Dynamics of multiple pendula

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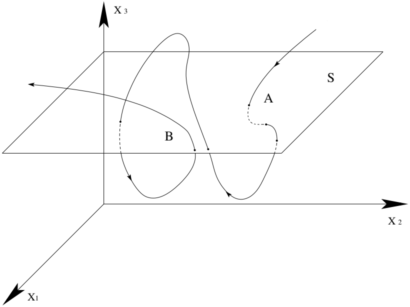
Let us consider a system of 3 autonomous, ordinary differential equations of first order

$$\frac{dx_1}{dt} = F_1(x_1, x_2, x_3),$$
$$\frac{dx_2}{dt} = F_2(x_1, x_2, x_3),$$
$$\frac{dx_3}{dt} = F_3(x_1, x_2, x_3).$$

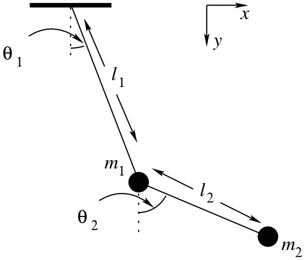


The plane S is the surface of cross section $x3 = constant$. Each time the trajectory pierces S in downward direction.

If the phase trajectory intersect the cross-section along a closed loop, then it lies on a two-dimensional invariant surface. This surface is a torus. If ω_1/ω_2 is

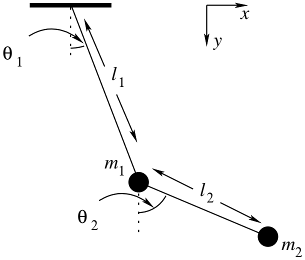
- **rational**
  - closed orbit,
  - the solution laying on that torus is periodic,
  - finite number of intersections.
- **irrational**
  - a single orbit covers the torus densely,
  - the motion is quasi-periodic,
  - the intersection points form continuous loops.
- A chaotic trajectory intersect the plane in scattered points.

The double penduum



$$x_1 = l_1 \sin\theta_1, \qquad y_1 = l_1 \cos\theta_1,$$
$$x_2 = x_1 + l_2 \sin\theta_2, \qquad y_2 = y_1 + l_2 \cos\theta_2. \tag{1}$$

The double penduum



$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1, & y_1 &= l_1 \cos\theta_1, \\
x_2 &= x_1 + l_2 \sin\theta_2, & y_2 &= y_1 + l_2 \cos\theta_2.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
H = &\frac{l_2^2 m_2 p_1^2 + l_1(m_1 + m_2)p_2^2 - 2m_2 l_1 l_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2[m_1 + \sin^2(\theta_1 - \theta_2)m_2]} \\
&- m_2 g l_2 \cos\theta_2 - (m_1 + m_2) g l_1 \cos\theta_1.
\end{aligned} \tag{2}
$$

Dynamics described by canonical equations takes place in four-dimensional phase space $(\theta_1, \theta_2, p_1, p_2)$. To visualise the dynamics using the Poincaré sections we need to eliminate one variable and us result we reduce dimension by one. To do this we use the conservation of energy low $H = E = const.$ to determine p_1 .

$$
\begin{aligned}
p_1 = \frac{1}{2l_2^2 m_2} \{ & (2 l_1 l_2 m_2 p_2 \cos(\theta_1 - \theta_2) \\
& + \sqrt{2}\sqrt{l_1^2 l_2^2 m_2 (2m_1 + m_2 - m_2 \cos[2(\theta_1 - \theta_2)])} \\
& \times \sqrt{[2El_2^2 m_2 - p_2^2 + 2gl_2^2 m_2 (l_1(m_1 + m_2)\cos\theta_1 + l_2 m_2 \cos\theta_2)]} \Bigg\}
\end{aligned}
\tag{3}
$$

We will analyse the dynamics of considered system for the following constant parameters:

$$m_1 = 3,\ m_2 = 1,\ l_1 = 2,\ l_2 = 1,\ g = 1. \tag{4}$$

and we choice the cross section plane: $\theta_1 = 0,\ p_1 > 0.$

We will analyse the dynamics of considered system for the following constant parameters:

$$m_1 = 3,\ m_2 = 1,\ l_1 = 2,\ l_2 = 1,\ g = 1. \tag{4}$$

and we choice the cross section plane: $\theta_1 = 0,\ p_1 > 0$.
Fixing values $(\theta_1, \theta_2, p_1, p_2) = (0, 0, 0, 0)$ into the Hamiltonian (2), we can easy to check that the energy minimum corresponding to a state of rest for the pendulum is equal to $E_0 = -9$.

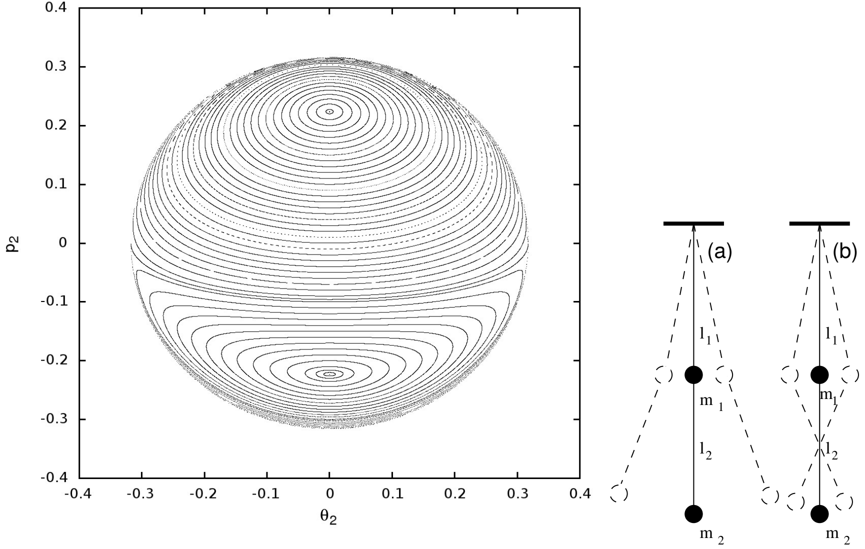


Figure: $E = -8.95$: regular behaviour.

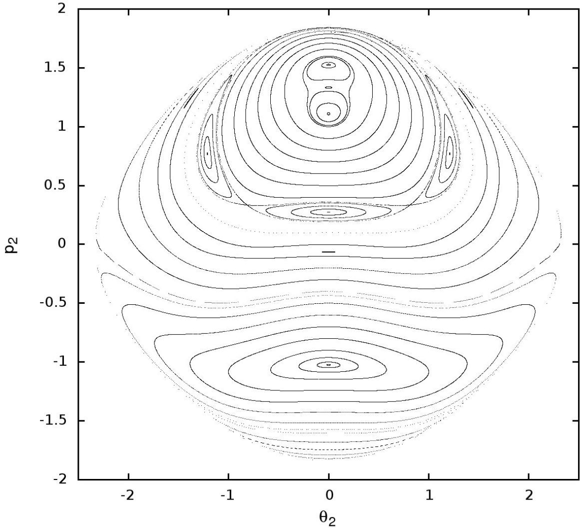


Figure: $E = -7.3$: The invariant tori becomes deformed.

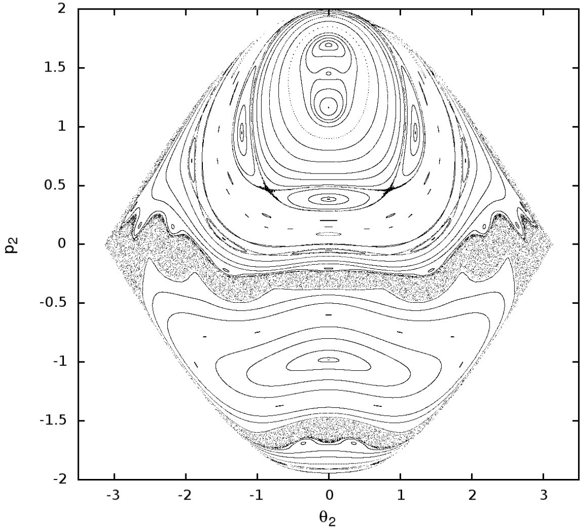


Figure: $E = -7.0$: The first appearance of chaotic region.

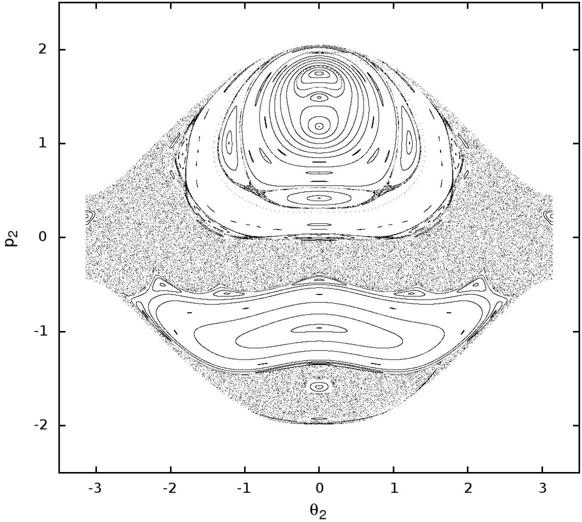


Figure: $E = -6.9$: The size of chaotic region increases.

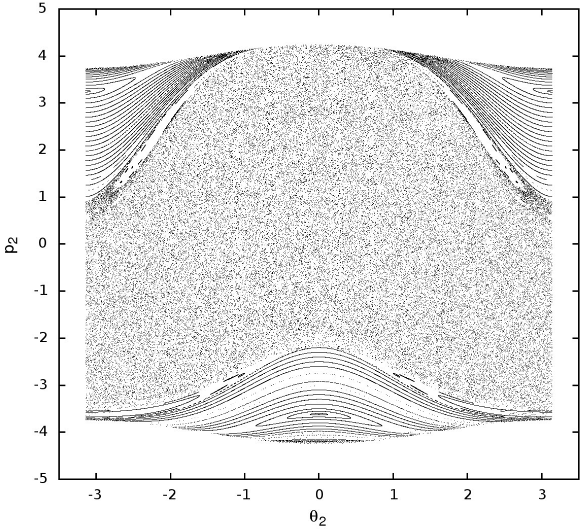


Figure: Poincare section for $E = 0$: the global chaos in the center.

$$E = 0,\ \theta_1 = 0,\ \theta_2 = 3.14,\ p_1 = 0.003,\ p_2 = 3.24. \qquad (5)$$

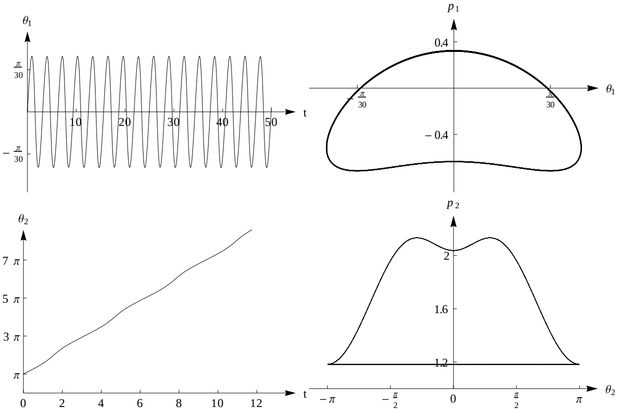
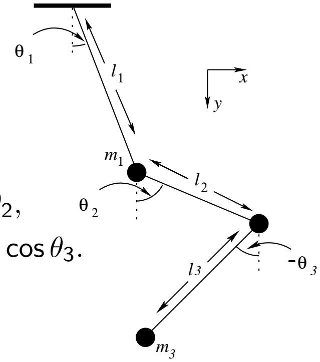


Figure: (a) Fast oscillations of the first pendulum around the equilibrium point, (b) rotations of the second pendulum around its fixed point.

The simple triple pendulum

The simple triple pendulum



$$x_1 = l_1 \sin \theta_1, \qquad y_1 = l_1 \cos \theta_1,$$
$$x_2 = x_1 + l_2 \sin \theta_2, \qquad y_2 = y_1 + l_2 \cos \theta_2,$$
$$x_3 = x_1 + x_2 + l_3 \sin \theta_3, \qquad y_3 = y_1 + y_2 + l_3 \cos \theta_3.$$

The Lagrange function for this system in the absence of gravity looks as follows

$$L = T = \frac{1}{2}\left\{\dot\theta_3^2 l_3^2 m_3 + \dot\theta_2^2 l_2^2 (m_2 + m_3) + \dot\theta_1^2 l_1^2 (m_1 + m_2 + 4m_3)\right.$$
$$+2\dot\theta_1 l_1 \left[\dot\theta_2 l_2 (m_2 + 2m_3)\cos(\theta_1 - \theta_2)\right.$$
$$\left.\left. +2\dot\theta_3 l_3 m_3 \cos(\theta_1 - \theta_3)\right] + 2\dot\theta_2\dot\theta_3 l_2 l_3 m_3 \cos(\theta_2 - \theta_3)\right\}. \tag{6}$$

$$L = T = \frac{1}{2}\left\{\dot{\theta}_3^2 l_3^2 m_3 + \dot{\theta}_2^2 l_2^2 (m_2 + m_3) + \dot{\theta}_1^2 l_1^2 (m_1 + m_2 + 4m_3)\right.$$
$$+2\dot{\theta}_1 l_1 \left[\dot{\theta}_2 l_2 (m_2 + 2m_3)\cos(\theta_1 - \theta_2)\right.$$
$$\left.\left.+2\dot{\theta}_3 l_3 m_3 \cos(\theta_1 - \theta_3)\right] + 2\dot{\theta}_2\dot{\theta}_3 l_2 l_3 m_3 \cos(\theta_2 - \theta_3)\right\}. \tag{6}$$

Because the Lagrangian in Eq.(6) depends on the cosine of the angles difference, we introduce new variables

$$\begin{aligned} \gamma_1 &= \theta_2 - \theta_1, \\ \gamma_2 &= \theta_3 - \theta_2, \\ \gamma_3 &= \theta_1. \end{aligned} \tag{7}$$

the Lagrange functions in the new variables looks as follows

$$L = \frac{1}{2}\left\{2\dot{\gamma}_3 l_1 [2 l_3 m_3 \cos(\gamma_1+\gamma_2)(\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3) + l_2 \cos\gamma_1 (\dot{\gamma}_1+\dot{\gamma}_3) \right.$$
$$\times\ (m_2+2m_3)] + 2 l_2 l_3 m_3 \cos\gamma_2 (\dot{\gamma}_1+\dot{\gamma}_3)(\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3) + l_3^2 m_3$$
$$\left.\times\ (\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3)^2 + l_2^2(\dot{\gamma}_1+\dot{\gamma}_3)^2(m_2+m_3) + \dot{\gamma}_3^2 l_1^2 (m_1+m_2+4m_3)\right\}.$$

the Lagrange functions in the new variables looks as follows

$$L = \frac{1}{2}\{2\dot{\gamma}_3 l_1[2l_3 m_3 \cos(\gamma_1+\gamma_2)(\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3) + l_2\cos\gamma_1(\dot{\gamma}_1+\dot{\gamma}_3)$$
$$\times\ (m_2+2m_3)] + 2l_2 l_3 m_3 \cos\gamma_2(\dot{\gamma}_1+\dot{\gamma}_3)(\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3) + l_3^2 m_3$$
$$\times\ (\dot{\gamma}_1+\dot{\gamma}_2+\dot{\gamma}_3)^2 + l_2^2(\dot{\gamma}_1+\dot{\gamma}_3)^2(m_2+m_3) + \dot{\gamma}_3^2 l_1^2(m_1+m_2+4m_3)\}\,.$$

Wee see that the Lagrangian in new variables does not depend explicitly on the variable γ_3, which leads $p_3 = b = const$. Then, the Hamiltonian depends only on four variables γ_1, γ_2. and p_1, p_2, and b is parameter. By means of these new variables we reduced the number of degrees of freedom from 3 to 2.

We are going to analyse the dynamics of the considered system for the following constant parameters:

$$p_3 = b = 1,\ m_1 = 2,\ m_2 = 1,\ m_3 = 1,\ l_1 = 2,\ l_2 = 1,\ l_3 = 1.$$

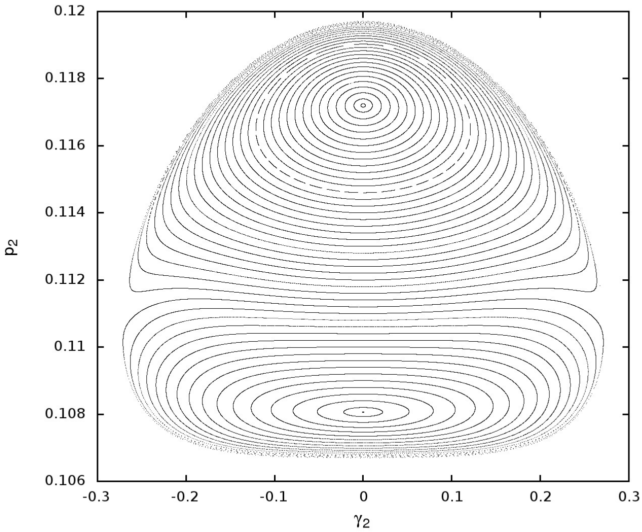


Figure: The Poincaré section for $E = 0.0095$: regular behaviour.

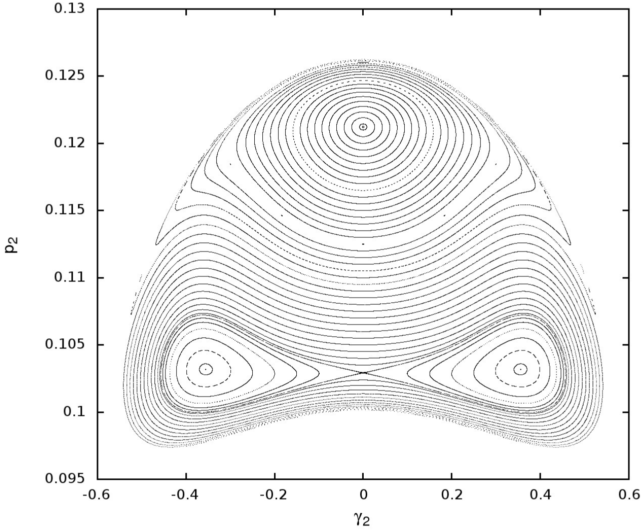


Figure: $E = 0.0097$: the first sign of chaotic motion.

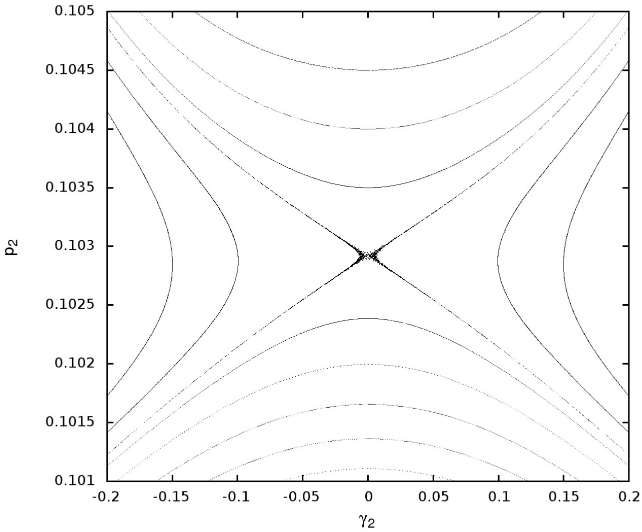


Figure: The enlargement of lower region of previous Poincaré section.

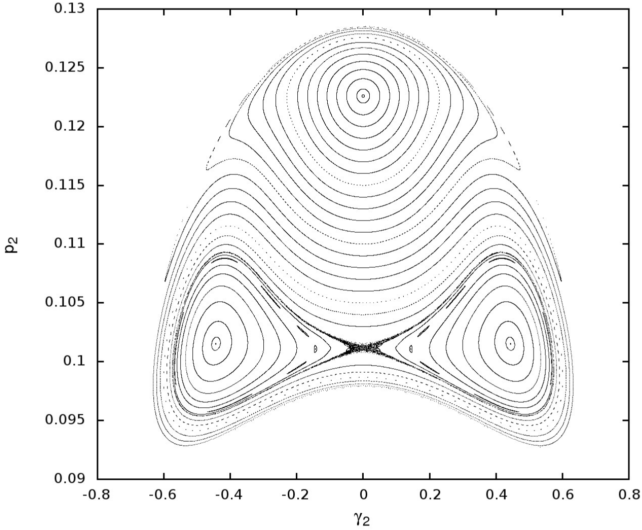


Figure: T $E = 0.0098$: invariant tori becomes deformed.

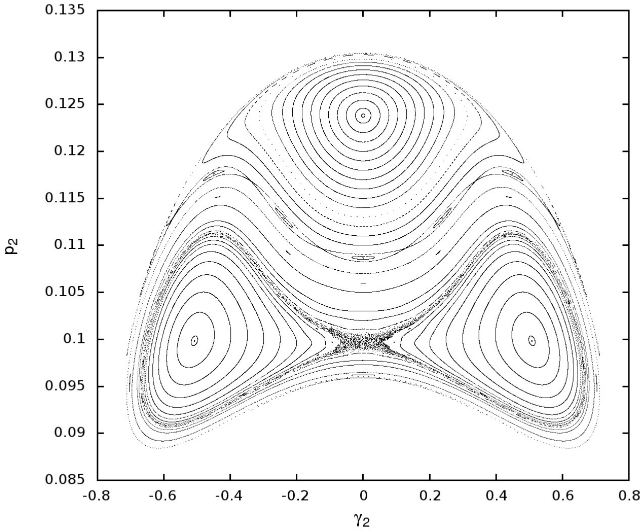


Figure: The Poincaré section for $E = 0.0099$: invariant tori becomes deformed.

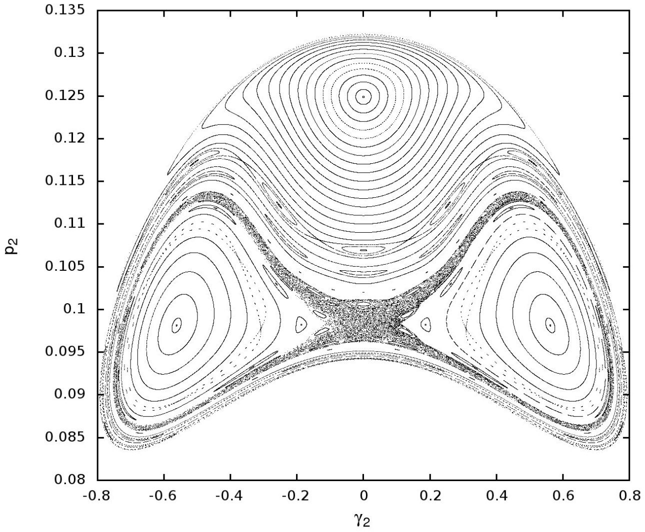


Figure: The Poincaré section for $E = 0.01$: invariant tori becomes deformed.

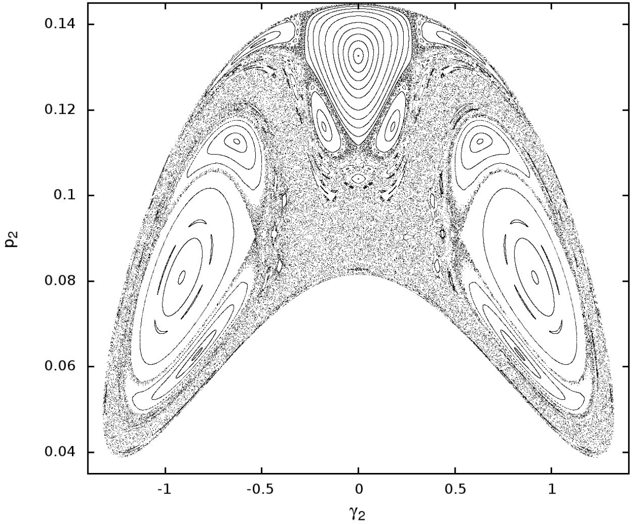


Figure: The Poincaré section for $E = 0.011$: invariant tori becomes deformed.

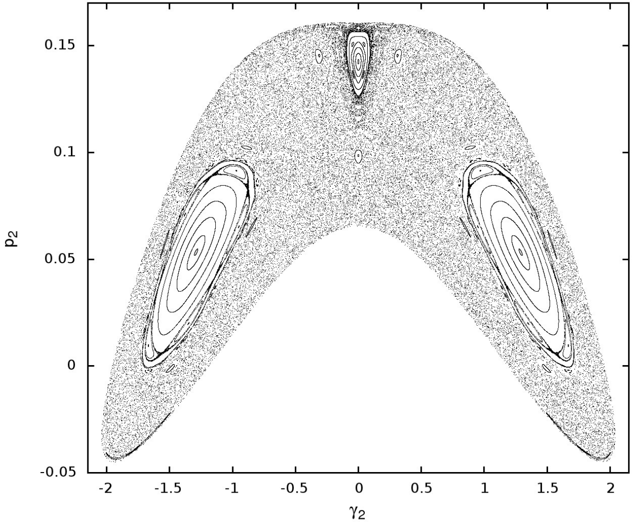
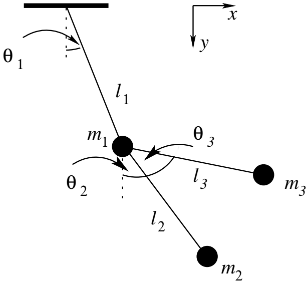


Figure: The Poincaré section for $E = 0.013$: invariant tori becomes deformed.

The "Flail"

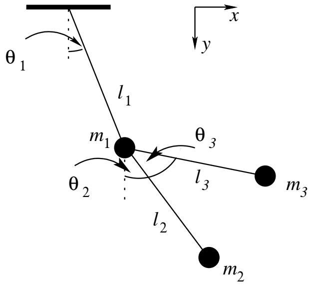
THE "FLAIL"



$$x_1 = l_1 \sin\theta_1, \qquad y_1 = l_1\cos\theta_1,$$
$$x_2 = x_1 + l_2 \sin\theta_2, \qquad y_2 = y_1 + l_2\cos\theta_2,$$
$$x_3 = x_1 + l_3 \sin\theta_3, \qquad y_3 = y_1 + l_3\cos\theta_3.$$

$$L = T = \frac{1}{2}\Big[\dot\theta_2^2 l_2^2 m_2 + \dot\theta_3^2 l_3^2 m_3 + \dot\theta_1^2 l_1^2$$
$$(m_1+m_2+m_3) + 2\dot\theta_1 l1\left(\dot\theta_2 l_2 m_2\cos(\theta_1-\theta_2) + \dot\theta_3 l_3 m_3 \cos(\theta_1-\theta_3)\right)\Big].$$

The "Flail"



$$x_1 = l_1 \sin\theta_1, \qquad y_1 = l_1 \cos\theta_1,$$
$$x_2 = x_1 + l_2 \sin\theta_2, \qquad y_2 = y_1 + l_2 \cos\theta_2,$$
$$x_3 = x_1 + l_3 \sin\theta_3, \qquad y_3 = y_1 + l_3 \cos\theta_3.$$

$$L = T = \frac{1}{2}\left[\dot{\theta}_2^2 l_2^2 m_2 + \dot{\theta}_3^2 l_3^2 m_3 + \dot{\theta}_1^2 l_1^2\right.$$
$$\left.(m_1 + m_2 + m_3) + 2\dot{\theta}_1 l1 \left(\dot{\theta}_2 l_2 m_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_3 l_3 m_3 \cos(\theta_1 - \theta_3)\right)\right].$$

Introducing the new variables

$$\begin{aligned} \gamma_1 &= \theta_2 - \theta_1, \\ \gamma_2 &= \theta_3 - \theta_2, \\ \gamma_3 &= \theta_1. \end{aligned} \tag{8}$$

Introducing the new variables

$$\begin{aligned} \gamma_1 &= \theta_2 - \theta_1, \\ \gamma_2 &= \theta_3 - \theta_2, \\ \gamma_3 &= \theta_1. \end{aligned} \tag{8}$$

Then, substituting into the Eq.(8) we get the Lagrange functions in the new variables

$$\begin{aligned} L = T = \frac{1}{2} \Big[& \left(\dot{\gamma}_1 + \dot{\gamma}_3\right)^2 l_2^2 m_2 + \left(\dot{\gamma}_1 + \dot{\gamma}_1 + \dot{\gamma}_3\right)^2 l_3^2 m_3 + \dot{\gamma}_3^2 l_1^2 \\ \times & \left(m_1 + m_2 + m_3\right) + 2\dot{\gamma}_3 l_1 \left(\left(\dot{\gamma}_1 + \dot{\gamma}_3\right) l_2 m_2 \cos\gamma_1 + \left(2 + \dot{\gamma}_2 + \dot{\gamma}_3\right)\right. \\ \times & \left. l_3 m_3 \cos\left(\gamma_1 + \gamma_2\right)\right) \Big] . \end{aligned} \tag{9}$$

Like previous, the Lagrange functions does not depend explicitly on variable γ_3. Therefore the momentum $p_3 = C$ is a parameter and the Hamiltonian depends only on four variables γ_1, γ_2, and p_1, p_2.

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$$p_3 = b = 1,\ m_1 = 1,\ m_2 = 3,\ m_3 = 2,\ l_1 = 1,\ l_2 = 2,\ l_3 = 3. \tag{10}$$

and we choice the cross section plane $\gamma_1 = 0,\ p_1 > 0$.

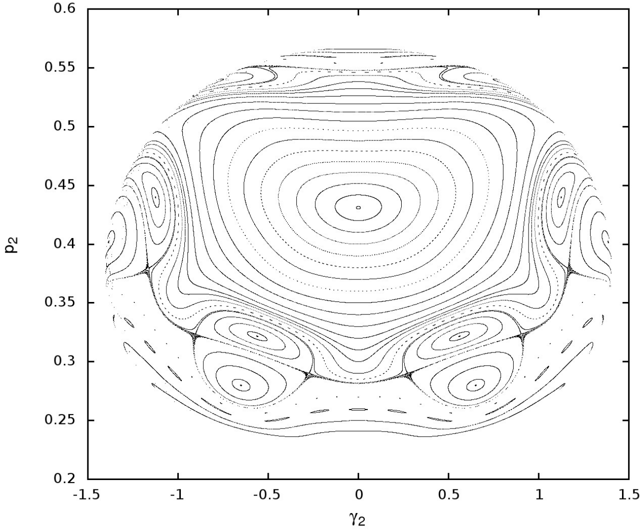

Figure: $E = 0.01$: detected three types of motion.

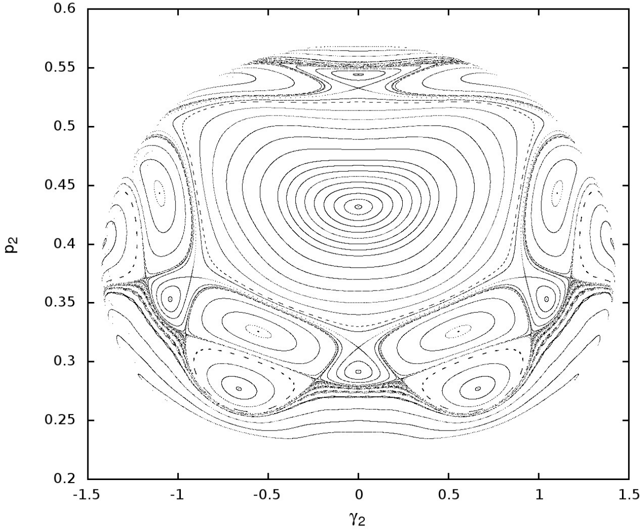

Figure: $E = 0.01005$: the newly rising stable periodic solution.

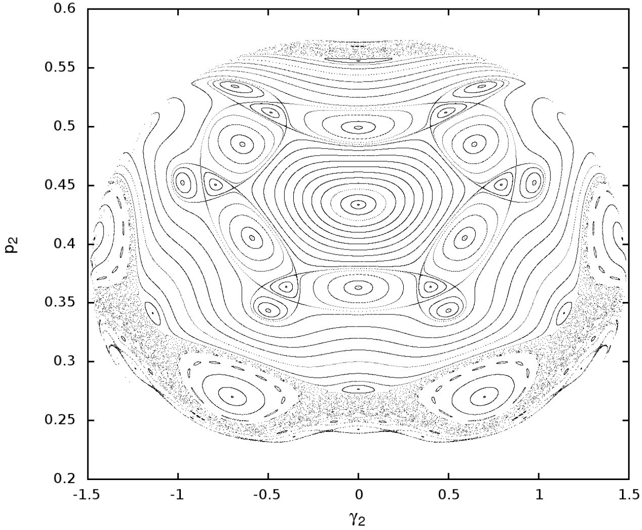

Figure: $E = 0.0102$: three clusters of stable periodic solutions.

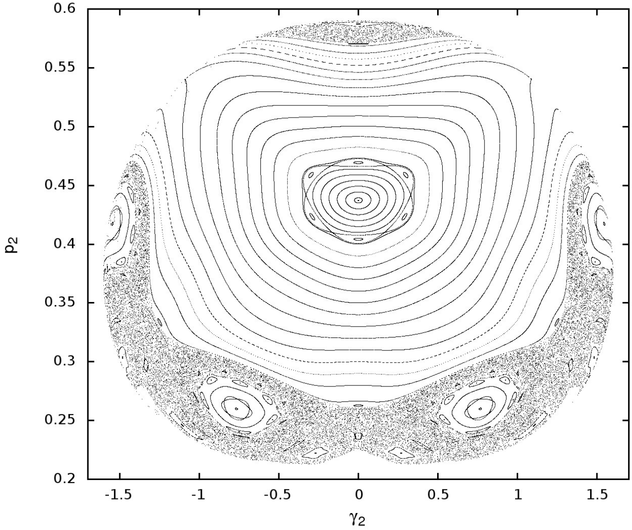

Figure: The Poincaré section for $E = 0.0105$.

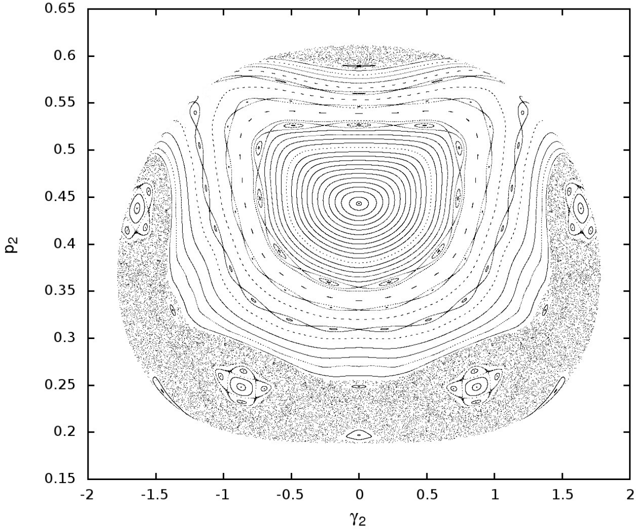

Figure: $E = 0.011$: the newly rising "neckles" formations .

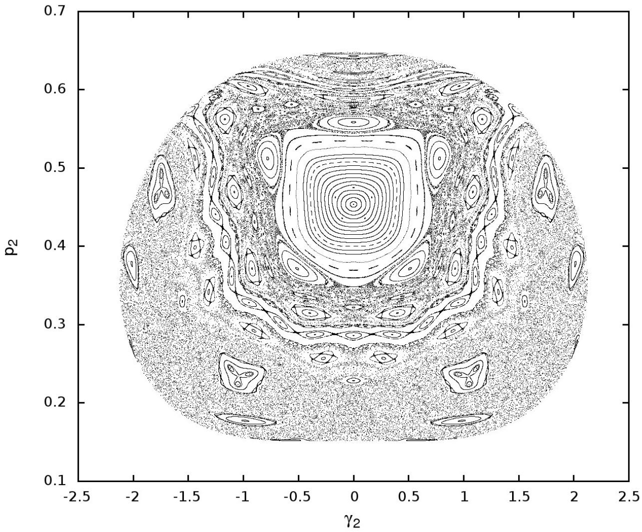

Figure: The Poincaré section for $E = 0.012$.

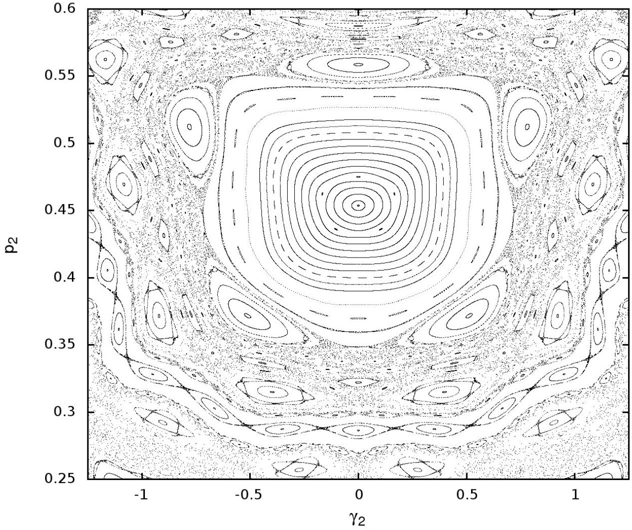

Figure: $E = 0.012$: the enlargement of the central region.

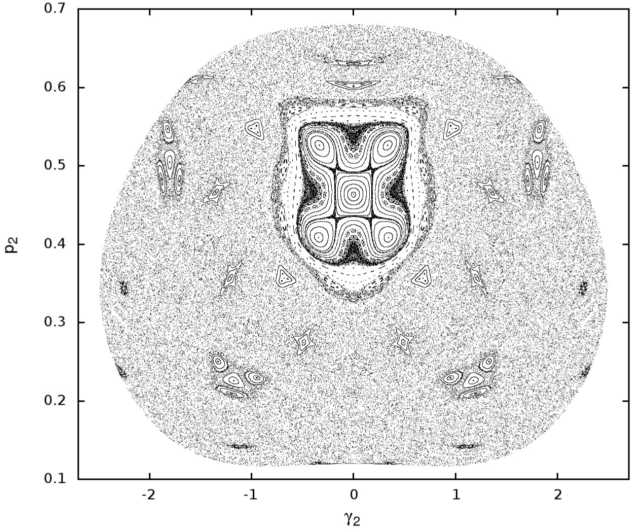

Figure: The Poincaré section for $E = 0.013$.

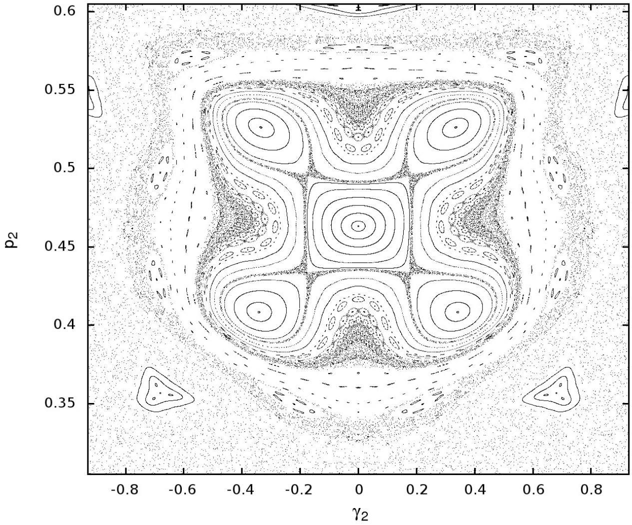

Figure: $E = 0.013$: the enlargement of the central region.

SOME OTHER EXAMPLES

$$p_3 = b = 1/2,\ m_1 = 1,\ m_2 = 2,\ m_3 = 2,\ l_1 = 2,\ l_2 = 1,\ l_3 = 1. \tag{11}$$

and we choice the cross section plane $\gamma_1 = 0,\ p_1 > 0$.

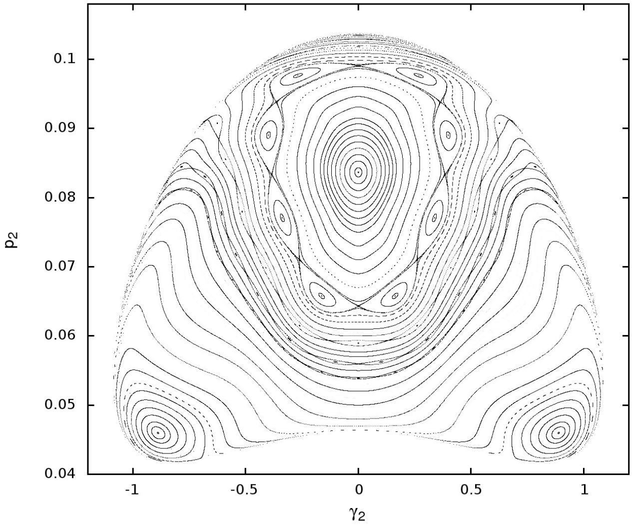

Figure: The Poincaré section for $E = 0.0035$.

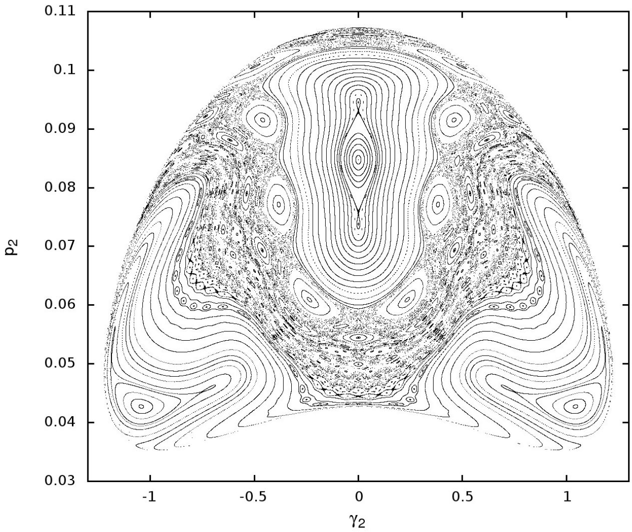

Figure: The Poincaré section for $E = 0.0036$.

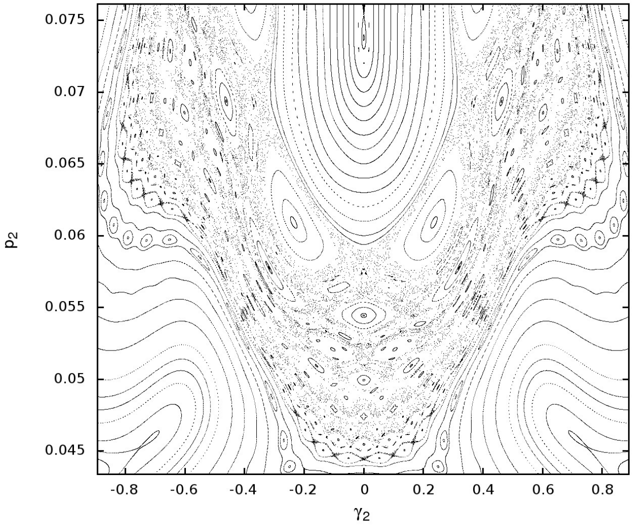

Figure: The enlargement of the central part of previous Poincaré section.

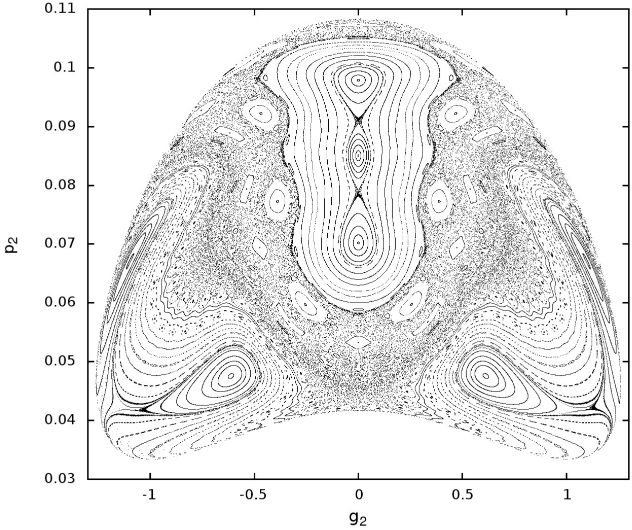

Figure: The Poincaré section for $E = 0.00363$.

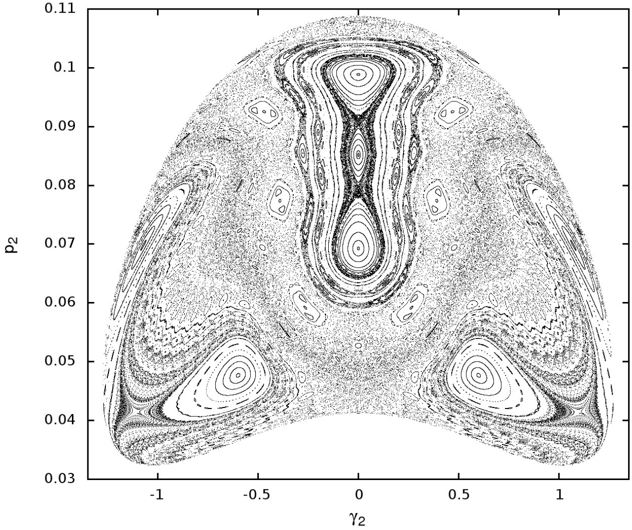

Figure: The Poincaré section for $E = 0.003645$.

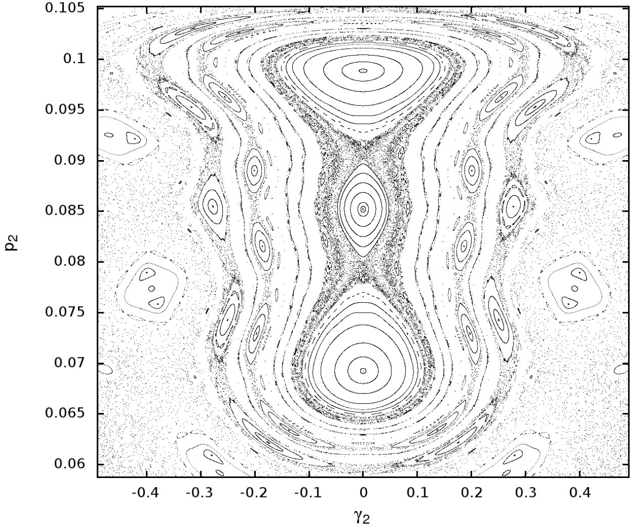

Figure: The enlargement of upper part of previous Poincaré section

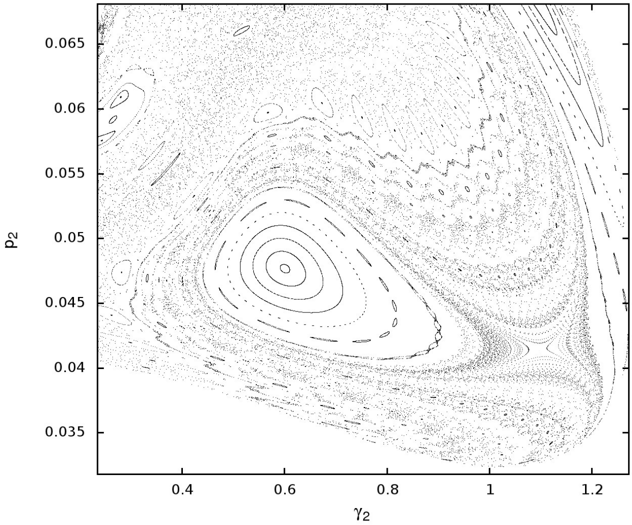

Figure: The enlargement of right corner of previous Poincaré section.

Triple "flail" pendulum without gravity

Problem

Problem: how to find values of parameters m_1, m_2, m_3 and l_1, l_2 and l_3 for that system is integrable?
Answer: To apply the Morales-Ramis theory!

When

$$m_2 l_2 = m_3 l_3$$

then exists particular solution $\theta_1 = p_{\theta_1} = 0$, $\theta_3 = -\theta_2$ and $p_{\theta_3} = -l_3 p_{\theta_2}/l_2$

$$(\dot{\theta}_1, \dot{\theta}_2, \dot{p}_{\theta_1}, \dot{p}_{\theta_2}) = \left(0, \frac{p_{\theta_2}}{l_2^2 m_2}, -\frac{p_{\theta_2}}{l_2^2 m_2}, 0, 0, 0\right). \qquad (12)$$

Main results

Theorem

Triple "flail" pendulum satisfying

$$m_2 l_2 = m_3 l_3$$

is non-integrable in the class of meromorphic functions except the case $m_1 = 0$.

Lemma

Differential Galois group of normal variational equations for $m_1 = 0$ *is a finite group or full* $\mathrm{SL}(2, \mathbb{C})$.

What for triple “flail’ pendulum in gravity field?

- In the case
$$l_2 = l_3, \qquad m_2 = m_3$$
a non-equilibrium particular solution is known.
- System of normal variational equations has dimension four.
- Normal variational equations can be transformed into a fourth order linear equation with rational coefficients.
- Non-trivial problem: how to find its differential Galois group?
- Work in progress.