \dot{\theta}_3^2 l_3^2 m_3 + \dot{\theta}_2^2 l_2^2 (m_2 + m_3) + \dot{\theta}_1^2 l_1^2 (m_1 + m_2 + 4m_3) \\
+ 2\dot{\theta}_1 l_1 \left[ \dot{\theta}_2 l_2 (m_2 + 2m_3) \cos(\theta_1 - \theta_2) + 2\dot{\theta}_3 l_3 m_3 \cos(\theta_1 - \theta_3) \right] + 2\dot{\theta}_2 \dot{\theta}_3 l_2 l_3 m_3 \cos(\theta_2 - \theta_3) \right\}. \]  

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\[ L = T = \frac{1}{2} \left\{ \dot{\theta}_3^2 l_3^2 m_3 + \dot{\theta}_2^2 l_2^2 (m_2 + m_3) + \dot{\theta}_1^2 l_1^2 (m_1 + m_2 + 4m_3) \right. \\
+ 2\dot{\theta}_1 l_1 \left[ \dot{\theta}_2 l_2 (m_2 + 2m_3) \cos(\theta_1 - \theta_2) \right. \\
\left. + 2\dot{\theta}_3 l_3 m_3 \cos(\theta_1 - \theta_3) \right] + 2\dot{\theta}_2 \dot{\theta}_3 l_2 l_3 m_3 \cos(\theta_2 - \theta_3) \} . \]  

(6)

Because the Lagrangian in Eq.(6) depends on the cosine of the angles difference, we introduce new variables

\[ \gamma_1 = \theta_2 - \theta_1, \]
\[ \gamma_2 = \theta_3 - \theta_2, \]
\[ \gamma_3 = \theta_1. \]  

(7)
the Lagrange functions in the new variables looks as follows

\[ L = \frac{1}{2} \left\{ 2\dot{\gamma}_3 l_1 [2l_3 m_3 \cos(\gamma_1 + \gamma_2)(\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3) + l_2 \cos \gamma_1 (\dot{\gamma}_1 + \dot{\gamma}_3) \right. \]
\[ \times (m_2 + 2m_3)] + 2l_2 l_3 m_3 \cos \gamma_2 (\dot{\gamma}_1 + \dot{\gamma}_3)(\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3) + l_3^2 m_3 \]
\[ \times (\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3)^2 + l_2^2 (\dot{\gamma}_1 + \dot{\gamma}_3)^2 (m_2 + m_3) + \dot{\gamma}_3^2 l_1^2 (m_1 + m_2 + 4m_3) \} \].
the Lagrange functions in the new variables looks as follows

\[
L = \frac{1}{2} \left\{ 2\dot{\gamma}_3 l_1 [2l_3 m_3 \cos(\gamma_1 + \gamma_2)(\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3) + l_2 \cos \gamma_1 (\dot{\gamma}_1 + \dot{\gamma}_3) \\
\times (m_2 + 2m_3)] + 2l_2 l_3 m_3 \cos \gamma_2 (\dot{\gamma}_1 + \dot{\gamma}_3)(\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3) + l_3^2 m_3 \\\n\times (\dot{\gamma}_1 + \dot{\gamma}_2 + \dot{\gamma}_3)^2 + l_2^2 (\dot{\gamma}_1 + \dot{\gamma}_3)^2 (m_2 + m_3) + \dot{\gamma}_3^2 l_1^2 (m_1 + m_2 + 4m_3) \right\}.
\]

We see that the Lagrangian in new variables does not depend explicitly on the variable \( \gamma_3 \), which leads \( p_3 = b = const \). Then, the Hamiltonian depends only on four variables \( \gamma_1, \gamma_2 \). and \( p_1, p_2 \), and \( b \) is parameter. By means of these new variables we reduced the number of degrees of freedom from 3 to 2.
We are going to analyse the dynamics of the considered system for the following constant parameters:

\[ p_3 = b = 1, \quad m_1 = 2, \quad m_2 = 1, \quad m_3 = 1, \quad l_1 = 2, \quad l_2 = 1, \quad l_3 = 1. \]
Figure: The Poincaré section for $E = 0.0095$: regular behaviour.
Figure: $E = 0.0097$: the first sign of chaotic motion.
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Figure: The enlargement of lower region of previous Poincaré section.
Figure: $T \ E = 0.0098$: invariant tori becomes deformed.
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Figure: The Poincaré section for $E = 0.0099$: invariant tori becomes deformed.
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Figure: The Poincaré section for $E = 0.01$: invariant tori becomes deformed.
Figure: The Poincaré section for $E = 0.011$: invariant tori becomes deformed.
Figure: The Poincaré section for $E = 0.013$: invariant tori becomes deformed.
The "Flail"

\[ x_1 = l_1 \sin \theta_1, \quad y_1 = l_1 \cos \theta_1, \]
\[ x_2 = x_1 + l_2 \sin \theta_2, \quad y_2 = y_1 + l_2 \cos \theta_2, \]
\[ x_3 = x_1 + l_3 \sin \theta_3, \quad y_3 = y_1 + l_3 \cos \theta_3. \]

\[ L = \frac{1}{2} \left[ \dot{\theta}_2^2 l_2^2 m_2 + \dot{\theta}_2^2 l_3^2 m_3 + \dot{\theta}_1^2 l_1^2 \left( m_1 + m_2 + m_3 \right) + 2 \dot{\theta}_1 l_1 \left( \dot{\theta}_2 l_2 m_2 \cos (\theta_1 - \theta_2) + \dot{\theta}_3 l_3 m_3 \cos (\theta_1 - \theta_3) \right) \right]. \]
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**The "Flail"**

\[
x_1 = \ell_1 \sin \theta_1, \quad y_1 = \ell_1 \cos \theta_1, \\
x_2 = x_1 + \ell_2 \sin \theta_2, \quad y_2 = y_1 + \ell_2 \cos \theta_2, \\
x_3 = x_1 + \ell_3 \sin \theta_3, \quad y_3 = y_1 + \ell_3 \cos \theta_3.
\]

\[
L = T = \frac{1}{2} \left[ \dot{\theta}_2^2 \ell_2^2 m_2 + \dot{\theta}_3^2 \ell_3^2 m_3 + \dot{\theta}_1^2 \ell_1^2 \\
(m_1 + m_2 + m_3) + 2 \dot{\theta}_1 \ell_1 \left( \dot{\theta}_2 \ell_2 m_2 \cos (\theta_1 - \theta_2) + \dot{\theta}_3 \ell_3 m_3 \cos (\theta_1 - \theta_3) \right) \right].
\]
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**The "Flail"**

\[
x_1 = l_1 \sin \theta_1, \quad y_1 = l_1 \cos \theta_1, \\
x_2 = x_1 + l_2 \sin \theta_2, \quad y_2 = y_1 + l_2 \cos \theta_2, \\
x_3 = x_1 + l_3 \sin \theta_3, \quad y_3 = y_1 + l_3 \cos \theta_3.
\]

\[
L = T = \frac{1}{2} \left[ \dot{\theta}_2^2 l_2^2 m_2 + \dot{\theta}_3^2 l_3^2 m_3 + \dot{\theta}_1^2 l_1^2 \\
(m_1 + m_2 + m_3) + 2\dot{\theta}_1 l_1 \left( \dot{\theta}_2 l_2 m_2 \cos (\theta_1 - \theta_2) + \dot{\theta}_3 l_3 m_3 \cos (\theta_1 - \theta_3) \right) \right].
\]
Introducing the new variables

\[ \gamma_1 = \theta_2 - \theta_1, \]
\[ \gamma_2 = \theta_3 - \theta_2, \]
\[ \gamma_3 = \theta_1. \] (8)
Introducing the new variables

\[
\begin{align*}
\gamma_1 &= \theta_2 - \theta_1, \\
\gamma_2 &= \theta_3 - \theta_2, \\
\gamma_3 &= \theta_1.
\end{align*}
\]  

(8)

Then, substituting into the Eq.(8) we get the Lagrange functions in the new variables

\[
L = T = \frac{1}{2} \left[ (\dot{\gamma}_1 + \dot{\gamma}_3)^2 l_2^2 m_2 + (\dot{\gamma}_1 + \dot{\gamma}_1 + \dot{\gamma}_3)^2 l_3^2 m_3 + \dot{\gamma}_3^2 l_1^2 \\
\times (m_1 + m_2 + m_3) + 2\dot{\gamma}_3 l_1 ((\dot{\gamma}_1 + \dot{\gamma}_3) l_2 m_2 \cos \gamma_1 + (2 + \dot{\gamma}_2 + \dot{\gamma}_3) \times l_3 m_3 \cos (\gamma_1 + \gamma_2)) \right].
\]  

(9)
Like previous, the Lagrange functions does not depend explicitly on variable $\gamma_3$. Therefore the momentum $p_3 = C$ is a parameter and the Hamiltonian depends only on four variables $\gamma_1, \gamma_2, \text{and } p_1, p_2$. 
Like previous, the Lagrange functions does not depend explicitly on variable $\gamma_3$. Therefore the momentum $p_3 = C$ is a parameter and the Hamiltonian depends only on four variables $\gamma_1, \gamma_2$, and $p_1, p_2$. We will analyse the dynamics of considered system for the following constant parameters:

$$p_3 = b = 1, \; m_1 = 1, \; m_2 = 3, \; m_3 = 2, \; l_1 = 1, \; l_2 = 2, \; l_3 = 3.$$  

(10)

and we choice the cross section plane $\gamma_1 = 0, \; p_1 > 0$. 
Figure: $E = 0.01$: detected three types of motion.
Figure: $E = 0.01005$: the newly rising stable periodic solution.
Figure: $E = 0.0102$: three clusters of stable periodic solutions.
Figure: The Poincaré section for $E = 0.0105$. 
Figure: $E = 0.011$: the newly rising "neckles" formations.
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Figure: The Poincaré section for $E = 0.012$. 
Figure: $E = 0.012$: the enlargement of the central region.
Figure: The Poincaré section for $E = 0.013$. 
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Figure: $E = 0.013$: the enlargement of the central region.
**SOME OTHER EXAMPLES**

\[ p_3 = b = 1/2, \ m_1 = 1, \ m_2 = 2, \ m_3 = 2, \ l_1 = 2, \ l_2 = 1, \ l_3 = 1. \]  

and we choice the cross section plane \( \gamma_1 = 0, \ p_1 > 0. \)
Figure: The Poincaré section for $E = 0.0035$. 
Figure: The Poincaré section for $E = 0.0036$. 
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Figure: The enlargement of the central part of previous Poincaré section.
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Figure: The Poincaré section for $E = 0.00363$. 
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**Figure:** The Poincaré section for $E = 0.003645$. 
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Figure: The enlargement of upper part of previous Poincaré section
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**Figure:** The enlargement of right corner of previous Poincaré section.
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Triple "flail" pendulum without gravity

**Problem**

Problem: how to find values of parameters $m_1$, $m_2$, $m_3$ and $l_1$, $l_2$ and $l_3$ for that system is integrable?

Answer: To apply the Morales-Ramis theory!

When

$$m_2 l_2 = m_3 l_3$$

then exists particular solution $\theta_1 = p_{\theta_1} = 0$, $\theta_3 = -\theta_2$ and $p_{\theta_3} = -l_3 p_{\theta_2} / l_2$

$$
(\dot{\theta}_1, \dot{\theta}_2, \dot{p}_{\theta_1}, \dot{p}_{\theta_2}) = \left(0, \frac{p_{\theta_2}}{l_2^2 m_2}, -\frac{p_{\theta_2}}{l_2^2 m_2}, 0, 0, 0\right).
$$

(12)
Main results

**Theorem**

*Triple “flail” pendulum satisfying*

\[ m_2 l_2 = m_3 l_3 \]

*is non-integrable in the class of meromorphic functions except the case* \( m_1 = 0 \).

**Lemma**

*Differential Galois group of normal variational equations for* \( m_1 = 0 \) *is a finite group or full* \( \text{SL}(2, \mathbb{C}) \).
What for triple “flail” pendulum in gravity field?

- In the case
  \[ l_2 = l_3, \quad m_2 = m_3 \]
  a non-equilibrium particular solution is known.
- System of normal variational equations has dimension four.
- Normal variational equations can be transformed into a fourth order linear equation with rational coefficients.
- **Non-trivial problem**: how to find its differential Galois group?
- Work in progress.