Existence of exact solutions of the Dirac equation

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- Introduction: Function fields, extensions, monodromy
- Differential Galois group solvability
- The case of the Dirac equation
- Summary

Field extensions...

$\mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{C}$

Extension of a field K to field F: F/K, $F \supset K$

eg. Adjoining an algebraic element

$x^2 + 1 = 0 \rightarrow \mathbb{R}(i) = \mathbb{C} \simeq \mathbb{R}[x]/(x^2 + 1)$

Symmetries – group of root permutations

$x \rightarrow \bar{x}: \ i \rightarrow -i, \ \mathbb{R} \rightarrow \mathbb{R}$

Field extensions...

Galois group:

$$Gal(F/K) = \{ g \in Aut(F) : g|_K = Id \}$$

Trivial action on the „smaller" field
or conservation of the equation coefficients

$$x^n + a_{n-1} x^{n-1} + \dots a_0 = (x - e_1)(x - e_2)\dots(x - e_n)$$

$$a_i \in \mathbb{R},\ e_i \in \mathbb{C},\ G \subset S_n$$

Field extensions...

Solvable group:

$$G = G_0 \supset G_1 \supset \ldots G_n = \{1\}$$

Normal subgroups and abelian G_i / G_{i-1}

$$F = F_0 \subset F_1 \subset \ldots F_n, \; p \in F[x], \; e_i \in F_n$$

Tower of fields obtained with simple extensions

$$F_{i+1} = F_i(\alpha), \; \alpha^m \in F_i$$

Field extensions...

Analogy of polynomials

$$z^n + a z^{n-1} + \ldots + c z + d = 0 \qquad a, c, d \in \mathbb{Q}$$

With linear differential equations

$$\frac{d^n}{dz^n} f + a(z) \frac{d^{n-1}}{dz^{n-1}} f + \ldots + d(z) f = 0$$

What kind of field and solvability?

Field extensions...

Differential field : $K = \{\mathbb{C}(z), \partial\}$

Eg. Rational functions, elliptic functions of the same periods.

Extension by adjoing functions as elements, not function composition.

$$F/K : F = K(f(z))$$

$$f(z) = \sin(z) \Rightarrow \cos(z), tg(z) \in F$$

$$\arcsin(z), \sin(1/z) \notin F$$

Field extensions...

Differential Galois group:

$$Gal_d(F/K)K=\{g\in Gal(F/K):[g,\partial]=0\}$$

The identity component is invariant with respect to finite coverings

$$f''(z)=(A\wp(z)+B)f(z),\ \wp(z)=\zeta$$

$$\downarrow$$

$$f''(\zeta)+q(\zeta)f'(\zeta)+p(\zeta)f(\zeta)=0,\ p,q\in\mathbb{C}(\zeta)$$

Field extensions...

What constitutes an „exact” solution?

1. All algebraic functions:

$$F_1 = K(y):\ P(y) = 0,\ P(x) \in K[x]$$

2. „Ordinary” integrals:

$$F_2 = K(y):\ \partial y \in K \Leftrightarrow y = \int v,\ v \in K$$

3. Solutions of first order equations:

$$F_3 = K(y):\ \partial y = w\, y,\ w \in K \Leftrightarrow y = \exp\left(\int w\right)$$

Field extensions...

Basic elements of new types

1. Radicals and polynomial solutions

$$\frac{\sqrt{z} + \sqrt[5]{x}}{\sqrt{x^2 + 1}}$$

2. Elliptic integrals, logarithms

$$\ln(x) = \int \frac{dx}{x}, \int \frac{dx}{\sqrt{x^4 + 1}}$$

3. Trigonometric functions

$$\sin(a z), \exp(-z^2)$$

Field extensions…

(Generalised) Liouvillian extension: adding a finite number of new elements of type 1, 2 or 3.

If the extension by adding the basis of solutions of L is (generalised) Liouvillian, then the (identity component of) differential Galois group is solvable.

$$L(y) = y'' + p(z)y' + q(z)y = 0,\ \ p, q \in \mathbb{C}(z)$$

$$G = Gal_D(K(y_1, y_2)/K)$$

Field extensions...

Galois group and analytic continuation

$$z^2 y'' - z y' + y = 0, \quad y = c_1 z \ln(z) + c_2 z$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Monodromy group is dense in the Galois group for Fuchsian equations

Dirac equation...

$$i\hbar\partial_t\psi=(\alpha p+m\beta)\psi$$

1 dimension + scalar coupling of a potential

$$E\psi=\left(-i\alpha\partial_x+(m+V)\beta\right)\psi$$

$$\alpha=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\ \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\psi_{1,2}''=\left(\pm V'+V^2-E^2\right)\psi_{1,2}$$

Dirac equation...

Potential in vector coupling: $m + V \rightarrow E + V$

Duality:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; V \rightarrow -iV, \; E \rightarrow -im, \; m \rightarrow iE$$

$$\boxed{\psi''(x) = r(x)\psi, \; r(x) \in \mathbb{C}[x]}$$

Solvability...

$$\frac{d}{dx}\begin{pmatrix}\psi\\ \psi'\end{pmatrix}=\begin{pmatrix}0 & 1\\ r(x) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix}$$

$$\begin{vmatrix}\psi_1 & \psi_2\\ \psi_1' & \psi_2'\end{vmatrix}=const \Rightarrow G\subset SL(2,\mathbb{C})$$

In fact, G is an algebraic subgroup consisting of separate connected components

Solvability...

If the identity component of G defined above is solvable, then G can be:

1. Finite
2. Triangularisable
3. Conjugate to a subgroup of:

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & d \\ d^{-1} & 0 \end{pmatrix},\ c, d \in \mathbb{C}_0 \right\}$$

Solvability...

Kovacic algorithm: check for invariants in each of 3 cases, eg.:

$$\sigma(y_1) = c\, y_{1,} \Rightarrow \theta = \frac{y_1{}'}{y_1} : \sigma(\theta) = \theta, \ \sigma \in G$$

Which leads to analysis of Riccati equation

$$\theta' + \theta^2 = r(x), \ \theta \in \mathbb{C}(x)$$

Dirac equation...

$r(x)$ is an even polynomial (the only pole lies at infinity) – only the simplest case is possible

$$\psi = P \exp\left(\int \omega\, dx\right),\ P \in \mathbb{C}[x],\ \omega \in \mathbb{C}(x)$$

For zero energy: $$\psi = \exp\left(\int V\, dx\right)$$

For deg(V) = 1: $$\psi = \exp(-z^2) H_n(z)$$

The rest unsolvable: *JMathPhys 52, 012301*

Summary

- Linear differential equations naturally fit in with the Galois theory
- There is an effective algorithm for order 2
- Liouvillian solutions correspond to intuitive solvability

- Possible to extend to other function fields
- Analysis of nonlinear systems, especially Hamiltonian