Some results concerning the constant astigmatism equation

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Abstract.

The constant astigmatism equation is

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0.$$

We provide:

– new interpretation of solutions as describing orthogonal equiareal patterns on the unit sphere

– relevance to two-dimensional plasticity

– the classical Bianchi superposition principle for the sine-Gordon equation extended to generate ∞ of solutions of the constant astigmatism equation

– slip line fields on the sphere described by sine-Gordon solutions

– equiareal patterns corresponding to classical Lipschitz surfaces of constant astigmatism.

Introduction.

The classical Bäcklund transformation for the sine-Gordon equation $u_{\xi\eta} = \sin u$ has been discovered in the context of pseudospherical surfaces.

Historical roots lie in another class of surfaces, characterised by the constancy of the difference $\rho_2 - \rho_1$ between the principal radii of curvature ρ_1, ρ_2.

The latter surfaces reemerged from the systematic search for integrable classes of Weingarten surfaces conducted by Baran and one of us (2009). We named them constant astigmatism surfaces. Connotation with the astigmatic interval of geometric optics, without suggesting any specific application.

The most important results about them are due to L. Bianchi.

L. Bianchi, Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante, *Ann. Scuola Norm. Sup. Pisa, I* **2** (1879) 285–341.

Bianchi 1879.

Evolutes of constant astigmatism surfaces are pseudospherical. Constant astigmatism surfaces correspond to parabolic geodesic systems on pseudospherical surfaces.

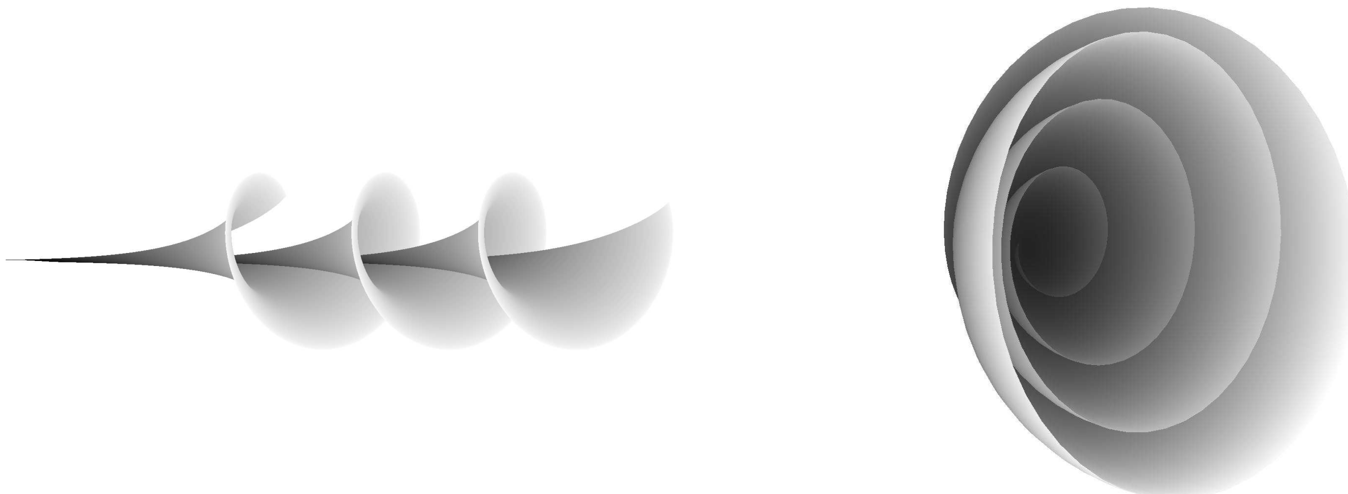

Figure 1: Dini's pseudospherical helicoid (left) and its constant astigmatism involute (right)

Lipschitz 1887

Obtained another class of surfaces of constant astigmatism. The full class is given in terms of elliptic integrals; a subclass of surfaces of revolution being further investigated by von Lilienthal.

Baran & Marvan, 2009

Surfaces of constant astigmatism referred to adapted parameterisation by lines of curvature $\longleftrightarrow$ solutions of the constant astigmatism equation

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0.$$

Zero-curvature representation with values in $\mathfrak{sl}(2)$.

Transformation to sine–Gordon equation = analytical representation of Bianchi's geometric picture.

Preliminaries

Under parameterisation by the lines of curvature (curvature coordinates), the fundamental forms can be written as

$$\mathbf{I} = u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2,\ \mathbf{II} = \frac{u^2}{\rho_1}\,\mathrm{d}x^2 + \frac{v^2}{\rho_2}\,\mathrm{d}y^2,\ \mathbf{III} = \frac{u^2}{\rho_1^2}\,\mathrm{d}x^2 + \frac{v^2}{\rho_2^2}\,\mathrm{d}y^2.$$

Here $\rho_1, \rho_2 =$ principal radii of curvature.

Definition 1. A surface is said to be of *constant astigmatism* if the difference $\rho_2 - \rho_1$ is a nonzero constant.

We assume the ambient space to be scaled so that $\rho_2 - \rho_1 = \pm 1$.

Definition 2. Curvature coordinates are said to be *adapted* if

$$uv\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \pm 1. \qquad (1)$$

(1) $\Rightarrow x, y$ are natural in the sense of Ganchev and Mihova (2010).

What is z?

Every constant astigmatism (more generally, Weingarten) surface can be equipped with an adapted parameterisation by lines of curvature.

Then the nonzero coefficients of the three fundamental forms of a surface of constant astigmatism can be expressed through a single function $z(x, y)$:

$$u = \frac{z^{\frac{1}{2}}(\ln z - 2)}{2}, \quad v = \frac{\ln z}{2z^{\frac{1}{2}}}, \quad \rho_1 = \frac{\ln z - 2}{2}, \quad \rho_2 = \frac{\ln z}{2}.$$

Obviously, $\rho_2 - \rho_1 = 1$.

Observation. The third fundamental form is

$$\mathbf{III} = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2.$$

But, $\mathbf{III} = \mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{n}$ coincides with the first fundamental form of the Gaussian sphere $\mathbf{n}(x, y)$.

Orthogonal equiareal patterns

Definition 3. An *orthogonal equiareal pattern* = parameterization x, y such that

$$\mathbf{I}_S = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2,$$

z being an arbitrary function of x, y (Sadowsky, 1941).

Implications:

1. Coordinate lines are orthogonal.
2. Local parameterisation $\mathbb{R}^2 \supset U \longrightarrow S$ is area preserving.

Consequently, evenly distributed coordinate lines cover the surface with curvilinear rectangles of equal area.

M.A. Sadowsky, Equiareal pattern of stress trajectories in plane plastic strain, *J. Appl. Mech.* **8** (1941) A-74–A-76.

M.A. Sadowsky, Equiareal patterns, *Amer. Math. Monthly* **50** (1943) 35–40.

History of orthogonal equiareal patterns

Can be traced back to Boussinesq (1872).

Seventy years later rediscovered by Sadowsky as configurations of the principal stress lines under the Tresca yield condition.

Hill 1966 gave a kinematic interpretation of these patterns.

Coburn 1950 established the same equiareal property for slip lines under a different yield condition.

Ament 1943 discovered a relation to the class of Weingarten surfaces, determined by the constancy of the difference between the principal curvatures (as opposed to the difference between the principal radii of curvature).

Fialkow 1945 observed relevance of orthogonal equiareal patterns to conformal geometry.

Example

The Archimedean projection = simplest example of an orthogonal equiareal pattern on the sphere.

The well-known Archimedean projection of the cylinder $(\cos y, \sin y, x)$ onto an inscribed sphere is

$$(x, y) \mapsto \big(\sqrt{1 - x^2}\, \cos y, \sqrt{1 - x^2}\, \sin y, x\big).$$

We have

$$\mathbf{I}_{\text{Arch}} = \frac{\mathrm{d}x^2}{1 - x^2} + (1 - x^2)\, \mathrm{d}y^2,$$

i.e., $z = 1/(1 - x^2)$.

This z corresponds to von Lilienthal surfaces of constant astigmatism.

Archimedean equiareal pattern

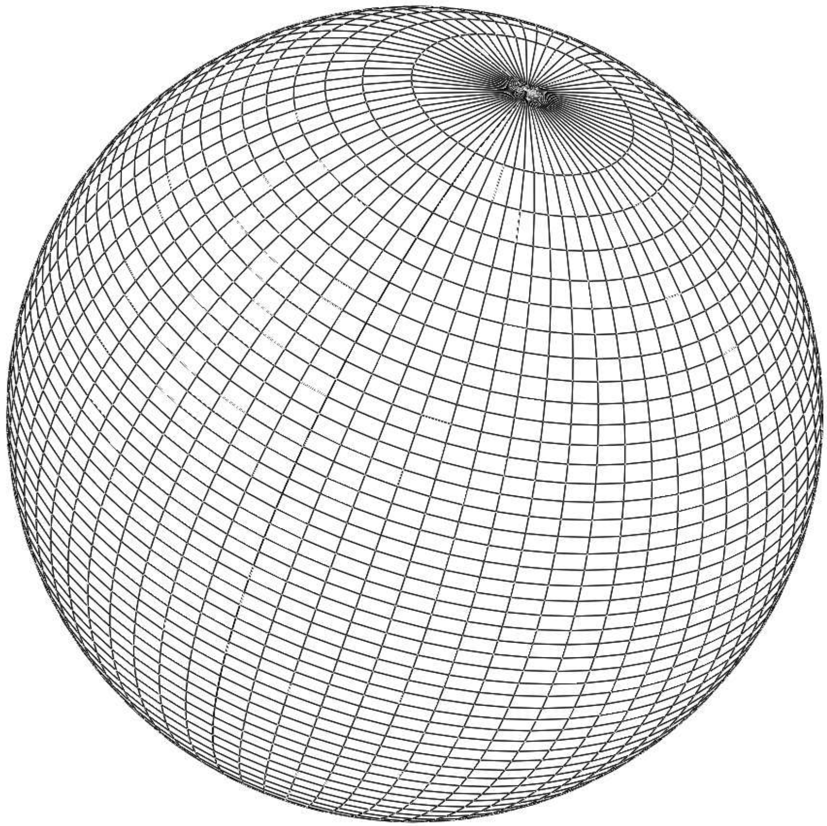

Figure 2: The Archimedean equiareal parameterisation of the sphere

Converse statement

As we have seen, every constant astigmatism surface generates an orthogonal equiareal parameterization of the unit sphere. Conversely,

Proposition 1. *Let* $z\,\mathrm{d}x^2 + (1/z)\,\mathrm{d}y^2$ *be an orthogonal equiareal pattern on the unit sphere* S*. Then* z *is a solution of the constant astigmatism equation.*

Proof. Using the well-known Brioschi formula, we compute the Gaussian curvature of the sphere, obtaining

$$1 = -\frac{1}{2}\,z_{yy} - \frac{1}{2}\left(\frac{1}{z}\right)_{xx}.$$

The constant astigmatism equation follows immediately. □

Relation to two-dimensional plasticity

Orthogonal equiareal patterns were introduced by Sadowski in the case of S being a plane model of plasticity.

Choosing the vectors ∂_x, ∂_y along the *principal stress directions* (i.e., eigenvectors of the stress tensor σ^i_j), Sadowski derived the equiareal property from the equilibrium condition $\operatorname{div} \sigma = 0$ and the Tresca yield condition $\sigma^1_1 - \sigma^2_2 = \text{const}$.

We reverse the line of reasoning. Reconstruct a two-dimensional stress tensor from a given orthogonal equiareal pattern $g = \mathrm{I}_S$.

In what follows, all components are taken with respect to the basis ∂_x, ∂_y of the tangent space and indices are raised and lowered with the metric.

A two-dimensional plasticity model

Proposition 2. *Consider an orthogonal equiareal pattern* $g = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ *such that*

$$g_{11} = z, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1/z.$$

Then the tensor σ *given by the components*

$$\sigma^1_1 = \tfrac{1}{2}\ln z, \quad \sigma^1_2 = \sigma^2_1 = 0, \quad \sigma^2_2 = \tfrac{1}{2}(\ln z - 2). \tag{2}$$

satisfies $\sigma^{ij}_{;j} = 0$ *(the equilibrium equation) and* $\sigma^1_1 - \sigma^2_2 = 1$ *(the Tresca yield condition).*

Proof. The yield condition $\sigma^1_1 - \sigma^2_2 = 1$ is obvious. Checking the equilibrium equation $\sigma^{ij}_{;j} = 0$ is routine. □

Slip lines

Tresca yield condition: Yielding occurs when the maximal shear stress magnitude achieves a threshold depending on the material.

The lines along the maximal shear stress directions are called slip lines.

Slip lines have a constant deviation of $\frac{1}{4}\pi$ from the principal stress directions.

Definition 4. By a *slip line field* associated with the orthogonal equiareal pattern on a surface S we shall mean a parameterization ξ, η such that the angle between ∂_x and ∂_ξ as well as the angle between ∂_y and ∂_η is equal to $\frac{1}{4}\pi$.

It follows that slip lines form an orthogonal net.

Remark. Planar slip lines satisfy Hencky conditions. These fail on surfaces of non-vanishing curvature.

Example

The net of slip lines corresponding to the Archimedean equiareal pattern is, by definition, formed by the $\pm 45°$ loxodromes. Observed by Zelin (1996, superplastic sheet stretched with a spherical punch).

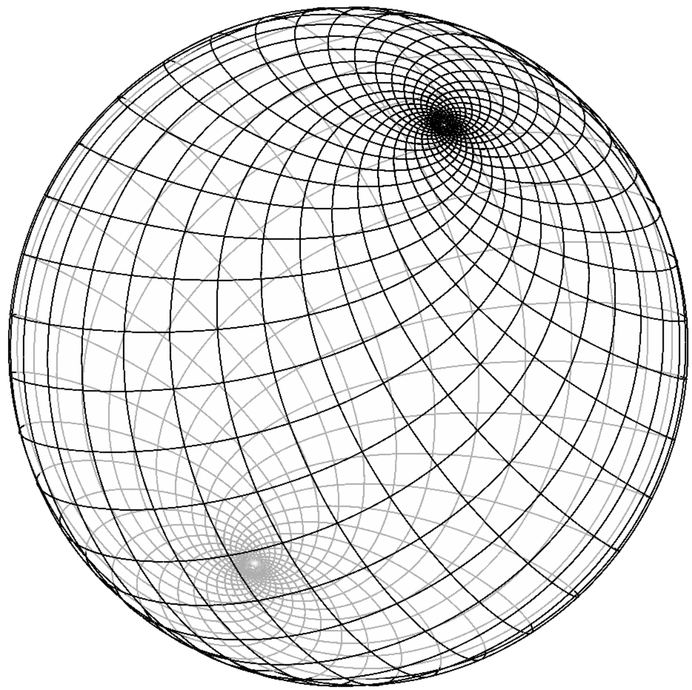

Figure 3: Sphere's slip line field composed of loxodromes

Bäcklund transformation

A powerful way to generate pseudospherical surfaces and solutions of the sine–Gordon equation.

Consider a pseudospherical surface $\mathbf{r}(\xi, \eta)$ parameterised so that

$$\mathbf{I} = \mathrm{d}\xi^2 + 2\cos(2\omega)\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{d}\eta^2, \qquad \mathbf{II} = 2\sin(2\omega)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

The Bäcklund transform of our surface is

$$\mathbf{r}^{(\lambda)} = \mathbf{r} + \frac{2\lambda}{1+\lambda^2}\left(\frac{\sin(\omega - \omega^{(\lambda)})}{\sin(2\omega)}\mathbf{r}_\xi + \frac{\sin(\omega + \omega^{(\lambda)})}{\sin(2\omega)}\mathbf{r}_\eta\right).$$

Here $\omega^{(\lambda)}$ is another sine-Gordon solution, obtained from the pair of compatible first-order equations

$$\omega_\xi^{(\lambda)} = \omega_\xi + \lambda\sin(\omega^{(\lambda)} + \omega), \qquad \omega_\eta^{(\lambda)} = -\omega_\eta + \frac{1}{\lambda}\sin(\omega^{(\lambda)} - \omega),$$

λ being a constant called the Bäcklund parameter.

Bianchi superposition principle

Write $\mathcal{B}_c^{(\lambda)}\omega$ to denote a solution $\omega^{(\lambda)}$ for a specified value of the integration constant c.

The Bianchi permutability theorem says that given a pair of Bäcklund parameters $\lambda_1 \neq \lambda_2$, then for every choice of integration constants c_1, c_2 there is a unique choice of integration constants c_1', c_2' such that

$$\mathcal{B}_{c_2'}^{(\lambda_2)}\mathcal{B}_{c_1}^{(\lambda_1)}\omega = \mathcal{B}_{c_1'}^{(\lambda_1)}\mathcal{B}_{c_2}^{(\lambda_2)}\omega$$

Moreover, denoting by $\omega^{(\lambda_1\lambda_2)}$ the common value, then $\omega^{(\lambda_1\lambda_2)}$ can be obtained from the *superposition principle*

$$\tan\frac{\omega^{(\lambda_1\lambda_2)} - \omega}{2} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\tan\frac{\omega^{(\lambda_1)} - \omega^{(\lambda_2)}}{2}. \tag{3}$$

No further integration, just purely algebraic manipulations.

Bianchi superposition principle continued

Assume now that for a fixed seed ω, the BT $\omega^{(\lambda)}$ is known for every value of the Bäcklund parameter λ.

Using the superposition principle, one can compute $\omega^{(\lambda_1\lambda_2)}$, $\omega^{(\lambda_1\lambda_2\lambda_3)}$, etc., $\omega^{(\lambda_1\lambda_2\cdots\lambda_s)}$ by purely algebraic manipulations.

Solutions depending on any finite number of Bäcklund parameters and integration constants.

Complementarity. In the particular case of $\lambda = 1$ the Bäcklund transformation $\mathcal{B}^{(1)}$ coincides with Bianchi's complementarity relation.

Consequently, the superposition formula yields a method to obtain abundant pairs of complementary sine-Gordon solutions $\omega^{(\lambda_1\lambda_2\cdots\lambda_s)}$ and $\omega^{(\lambda_1\lambda_2\cdots\lambda_s 1)}$. Hence also abundant pairs of complementary pseudospherical surfaces $\mathbf{r}^{(\lambda_1\lambda_2\cdots\lambda_s)}$ and $\mathbf{r}^{(\lambda_1\lambda_2\cdots\lambda_s 1)}$.

Generation of constant astigmatism surfaces

Surfaces of constant astigmatism are easier to obtain from a pair of complementary pseudospherical surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ rather than from a single pseudospherical surface.

Denote

$$\tilde{\mathbf{n}} = \mathbf{r}^{(1)} - \mathbf{r} = \frac{\sin(\omega - \tilde{\omega})}{\sin(2\omega)}\mathbf{r}_\xi + \frac{\sin(\omega + \tilde{\omega})}{\sin(2\omega)}\mathbf{r}_\eta.$$

Then $\tilde{\mathbf{n}}$ is a unit vector tangent to both surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ and determines what is called a pseudospherical congruence.

Normal surfaces of this congruence are the constant astigmatism surfaces sought.

Two important observations due to Bianchi

Differentiating with respect to the integration constant c, we denote $f = \ln(\mathrm{d}\omega^{(1)}/\mathrm{d}c)$. Then

$$f_\xi = \cos(\omega^{(1)} + \omega), \qquad f_\eta = \cos(\omega^{(1)} - \omega).$$

Similarly, taking one more derivative $f' = \mathrm{d}f/\mathrm{d}c$, we get

$$f'_\xi = -\mathrm{e}^f \sin(\omega^{(1)} + \omega), \qquad f'_\eta = -\mathrm{e}^f \sin(\omega^{(1)} - \omega).$$

Proposition 3. *Let $\omega^{(1)}(\xi, \eta, c)$ be a general solution of the Bäcklund system, c being the integration constant. Then $\tilde{\mathbf{r}} = \mathbf{r} - f\tilde{\mathbf{n}}$, where $f = \ln(\mathrm{d}\omega^{(1)}/\mathrm{d}c)$ is a surface of constant astigmatism having complementary surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ as evolutes.*

Proposition 3 shows that the constant astigmatism surfaces $\tilde{\mathbf{r}} = \mathbf{r} - f\tilde{\mathbf{n}}$ can be found by purely algebraic manipulations and differentiation once a one-parameter family of pseudopotentials $\omega^{(1)}$ is known.

Another proposition and its corollary

Proposition 4. *Let* $\omega^{(1)}(\xi, \eta, c)$ *be a general solution of the Bäcklund system, let* $f = \ln(\mathrm{d}\omega^{(1)}/\mathrm{d}c)$ *and* $x = \mathrm{d}f/\,\mathrm{d}c$*. Let* $y(\xi, \eta)$ *be a solution of the system*

$$y_\xi = \mathrm{e}^{-f} \sin(\omega + \omega^{(1)}), \qquad y_\eta = \mathrm{e}^{-f} \sin(\omega - \omega^{(1)}).$$

Then x, y *are adapted curvature coordinates on the surface* $\tilde{\mathbf{r}}$*.*

Moreover, if $z = \mathrm{e}^{-2f}$*, then* $z(x, y)$ *is a solution of the constant astigmatism equation.*

Finally, $z\,\mathrm{d}x^2 + \mathrm{d}y^2/z$ *is an orthogonal equiareal pattern on the unit sphere* $\tilde{\mathbf{n}}$*, while* ξ, η *constitute an associated slip line field.*

Corollary 1. *If* S *is a constant astigmatism surface, then the asymptotic coordinates on the focal surfaces of* S *correspond to slip line fields on the Gaussian image of* S*.*

The superposition principle extended

One of the adapted curvature coordinates is obtained by purely algebraic manipulations and differentiation, while the other curvature coordinate has to be obtained by integration.

It is therefore natural to ask whether one could obtain superposition formulas for f, x, y.

The answer is positive, even for arbitrary λ.

Associated potentials

Definition 5. Given two sine-Gordon solutions ω and $\omega^{(\lambda)}$ related by the Bäcklund transformation $\mathcal{B}^{(\lambda)}$, let $f^{(\lambda)}$, $x^{(\lambda)}$, $y^{(\lambda)}$ denote functions satisfying the compatible equations

$$\begin{aligned}
f_\xi^{(\lambda)} &= \lambda \cos(\omega^{(\lambda)} + \omega), & f_\eta^{(\lambda)} &= \frac{1}{\lambda} \cos(\omega^{(\lambda)} - \omega),\\
x_\xi^{(\lambda)} &= \lambda \mathrm{e}^{f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & x_\eta^{(\lambda)} &= \frac{1}{\lambda} \mathrm{e}^{f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega),\\
y_\xi^{(\lambda)} &= \lambda \mathrm{e}^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & y_\eta^{(\lambda)} &= -\frac{1}{\lambda} \mathrm{e}^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega).
\end{aligned}$$

The quantities $f^{(\lambda)}$, $x^{(\lambda)}$, $y^{(\lambda)}$ will be called *associated potentials* corresponding to the pair $\omega, \omega^{(\lambda)}$.

Superposition principle for the associated potentials

Proposition 5. *Let* $\omega, \omega^{(\lambda_1)}, \omega^{(\lambda_2)}, \omega^{(\lambda_1\lambda_2)}$ *be four sine-Gordon solutions related by the Bianchi superposition principle. Then the associated potentials* $f^{(\lambda_1\lambda_2)}$, $x^{(\lambda_1\lambda_2)}$, $y^{(\lambda_1\lambda_2)}$ *corresponding to the pair* $\omega^{(\lambda_1)}, \omega^{(\lambda_1\lambda_2)}$ *are related to the associated potentials* $f^{(\lambda_2)}$, $x^{(\lambda_2)}$, $y^{(\lambda_2)}$ *corresponding to the pair* $\omega, \omega^{(\lambda_2)}$ *by*

$$f^{(\lambda_1\lambda_2)} = f^{(\lambda_2)} - \ln\left(2\cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) - \frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right),$$

$$x^{(\lambda_1\lambda_2)} = \frac{\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\left(x^{(\lambda_2)} - \frac{2\lambda_1\lambda_2\sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)})}{\lambda_1^2 - 2\lambda_1\lambda_2\cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) + \lambda_2^2}\mathrm{e}^{f^{(\lambda_2)}}\right),$$

$$y^{(\lambda_1\lambda_2)} = \left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)y^{(\lambda_2)} - 2\mathrm{e}^{-f^{(\lambda_2)}}\sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}),$$

up to an additive constant.

One soliton solutions

As is well known, the one-soliton solutions $\omega^{(\lambda)} = \mathcal{B}_0^{(\lambda)}(0)$ of the sine-Gordon equation correspond to the Dini surfaces (helicoids of the tractrix)

The complementary surfaces of the Dini surfaces correspond to the nonlinear superposition of $\omega^{(\lambda)}$ and $\omega^{(1)}$.

One obtains $x = x^{(\lambda 1)}$, $y = y^{(\lambda 1)}$, $z = \mathrm{e}^{-2f^{\lambda 1}}$ as

$$x = \frac{\lambda}{\lambda^2 - 1} \times \frac{(\lambda-1)^2(c_2A^2B^2 - c_1) - (\lambda+1)^2(c_1B^2 + c_2A^2) + 4(c_1 - c_2)\lambda AB}{(\lambda-1)^2(A^2B^2+1) + (\lambda+1)^2(B^2+A^2) - 8\lambda AB},$$

$$y = \frac{4\ln B}{c_1 + c_2} - 2\frac{(\lambda^2+1)\ln A}{(c_1+c_2)\lambda} + \frac{4\lambda(AB+1)(A-B) + c_3(c_1+c_2)(\lambda^2-1)A(1+B^2)}{(c_1+c_2)\lambda \mathrm{e}^s(1+\mathrm{e}^{2p_\lambda})},$$

$$z = \left(\frac{(\lambda-1)^2(A^2B^2+1) + (\lambda+1)^2(B^2+A^2) - 8\lambda AB}{(c_1+c_2)\lambda A(1+B^2)}\right)^2,$$

where $A = \mathrm{e}^{\xi+\eta}$ and $B = \mathrm{e}^{\lambda\xi+\eta/\lambda}$, while c_1, c_2, c_3 are arbitrary constants.

An implicit formula for one soliton solution

Eliminating ξ, η, one obtains

$$y = \frac{1}{c_1 + c_2}\left(\frac{4(AB+1)(A-B)}{(B^2+1)A} - 2\frac{\lambda^2+1}{\lambda}\ln A + 4\ln B\right) + \frac{(\lambda^2-1)c_3}{\lambda},$$

where

$$A = \frac{\lambda(\lambda^2+1)(c_1+c_2)\sqrt{z} - \sqrt{k}}{(\lambda^2-1)^2 + (\lambda^2 x - \lambda c_2 - x)^2 z},$$

$$B = \frac{2\lambda^2(c_1+c_2)\sqrt{z} + \sqrt{k}}{(\lambda^2-1)^2 + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z},$$

$$k = -[(\lambda^2-1)^2 + 2(c_1+c_2)\lambda^2\sqrt{z} + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z]$$
$$\times [(\lambda^2-1)^2 - 2(c_1+c_2)\lambda^2\sqrt{z} + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z].$$

Yields $y(x, z)$.

Slip line net on the Gaussian sphere

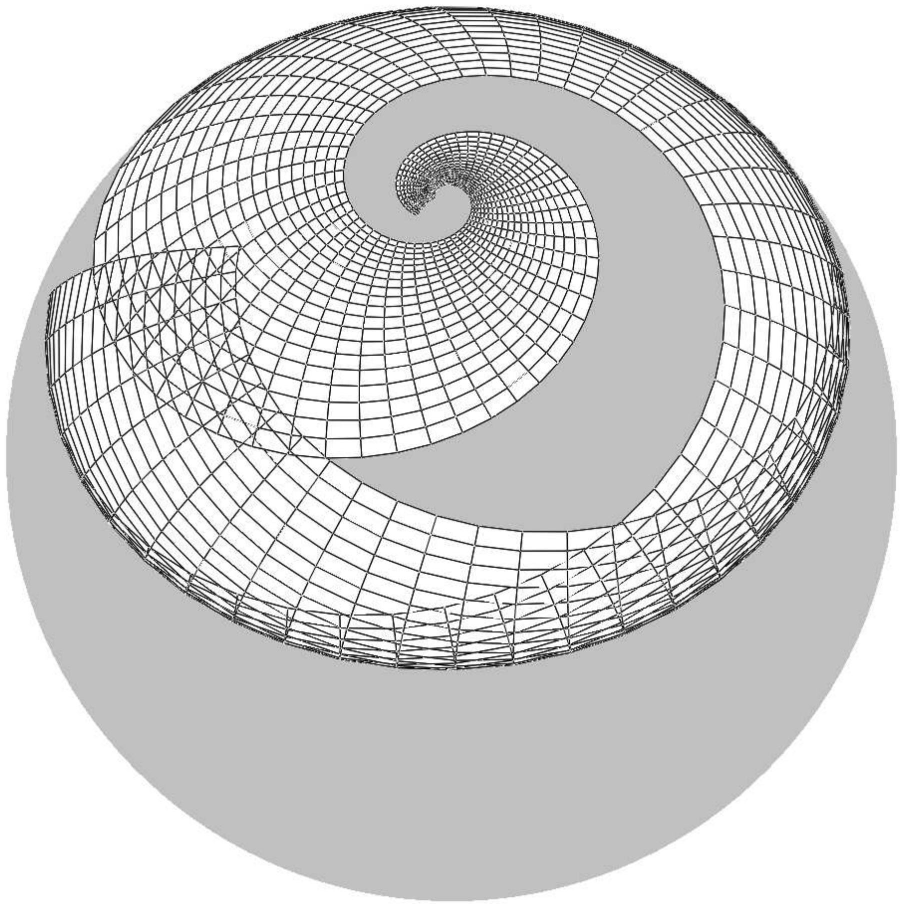

Figure 4: Sphere's slip line field corresponding to one soliton solution (a part; the sphere is multiply covered)

Lipschitz surfaces in principal coordinates

Redoing the computation is easier than transforming the Lipschitz 1887 result.

Consider the unit sphere $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ parameterised by the latitude θ and longitude ϕ.

To specify an orthogonal equiareal pattern we let θ, ϕ denote yet unknown functions of parameters x, y.

Lipschitz defines a *Stellungswinkel* to be the angle ω between $\mathbf{n}_\theta$ and $\mathbf{n}_x = \phi_x \mathbf{n}_\phi + \theta_x \mathbf{n}_\theta$.

The Lipschitz class is specified by allowing the Stellungswinkel to depend solely on the latitude θ.

The general Lipschitz solution

Theorem 1. *The general Lipschitz solution of the constant astigmatism equation depends on four constants* $h_{11}, h_{10}, h_{01}, h_{00}$ *and consists of functions*

$$z = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(h_{11}x + h_{01})^2},$$

where

$$h = h_{11}xy + h_{10}x + h_{01}y + h_{00},$$
$$H_1 = h_{11}, \qquad H_2 = h_{11}h_{00} - h_{10}h_{01}.$$

Formula covers all Lipshitz solutions except a particular solution

$$z = \frac{1}{c_1 - (x - c_0)^2},$$

c_1, c_0 *being arbitrary constants.*

The Lipschitz orthogonal equiareal pattern

Theorem 2. *The orthogonal equiareal pattern corresponding to the general Lipschitz solution is* $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, *where*

$$\theta = \arccos h,$$

$$\phi = -\frac{\ln(h_{11}x + h_{01})}{h_{11}} + \int \frac{1 - h^2 + \sqrt{(1-h^2)^2 - 4(H_1 h - H_2)^2}}{2(H_1 h - H_2)(1-h^2)}\,\mathrm{d}h,$$

where $h = h_{11}xy + h_{10}x + h_{01}y + h_{00}$ *and* $H_1 = h_{11}$, $H_2 = h_{11}h_{00} - h_{10}h_{01}$.

The Stellungswinkel ω is a function of the lattitude θ as required:

$$\cos^2\omega = \frac{1}{\Theta^2 + 1} = \frac{1 - h^2 + \sqrt{(1-h^2)^2 - 4(H_1 h - H_2)^2}}{2(1-h^2)}$$

$$= \frac{\sin^2\theta + \sqrt{\sin^4\theta - 4(H_1\cos\theta - H_2)^2}}{2\sin^2\theta}.$$

Invariance

It is easy to check that the general Lipschitz solution satisfies

$$h_{11}\mathfrak{s} + h_{01}\mathfrak{t}^x - h_{10}\mathfrak{t}^y = 0,$$

where

$$\mathfrak{t}^x = z_x,$$
$$\mathfrak{t}^y = z_y,$$
$$\mathfrak{s} = xz_x - yz_y + 2z$$

are generators of the Lie symmetries of the constant astigmatism equation.

This means that the general Lipschitz solution is a symmetry-invariant solution of the constant astigmatism equation.

Longitude ϕ in terms of elementary functions

The lattitude θ is $\arccos h$, while the longitude ϕ is given by an elliptic integral.

Assuming that h_{11} is nonzero, h_{10} and h_{01} can be removed by shifts, so we set $h_{10} = h_{01} = 0$.

Then the discriminant with respect to h is proportional to

$$(1 + H_1^2 + 2H_2)(1 + H_1^2 - 2H_2)(H_1 - H_2)^2(H_1 + H_2)^2$$
$$= h_{11}^4(1 + h_{11}^2 + 2h_{11}h_{00})(1 + h_{11}^2 - 2h_{11}h_{00})(1 - h_{00})^2(1 + h_{00})^2.$$

Is zero if and only if

$$h_{00} = \pm 1 \quad \text{or} \quad h_{00} = \pm\frac{1 + h_{11}^2}{2h_{11}}.$$

In these cases, ϕ can be expressed in terms of elementary functions.

Longitude ϕ in terms of elementary functions continued

For $h_{00} = \pm 1$ we have

$$\phi = \mp \frac{\sqrt{1-C^2}}{2C} \ln\left(\frac{4\big(1 - 2C^2 \pm h + \sqrt{1-C^2}\,\sqrt{(h \mp 1)^2 - 4(C^2 \mp h)}\,\big)}{h \mp 1} \right)$$

$$- \frac{\ln(Cx)}{C} + \frac{\ln\big[h^2 - 1 + (h \mp 1)\sqrt{(h \mp 1)^2 - 4(C^2 \mp h)}\,\big]}{2C}$$

$$\pm \frac{1}{2} \arctan\left(\frac{\sqrt{(h \pm 1)^2 - 4C^2}}{2C} \right),$$

where $C = H_1 = h_{11}$ is a constant and $h = Cxy + 1$. The orthogonal equiareal pattern corresponding to $h_{00} = 1$ and $h_{11} = 1/4$ can be seen below.

In the second case, when $h_{00} = \pm(1 + h_{11}^2)/2h_{11}$, we obtain ϕ and θ that cannot be simultaneously real.

A picture

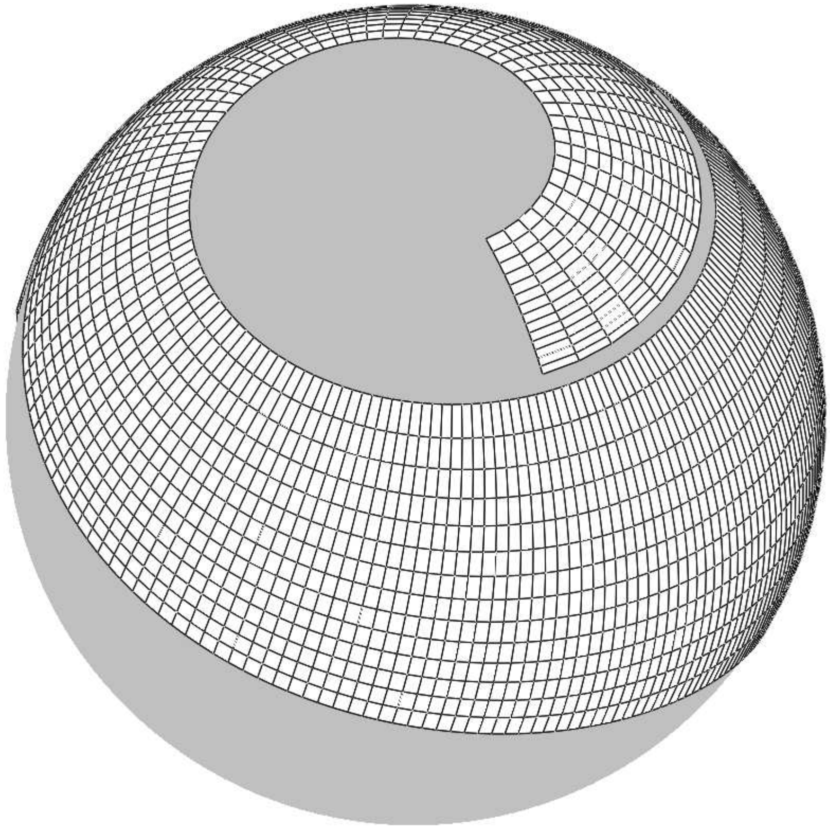

Figure 5: The orthogonal equiareal pattern on the sphere corresponding to one of the Lipschitz solutions