Application of residue calculus to integrability analysis of rational potentials

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Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

$$V \in \mathbb{C}(\boldsymbol{q}), \qquad \deg V = k \in \mathbb{Z},$$

V — homogeneous function i.e.

$$V(\lambda q_1, \ldots, \lambda q_n) = \lambda^k V(q_1, \ldots, q_n).$$

Search of integrable potentials

Problem

$$V = \sum_{i_1,\ldots,i_n} v_{i_1\cdots i_n} q_1^{i_1}\cdots q_n^{i_n},$$

where $i_1,\ldots,i_n \in \{0,1,\ldots,k\}$ and the sum is taken over such elements that $i_1+\cdots+i_n = k$.
How to find coefficients $v_{i_1\cdots i_n}$ for which potential is integrable?

Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ*. Then the identity component of the differential Galois group of the variational equations along* Γ *is Abelian.*

Other Morales-Ramis Theorem

If the Hamiltonian system with homogeneous potential is meromorphically integrable then each (k, λ_i) belong to the following list:

case	k	λ	
1.	± 2	λ	
2.	k	$p + \frac{k}{2}p(p-1)$	
3.	k	$\frac{1}{2}\left(\frac{k-1}{k} + p(p+1)k\right)$	
4.	3	$-\frac{1}{24} + \frac{1}{6}(1+3p)^2$,	$-\frac{1}{24} + \frac{3}{32}(1+4p)^2$
		$-\frac{1}{24} + \frac{3}{50}(1+5p)^2$,	$-\frac{1}{24} + \frac{6}{25}(1+5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1+3p)^2$	

Morales-Ramis table

case	k	λ	
6.	5	$-\frac{9}{40}+\frac{5}{18}(1+3p)^2$,	$-\frac{9}{40}+\frac{2}{5}(1+5p)^2$
7.	-3	$\frac{25}{24}-\frac{1}{6}(1+3p)^2$,	$\frac{25}{24}-\frac{3}{32}(1+4p)^2$
		$\frac{25}{24}-\frac{3}{50}(1+5p)^2$,	$\frac{25}{24}-\frac{6}{25}(1+5p)^2$
8.	-4	$\frac{9}{8}-\frac{2}{9}(1+3p)^2$	
9.	-5	$\frac{49}{40}-\frac{5}{18}(1+3p)^2$,	$\frac{49}{40}-\frac{2}{5}(1+5p)^2$

where p is an integer and λ an arbitrary complex number.

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, Birkhäuser, 1999.

Weakness of this theorem in applications

$$V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{3}cq_2^3.$$

$$\lambda_1 = \frac{1}{c}, \qquad \lambda_{2,3} = \frac{2c-1}{1+a^2 \mp \Delta}, \qquad \Delta = \sqrt{a^2(2+a^2-2c)}$$

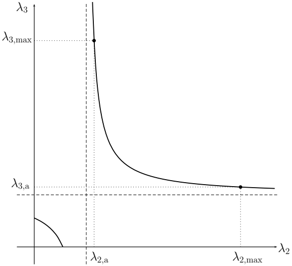
$$\lambda_1, \lambda_2, \lambda_3 \in \mathcal{M}_3 := \left\{p + \frac{3}{2}p(p-1)\right\} \cup \left\{\frac{1}{2}\left(\frac{2}{3} + 3p(p+1)\right)\right\}$$
$$\cup \left\{-\frac{1}{24} + \frac{1}{6}(1+3p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{32}(1+4p)^2\right\}$$
$$\cup \left\{-\frac{1}{24} + \frac{3}{50}(1+5p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{50}(2+5p)^2\right\}.$$

$$c = \frac{1}{\lambda_1}, \quad a = \pm\sqrt{\frac{(\lambda_1 + \lambda_1\lambda_i - 2)^2}{2\lambda_1\lambda_i(2-\lambda_1-\lambda_i)}}, \qquad i = 2, 3.$$

'A parametric problem'

Observation

$$\frac{1}{\lambda_1 - 1} + \frac{1}{\lambda_2 - 1} + \frac{1}{\lambda_3 - 1} = -1$$



Observation

There is at most a finite number of choices for λ_1, λ_2 and λ_3 from the Morales-Ramis table!!!

Our observations for $n = 2$

For $n = 2$, each $[\boldsymbol{d}] \in \mathcal{D}^\star(V)$ gives one non-trivial eigenvalue $\lambda(\boldsymbol{d})$. Set $\Lambda(\boldsymbol{d}) = \lambda(\boldsymbol{d}) - 1$.

Theorem

Assume that a polynomial homogeneous potential $V(q_1, q_2)$ of degree $k > 2$ satisfies two conditions:

- **C1** *it has $0 < l \leq k$ Darboux points and all of them are simple;*
- **C2** *if W or U is factorisable by $(q_2 \pm \mathrm{i} q_1)$, then multiplicity of this factor is one.*

Then the following relation holds

$$\sum_{[\boldsymbol{d}] \in \mathcal{D}^\star(V)} \frac{1}{\Lambda(\boldsymbol{d})} = -1.$$

Our results for an arbitrary n

$$\mathcal{D}^\star(V) \ni [\boldsymbol{d}] \longmapsto \boldsymbol{\Lambda}(\boldsymbol{d}) = (\Lambda_1(\boldsymbol{d}), \dots, \Lambda_{n-1}(\boldsymbol{d}))$$

where $\lambda_i(\boldsymbol{d}) := \Lambda_i(\boldsymbol{d}) + 1$, are the non-trivial eigenvalues of $V''(\boldsymbol{d})$.
τ_i is the elementary symmetric polynomial of degree i in $(n-1)$ variables.

Our results for an arbitrary n

Theorem

Let $V \in \mathbb{C}_k[\boldsymbol{q}]$ be a homogeneous potential of degree $k > 2$ and let all its Darboux points be proper and simple. Then

$$\sum_{[\boldsymbol{d}]\in\mathcal{D}^\star(V)} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n-1}(-n-(k-2))^r,$$

and

$$\sum_{[\boldsymbol{d}]\in\mathcal{D}^\star(V)} \frac{\tau_r(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{r+n-1}\sum_{i=0}^{r}\binom{n-i-1}{r-i}(k-1)^i,$$

for $r = 0, \ldots, n-1$.

Our results for an arbitrary n

Theorem

For a generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ of degree k set of admissible $\{ \boldsymbol{\Lambda}(\boldsymbol{d}) \mid [\boldsymbol{d}] \in \mathcal{D}^\star(V) \} =: \mathcal{I}_{n,k}$ is finite.

$\Downarrow$ +many other things

New integrable potentials for $k = n = 3$

Do exist relations for rational potentials?

$$V = \frac{W(\boldsymbol{q})}{U(\boldsymbol{q})},$$

- $W, U \in \mathbb{C}[\boldsymbol{q}], \quad \deg W = r, \quad \deg U = s,$
- Assumption: W and U are relatively prime,
- V homogeneous function of degree $k = r - s \in \mathbb{Z}$,

Question: Do exist relations for rational potentials?

Darboux points for rational potentials

Definition

$[\boldsymbol{d}] \in \mathbb{CP}^{n-1}$ is a Darboux point of the potential $V = W/U$ iff

1. $$\boldsymbol{d} \wedge V'(\boldsymbol{d}) = \frac{1}{U^2(d)} \boldsymbol{d} \wedge \left(W'(\boldsymbol{d}) U(\boldsymbol{d}) - U'(\boldsymbol{d}) W(\boldsymbol{d}) \right) = 0$$

   or

   $$V'(\boldsymbol{d}) = \gamma \boldsymbol{d},$$

   for some $\gamma \in \mathbb{C}$
2. $V(\boldsymbol{q})$ is well defined at $\boldsymbol{q} = \boldsymbol{d}$
3. all components of $V'(\boldsymbol{q})$ are well defined at $\boldsymbol{q} = \boldsymbol{d}$

Integrability obstructions due to improper Darboux points

Theorem

Assume that homogeneous potential $V \in \mathbb{C}(\boldsymbol{q})$ of degree $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ possesses an improper Darboux point $\boldsymbol{d}$. If V is integrable in Liouville sense with rational first integrals, then all eigenvalues of $V''(\boldsymbol{d})$ vanish.

Relation for $n = 2$

Theorem

Assume that a rational homogeneous potential $V(q_1, q_2)$ of degree $k \in \mathbb{Z}$ satisfies three conditions:

- **C1** *it has $0 < l \leq r + s$ proper Darboux points and all of them are simple;*
- **C2** *U is not factorisable neither by $(q_2 + \mathrm{i}q_1)$, nor by $(q_2 - \mathrm{i}q_1)$;*
- **C3** *if W is factorisable by $(q_2 + \mathrm{i}q_1)$, or by $(q_2 - \mathrm{i}q_1)$ then multiplicity of these factor is one.*

Then

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1.$$

Generalised Relation for $n = 2$. Notation

$$\theta_{x,y} := \begin{cases} 0 & \text{for} \quad x < y, \\ 1 & \text{for} \quad x \geq y. \end{cases}$$

Let $r_\pm$ and $s_\pm$ be the respective multiplicities of linear factors $(q_2 \pm \mathrm{i} q_1)$ of W and U, respectively.

Generalised Relation for $n = 2$

Theorem

Assume that a rational homogeneous potential $V(q_1, q_2)$ of degree $k = r - s \in \mathbb{Z}$ satisfies three conditions:

- **C1** *it has $0 < l \leq r + s$ proper Darboux points and all of them are simple;*
- **C2** *neither r_+, nor r_- is equal to $k/2$;*
- **C3** *neither s_+, nor s_- is equal to $-k/2$.*

Then

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1 - \theta_{r_+,2} \frac{r_+}{k - 2r_+} - \theta_{r_-,2} \frac{r_-}{k - 2r_-} + \theta_{s_+,1} \frac{s_+}{k + 2s_+} + \theta_{s_-,1} \frac{s_-}{k + 2s_-}.$$

Darboux points in affine coordinates

affine coordinate $z = q_2/q_1$

conditions on Darboux point

$$\big(\boldsymbol{q} \wedge V'(\boldsymbol{q})\big)_1 = q_1 \frac{\partial V}{\partial q_2} - q_2 \frac{\partial V}{\partial q_1} = \frac{q_1^{r-s}}{u(z)^2} g(z),$$

$$g(z) = (1 + z^2) \left[w'(z)u(z) - u'(z)w(z)\right] - kzw(z)u(z),$$

$$\frac{\partial V}{\partial q_1} = \frac{q_1^{r-s-1}}{u^2} h(z), \; h(z) = kw(z)u(z) - z \left[w'(z)u(z) - u'(z)w(z)\right]$$

$$\frac{\partial V}{\partial q_2} = \frac{q_1^{r-s-1}}{u^2} \left[w'(z)u(z) - u'(z)w(z)\right]$$

Sketch of the proof of theorem on relation

- nontrivial eigenvalue of Hessian

$$\Lambda = \frac{g'}{h}.$$

- meromorphic form that takes form in affine part of $\mathbb{CP}^1$

$$\omega = \frac{h(z)}{g(z)}\mathrm{d}z.$$

- Application of global residue theorem for meromorphic one-form ω on $\mathbb{CP}^1$
- residues calculated at Darboux points are $1/\Lambda_i$
- residue at the infinity is 1.

Finiteness theorem

Theorem

Let us consider relation

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1, \quad \textit{or}$$

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1 - \theta_{r_+,2}\frac{r_+}{k-2r_+} - \theta_{r_-,2}\frac{r_-}{k-2r_-} + \theta_{s_+,1}\frac{s_+}{k+2s_+} + \theta_{s_-,1}\frac{s_-}{k+2s_-}.$$

as an equation for $(\Lambda_1, \ldots, \Lambda_l) \in \underbrace{\mathbb{C}^\star \times \cdots \times \mathbb{C}^\star}_{l}$*. Then, for* $k \in \mathbb{Z} \setminus \{-2, 2\}$*, it has at most a finite number of solutions contained in* $\underbrace{\mathcal{I}_k \times \cdots \times \mathcal{I}_k}_{l}$*.* $\quad \mathcal{I}_k = \{\Lambda \in \mathbb{Q} \mid \Lambda + 1 \in \mathcal{M}_k\}$

What for $n > 2$?

Problem with calculation of # proper Darboux points

- for polynomial potentials

$$f_i = \frac{\partial V}{\partial q_i} - q_i, \qquad \deg f_i = k-1, \quad i = 1, \ldots, n,$$

$$\#D \leq \frac{(k-1)^n - 1}{k-2}$$

- for rational potentials

$$f_i = \frac{\partial W}{\partial q_i} U - \frac{\partial U}{\partial q_i} W - q_i U^2.$$

- from the above solutions remove

$$U = W = 0, \quad \text{and} \quad \left\{ U = 0, \quad \frac{\partial U}{\partial q_i} = 0, \quad i \in 1, \ldots, n \right\}$$

What for $n > 2$?

Conjecture

Let $V \in \mathbb{C}_k(\boldsymbol{q})$ be a homogeneous potential of degree $k \in \mathbb{Z}$ and let all its Darboux points be proper and simple. Then

$$\sum_{[\boldsymbol{d}] \in \mathcal{D}^\star(V)} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^i}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n-1}(-n-(r+s-2))^i,$$

for $i = 0, \ldots, n-1$.

Problems with application of the residue calculus

$$f_i = \frac{\partial W}{\partial q_i} U - \frac{\partial U}{\partial q_i} W - q_i U^2.$$

form

$$\omega = \frac{p(\boldsymbol{q})}{f_1(\boldsymbol{q}) \cdots f_n(\boldsymbol{q})}$$

polar loci of this form – elements of $\mathcal{V}(f_1, \ldots, f_n)$.
But $\mathbf{0} \in \mathcal{V}(f_1, \ldots, f_n)$ is not a simple point because $\boldsymbol{f}'(\mathbf{0}) = \mathbf{0}$ and not isolated because $\mathbf{0} \in \mathcal{V}(W, U)$
Problem: How to calculate residue for non-isolated point?

Example

$$V = q_1^{-s} \sum_{i=0}^{r} v_{r-i} q_1^{r-i} q_2^i, \quad v_i \in \mathbb{C}, k = r - s \in \mathbb{Z}_- \setminus \{-2\}. \quad (1)$$

We chose

$$\Lambda := \lambda - 1 = \frac{1}{2}\left(-\kappa p^2 + (\kappa + 2)p - 2\right), \quad k = -\kappa. \quad (2)$$

and assume maximal number of $r + 1$ simple proper Darboux points different from $\pm \mathrm{i}$ such that

$$r + 1 = -\frac{1}{2}\left(-\kappa p^2 + (\kappa + 2)p - 2\right), \quad (3)$$

$$\sum_{i=1}^{r+1} \frac{1}{\Lambda} = -1 \quad (4)$$

Example

Reconstruction gives potential

$$V(q_1, q_2) = q_1^{-\kappa} v(\mathrm{i} z) = q_1^{-\kappa} P_r^{(\alpha,\beta)} \left(\mathrm{i} \frac{q_2}{q_1} \right),$$

where

$$P_r^{(\alpha,\beta)}(x) = 2^{-r} \sum_{i=0}^{r} \binom{r+\alpha}{i} \binom{r+\beta}{r-i} (x-1)^{r-i} (x+1)^i .$$

are Jacobi polynomials with parameters

$$\alpha = \beta = \frac{2(p-1) + \kappa(2 + p - p^2)}{4}, \qquad r = \frac{p\,(\kappa(p-1) - 2)}{2}$$

$\kappa > 2/(p-1)$.

Integrable family

for $p = 2$, $r = \kappa - 2$ and $\alpha = \beta = 1/2$

$$P_r^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) = \frac{1}{(r+1)!}\left(\frac{3}{2}\right)_r U_r(z),\ \ U_r(z) = \sum_{i=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^i (r-i)!(2z)^{r-2i}}{i!(r-2i)!}$$

$$V(q_1,q_2) = q_1^{-r-2} P_r^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{C}{q_1^{2r+2}} \sum_{i=0}^{\left[\frac{r}{2}\right]} 2^{-2i} \frac{(r-i)!}{i!(r-2i)!} q_1^{2i} q_2^{r-2i}$$

$$= \frac{C}{q_1^{2r+2}\rho}\left[\left(\frac{\rho+q_2}{2}\right)^{r+1} + (-1)^r \left(\frac{\rho-q_2}{2}\right)^{r+1}\right]$$

where $\rho = \sqrt{q_1^2 + q_2^2}$

Integrable family

$$V_n = \frac{1}{\rho}\left[\left(\frac{\rho+q_2}{2}\right)^{n+1} + (-1)^n\left(\frac{\rho-q_2}{2}\right)^{n+1}\right], \qquad (5)$$

for negative $n = -r-2$.

$$V_{-3}(q_1,q_2) = q_1^{-3}P_1^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{q_2}{q_1^4},$$

$$V_{-4}(q_1,q_2) = q_1^{-4}P_2^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\mathrm{i}\frac{q_2}{q_1}\right) = \frac{q_1^2+4q_2^2}{q_1^6}.$$

with the corresponding first integrals

$$I_{-3}(q_1,q_2,p_1,p_2) = p_1(q_2p_1 - q_1p_2) + \frac{q_1^2+4q_2^2}{2q_1^4},$$

$$I_{-4}(q_1,q_2,p_1,p_2) = p_1(q_2p_1 - q_1p_2) + \frac{4q_2(q_1^2+2q_2^2)}{q_1^6}.$$