Lie-Poisson pencils related to semisimple Lie agebras: towards classification

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Motivation: algebraic mechanisms for bihamiltonian structures

Definition

A *bihamiltonian* structure on a manifold M is a pair $\eta_1, \eta_2 \in \Gamma(TM)$ such that $\eta_1, \eta_2, \eta_1 + \eta_2$ are Poisson.

Historically: Particular hamiltonian integrable system on a Poisson manifold $(M, \eta_1) \rightarrow$ second hamiltonian structure $\eta_2 \hookrightarrow$ general algebraic mechanism relating η_1, η_2

Example

KdV $\rightarrow$ Magri's second hamiltonian structure $\hookrightarrow$ "argument translation method": η_2 is canonical linear, $\eta_1 = \eta_2(a)$ ($M = \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the Virasoro Lie algebra, $a \in \mathfrak{g}^*$ a particular point)

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Hierarchy of mechanisms (by complexity of structures):

- constant+constant (rather not interesting)
- constant+linear (proved to be powerful, eg. "argument translation")
- linear+linear (topic of present talk)
- linear+quadratic (eg. argument translation of quadratic bracket towards "vanishing direction")
- etc.

Motivation: pairs of linear structures

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [,], [,]')$, where $\mathfrak{h}$ is a vector space and $[,], [,]'$ are two Lie brackets on $\mathfrak{h}$ which are *compatible*, i.e. so that $[,] + [,]'$ is a Lie bracket.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure, ($[,]$ the standard commutator).

Main motivating example

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure.

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Semisimple case

Applications of the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure:

- Manakov top (n-dimensional free rigid body), here A is diagonal, the "inertia tensor" of the body (due to Bolsinov 1992)
- Landau-Livshits PDE ($n = 3$) (due to Holod 1987)

Nonsemisimple case

Works of Golubchik, Odesskii, Sokolov $\sim$ 2004-2006

- Matrix integrable ODE's
- Classification of "bi-associative structures" $(\cdot, \circ)$ on $\mathfrak{gl}(n, \mathbb{K}) \Longrightarrow$ Examples of bi-Lie structures on $\mathfrak{gl}(n, \mathbb{K})$ (which do not restrict to $\mathfrak{sl}(n, \mathbb{K})$)

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Motivation: Kantor-Persits theorem

Kantor-Persits 1988 (announced only)

The list of irreducible closed (*technical assumptions*) vector spaces of Lie structures:

- $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathrm{Symm}(n,\mathbb{K})}$
- $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathfrak{m}(n,\mathbb{K})}$
- several *nonsemisimple* cases

here

$$[X,_A Y] := XAY - YAX,$$

$\mathfrak{sp}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ + JX^T = 0\}$ the symplectic Lie algebra, $\mathfrak{m}(n, \mathbb{K}) := \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ - JX^T = 0\}$ its orthogonal complement in $\mathfrak{gl}(2n, \mathbb{K})$ w.r.t. "trace form"

Semisimple bi-Lie structures and operators

Useful notations

Let $\mathfrak{g}$ be a Lie algebra and $N : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y],$$

$$T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N.$$

Obvious or Easy:

Let $(\mathfrak{g}, [,])$ be a semisimple Lie algebra, $[,]'$ a bilinear skew-symmetric bracket. Then

$(\mathfrak{g}, [,], [,]')$ is a bi-Lie str. $\Longleftrightarrow$ $[,]' = [,]_W$ for some $W \in \mathrm{End}(\mathfrak{g})$ and $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$ for some $P \in \mathrm{End}(\mathfrak{g})$.

Moreover, the operators W, P are defined up to adding of inner differentiations $\mathrm{ad}\, x$.

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Semisimple bi-Lie structures: two more examples

Definition

- A *semisimple* bi-Lie structure $(\mathfrak{g}, [,], [,]')$;
- the *leading operator* W;
- the *primitive operator* P;
- the *main identity* $T_W(\cdot,\cdot) = [\cdot,\cdot]_P$.

Example

(Golubchik–Sokolov) Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ *be a* $\mathbb{Z}/n\mathbb{Z}$*-grading on* $\mathfrak{g}$*. Put* $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ *and* $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i)\mathrm{Id}_{\mathfrak{g}_i}$*. One checks MI directly.*

Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (sum of subalgebras). Put $W|_{\mathfrak{g}_i} = \lambda_i \mathrm{Id}_{\mathfrak{g}_i}, i = 1, 2$, where $\lambda_{1,2}$ are any scalars. Then $T_W = 0$ (so put $P = 0$ in the MI). Important example: $\mathfrak{g}$ simple, $\mathfrak{g}_1$ a parabolic subalgebra and $\mathfrak{g}_2$ its "complement".

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Principal leading operator

Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the direct complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

Theorem

1. *There exists a unique principal operator W with the property $[,]' = [,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [,], [,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the Killing form B on $\mathfrak{g}$.*

Example

For the $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

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We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *isomorphic* if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$.

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Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

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Gradings and Main assumptions

Definition

Let $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$ be a *grading* of a Lie algebra $(\mathfrak{g}, [,])$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, Γ an abelian group. We say that a linear operator $W : \mathfrak{g} \to \mathfrak{g}$ *preserves* the grading if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

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Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure and let $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$ be a grading. Then, if the principal operator $W : \mathfrak{h} \to \mathfrak{h}$ preserves the grading, so does its symmetric primitive P.

Main assumptions:

$(\mathbb{K} = \mathbb{C})$

- The principal operator $W \in \mathrm{End}(\mathfrak{g})$ preserves the root grading $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$.
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Consequences of the Main assumptions

Bi-Lie structure $(\mathfrak{g}, [,], [,]') \Longrightarrow$ Pencil of Lie brackets $(\mathfrak{g}, [,]^t), [,]^t := [,]' - t[,], t \in \mathbb{C}$.

Definition

The elements of the finite set $T := \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ are called the *times* of the bi-Lie structure, here B^t is the Killing form of $(\mathfrak{g}, [,]^t)$.

Theorem

- *Given an element $E_\alpha \in \mathfrak{g}_\alpha, \alpha \in R$, such that $B(E_\alpha, E_{-\alpha}) = 1$, there exist exactly two (up to multiplicity) times $t_{1,\alpha}, t_{2,\alpha}$ such that $E_\alpha \in \ker B^{t_{i,\alpha}}$.*
- *Put $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$. Then $T_\alpha = T_{-\alpha}$.*
- *The element $H_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{h}$ belongs to $\ker(W - t_{1,\alpha} I)(W - t_{2,\alpha} I)$.*

"Times selection rules" and times diagrams

Theorem

Let $\alpha, \beta, \gamma \in R$ be such that $\alpha + \beta + \gamma = 0$. Then only the following possibilities can occur:

1. *either there exist $t_1, t_2, t_3 \in \mathbb{C}$ such that*

$$T_\alpha = \{t_1, t_2\}, T_\beta = \{t_2, t_3\}, T_\gamma = \{t_3, t_1\};$$

2. *or there exist $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$, such that*

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}.$$

A collection $\{T_\alpha\}_{\alpha \in R}$ of unordered pairs $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$ of complex numbers with $T_\alpha = T_{-\alpha}$ and the properties mentioned is called a times diagram.

Times diagrams

Examples: $t_1\,t_2$ $\quad t_1\,t_3 \quad$ $t_2\,t_3$, $\quad t_1\,t_2 \quad t_1\,t_3 \quad t_2\,t_3 \quad t_1(t_4) \quad t_2(t_4) \quad t_3(t_4)$,

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Theorem

If $(\mathfrak{g}, [,])$ *is simple, the set times diagrams splits to two disjoint classes:* class I *and* class II.

Definition

An operator $U : \mathfrak{h} \to \mathfrak{h}$ *is* subject to a times diagram $\{T_\alpha\}_{\alpha \in R}$, $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, *if* $H_\alpha = [E_\alpha, E_{-\alpha}] \in \ker(W - t_{1,\alpha} I)(W - t_{2,\alpha} I)$ *for any* $\alpha \in R$.

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,
$$

$$
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Bi-Lie structures of Class I

Theorem

Given a pair $(U, \mathcal{T}), \mathcal{T} := \{T_\alpha\}_{\alpha \in R}$, where $\mathcal{T}$ is of class I and U is subject to $\mathcal{T}$,

- *there exists a unique operator $W : \mathfrak{g} \to \mathfrak{g}$ such that $W|_{\mathfrak{h}} = U$ and W is a principal leading operator for a bi-Lie structure.*
- *It is of the form $W|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}} = [(t_{1,\alpha} + t_{2,\alpha})/2]\mathrm{Id}_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ and is symmetric iff so is U.*

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Each times diagram of class I induces a specific type of $\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$-grading on the Lie algebra $(\mathfrak{g}, [,])$.

Conjecture-hope

It is possible to classify all bi-Lie structures of class I (at least with symmetric U).

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Bi-Lie structures of Class II

Theorem

Given a pair $(U, \mathcal{T}), \mathcal{T} := \{T_\alpha\}_{\alpha \in R}$*, where* $\mathcal{T}$ *is of class II and* U *is subject to* $\mathcal{T}$*, and some extra data* D *(*$\alpha, \beta, \gamma \in R, \alpha + \beta + \gamma = 0 \;\rightsquigarrow\; 0, +1, -1$*),*

- *there exists a uniquely defined operator* $W : \mathfrak{g} \to \mathfrak{g}$ *such that* $W|_{\mathfrak{h}} = U$ *and* W *is a principal leading operator for a bi-Lie structure.*
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- *Its antisymmetric part depends also on* D*.*

New examples

Related to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$-grading and other gradings on $(\mathfrak{g}, [,])$

Conjecture

Any bi-Lie structure of Class II is isomorphic to one of the bi-Lie structures related to $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$-grading with $n_i > 2$.

Bi-Lie structures of Class II

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Many thanks!