Elimination of the node in the Kinoshita problem

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Kinoshita problem

[Kinoshita(1970)]

The problem of a point mass and the rigid body.

The configuration space:
—relative position of the mass centres of the bodies (ρ, z, v);
—orientation of the body frame by the Euler angles (φ, θ, ψ);

After reduction the system has four degrees of freedoms

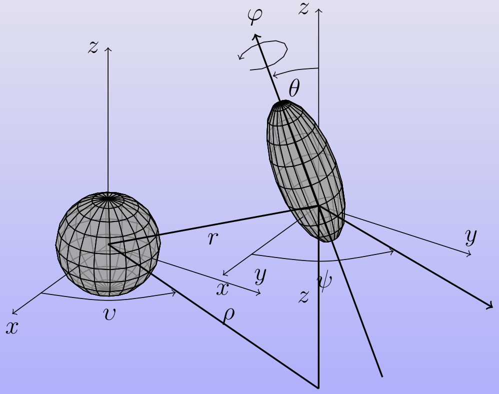

Figure : The geometry of the problem

Kinoshita's results

[Kinoshita(1970)]

1. "Spoke" motion
   The axis of symmetry lies along the relative position vector.
2. "Float" motion
   The axis of symmetry is always perpendicular to the orbital plane.
3. "Arrow" motion
   The axis of symmetry lies in the plane formed by the tangent and normal to the orbital plane vector.

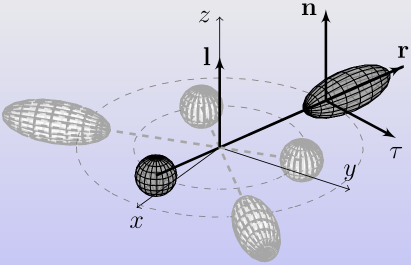

Figure : Spoke motion

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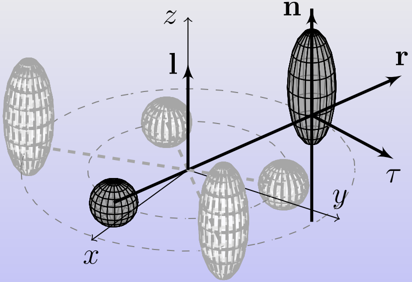

Figure : Float motion

Kinoshita's results

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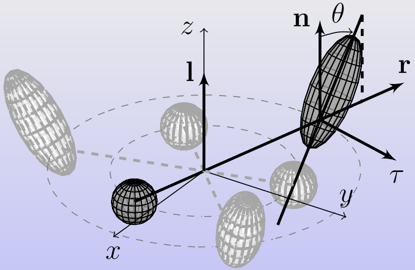

Figure : Arrow motion

Missing points

1. where are the non great circle solutions? (Aboelnaga, Barkin, 1979, 1980)
2. ...

Equations of motion in rotating reference frame

[Wang et al.(1991)Wang, Krishnaprasad, and Maddocks]

- The orbital motion

  $\mathbf{R}$ is the relative position vector
  $\mathbf{P}$ is the linear momentum of the system
  $\mathbf{G}$ is angular momentum of the body

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{R} &= \mathbf{R}\times\boldsymbol{\Omega}+\mathbf{P}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P} &= \mathbf{P}\times\boldsymbol{\Omega}-\frac{\partial U}{\partial \mathbf{R}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} &= \mathbf{G}\times\boldsymbol{\Omega}+\mathbf{R}\times\frac{\partial U}{\partial \mathbf{R}}, \end{aligned}\right\} \quad (1)$$

  where $\boldsymbol{\Omega}=\mathbf{I}^{-1}\mathbf{G}$ is the angular velocity of the body
  $\mathbf{I}$ is the tensor of inertia of the body

- The rotational motion

$$\dot{\mathbf{A}} = \mathbf{A}\widehat{\boldsymbol{\Omega}} \quad (2)$$

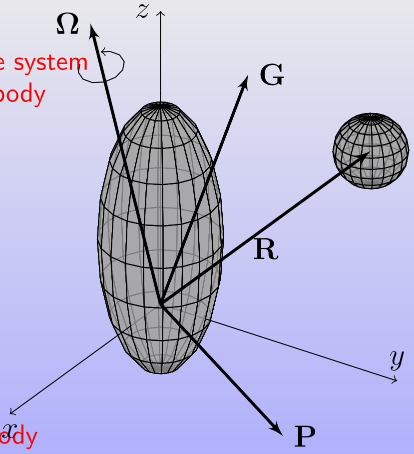

Figure : Geometry of the problem

Hamiltonian form of the system

Proposition

For 2 arbitrary smooth functions f_1 *and* f_2 *of dynamical variables* $\mathbf{z} = \left[\mathbf{R}^T, \mathbf{P}^T, \mathbf{G}^T\right] \in \mathbb{R}^9$ *we define*

$$\{f_1, f_2\} = \left[\frac{\partial f_1}{\partial \mathbf{z}}\right]^T \mathbf{\Lambda}(\mathbf{z}) \left[\frac{\partial f_2}{\partial \mathbf{z}}\right], \qquad (3)$$

where

$$\mathbf{\Lambda}(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & \mathbf{E} & \widehat{\mathbf{R}} \\ -\mathbf{E} & \mathbf{0} & \widehat{\mathbf{P}} \\ -\widehat{\mathbf{R}} & \widehat{\mathbf{P}} & \widehat{\mathbf{G}} \end{bmatrix} \qquad (4)$$

and $\{\cdot,\cdot\}$ *has properties of Poisson bracket (i.e., satisfies the Jacobi and Leibniz identities).*

Hamiltonian form of the system

Proposition

Then the equations of motion can be written as Hamilton's equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \{\mathbf{z}, H(\mathbf{z})\}\,, \tag{3}$$

where $H(\mathbf{z})$ *is equal to the total energy of the system and is given by:*

$$H = \frac{1}{2}\langle \mathbf{P}, \mathbf{P}\rangle + \frac{1}{2}\langle \mathbf{G}, \mathbf{I}^{-1}\mathbf{G}\rangle + U(\mathbf{R}) \tag{4}$$

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The introduced Poisson bracket is degenerated and the magnitude of the total angular momentum is a Casimir function, i.e., the first integral of motion.

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mathbf{G}, \quad L = \langle \mathbf{L}, \mathbf{L}\rangle = const \tag{5}$$

Reduction for axially symmetric case

An additional first integral of motion:

$$\mathbf{G}_3 = \mathbf{const} \quad (\boldsymbol{\Omega}_3 = \mathbf{const}) \tag{6}$$

The symmetry:

$$U(\mathbf{R}) = U\left(\mathbf{A}_3(\varphi)\mathbf{R}\right) \text{ for all } \varphi \in [0, 2\pi]$$

where

$$\mathbf{A}_3(\varphi) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(2, \mathbb{R})$$

Diagonal group action

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{P} \\ \mathbf{G} \end{bmatrix} \xrightarrow{\mathrm{SO}(2,\mathbf{R})} \begin{bmatrix} \widetilde{\mathbf{R}} \\ \widetilde{\mathbf{P}} \\ \widetilde{\mathbf{G}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3(\varphi)\mathbf{R} \\ \mathbf{A}_3(\varphi)\mathbf{P} \\ \mathbf{A}_3(\varphi)\mathbf{G} \end{bmatrix}$$

see Vereshchagin, Maciejewski, Goździewski, MNRAS, 2010.

Solution 1: Cylindrical precession

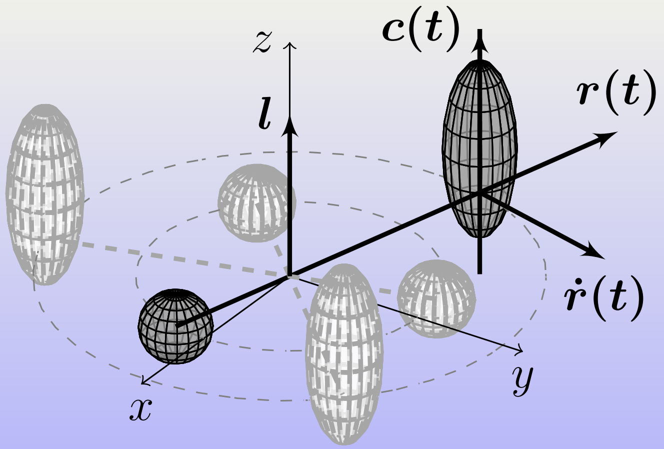

Figure : Cylindrical precession

Solution 2: Inclined planar precession

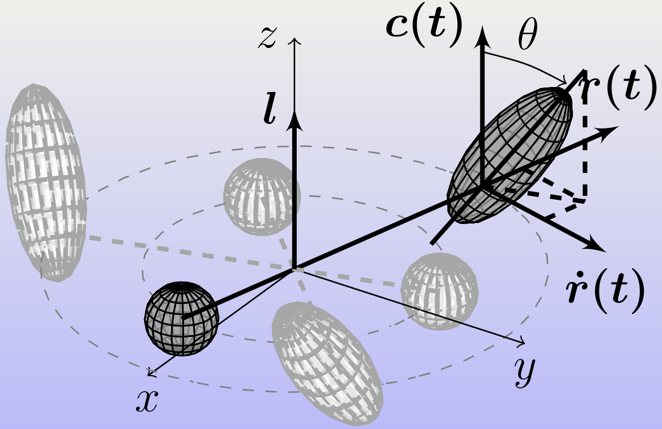

Figure : Inclined planar precession

Solution 3: Conic precession

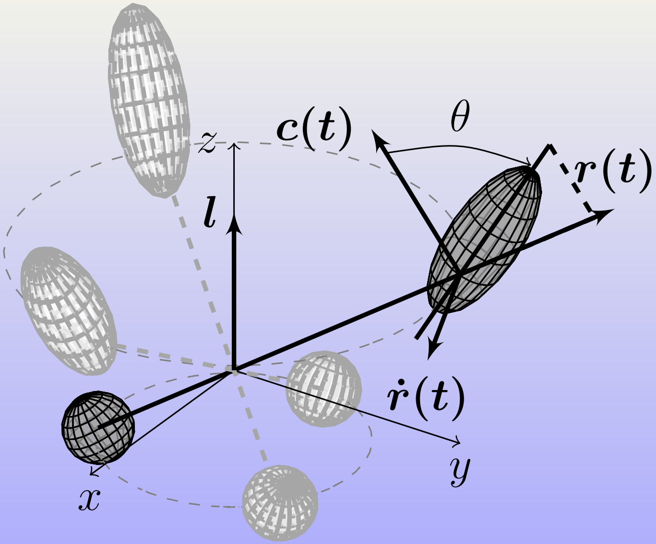

Figure : Conic precession

Is there anything left to do?

- Kinoshita: four degrees of freedom;
- we: seven variable and one additional integral: the total angular momentum;
- Question: can we reduce to three degrees of freedom!

Standard and non-standard coordinates

Euler angles (q_1, q_2, q_3) and conjugated momenta (p_1, p_2, p_3)

$$\mathcal{H} = \frac{1}{2}(A\Omega_1^2 + B\Omega_2^2 + C\Omega_3^2)$$

where

$$\begin{aligned}
\Omega_1 =& \frac{1}{A}\left[p_2 \cos q_3 + \frac{\sin q_3}{\sin q_2}\left(p_1 - p_3 \cos q_2\right)\right], \\
\Omega_2 =& \frac{1}{B}\left[p_2 \sin q_3 - \frac{\cos q_3}{\sin q_2}\left(p_1 + p_3 \cos q_2\right)\right], \\
\Omega_3 =& \frac{1}{C} p_3.
\end{aligned}$$

The Androyer-Deprit variables (l, g, h) and conjugated momenta (L, G, H)

$$\mathcal{H} = \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) + \frac{1}{2}\frac{L^2}{C}.$$

New variables

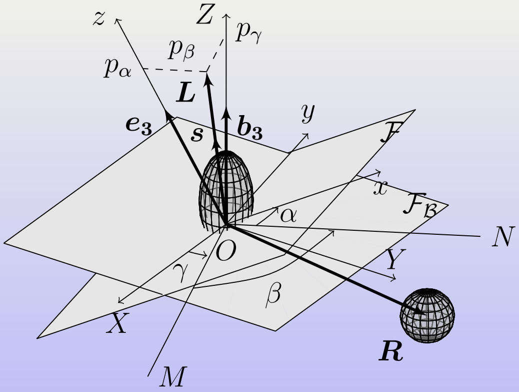

Figure : Reference frame

α, β, γ are Depri-Andoyer angles. p_α, p_β and p_γ are their generalized momentums correspondingly. $\mathbf{R}$ is a relative position vector.

Attitude of the rigid body

$$p_\alpha = \langle \frac{\partial \mathbf{\Omega}}{\partial \dot\alpha} \times \mathbf{R}, \mathbf{P} \rangle + \langle \frac{\partial \mathbf{\Omega}}{\partial \dot\alpha}, \mathbf{G} \rangle = \langle \frac{\partial \mathbf{\Omega}}{\partial \dot\alpha}, \mathbf{R} \times \mathbf{P} + \mathbf{G} \rangle = \langle \mathbf{E}_3, \mathbf{L} \rangle = l_3,$$
$$p_\beta = \langle \mathbf{S}, \mathbf{L} \rangle = L = l, \qquad p_\gamma = \langle \mathbf{B}_3, \mathbf{L} \rangle = L_3,$$

The total angular momentum vector:

$$\mathbf{L} = \left(\sqrt{p_\beta^2 - p_\gamma^2} \sin\gamma, \sqrt{p_\beta^2 - p_\gamma^2} \cos\gamma, p_\gamma \right)^T \tag{7}$$

The angular momentum of the rigid body:

$$\mathbf{G} = \mathbf{L} - \mathbf{R} \times \mathbf{P} = \begin{bmatrix} \sqrt{p_\beta^2 - p_\gamma^2} \sin\gamma + R_3 P_2 - R_2 P_3 \\ \sqrt{p_\beta^2 - p_\gamma^2} \cos\gamma + R_1 P_3 - R_3 P_1 \\ p_\gamma + R_2 P_1 - R_1 P_2 \end{bmatrix} \tag{8}$$

Reduced Hamiltonian for non-symmetric body

$$H = \frac{P_1^2}{2} + \frac{P_2^2}{2} + \frac{P_3^2}{2} + \frac{\left(\sqrt{p_\beta^2 - p_\gamma^2}\sin\gamma + R_3P_2 - R_2P_3\right)^2}{2A} + \frac{\left(\sqrt{p_\beta^2 - p_\gamma^2}\cos\gamma + R_1P_3 - R_3P_1\right)^2}{2B} + \frac{(p_\gamma + R_2P_1 - R_1P_2)^2}{2C} + U(R_1, R_2, R_3) \tag{9}$$

Properties of the Hamiltonian

1. Hamiltonian depends only on four non-cyclic variables R_1, R_2, R_3 and γ, the others are cyclic. Thus, Hamiltonian has been reduced to a one of eight degrees of freedom.
2. The potential is independent on the attitude of the rigid body.
3. The following momentums of the cyclic variables correspond to the first integrals of motion:
   3.1 p_α is the third component of total angular momentum vector in

Reduction

First canonical change of variables:

$$\rho = \sqrt{R_1^2 + R_2^2}, \quad \upsilon = \arctan \frac{R_2}{R_1}, \quad z = R_3$$
$$p_\rho = P_1 \cos \upsilon + P_2 \sin \upsilon, \quad p_\upsilon = -P_1 \rho \sin \upsilon + P_2 \rho \cos \upsilon, \quad p_z = P_3, \tag{10}$$

Second canonical change of variables:

$$\chi = \gamma + \upsilon, \quad f = \gamma,$$
$$p_\chi = p_\upsilon, \quad p_f = p_\gamma - p_\upsilon$$

The reduced Hamiltonian:

$$H = \frac{p_\rho^2}{2} + \frac{p_\chi^2}{2\rho^2} + \frac{p_z^2}{2} + \frac{p_f^2}{2C} + \frac{\left(\sqrt{p_\beta^2 - (p_f + p_\chi)^2} \cos \chi + \rho p_z - z p_\rho\right)^2}{2A} + \frac{\left(\sqrt{p_\beta^2 - (p_f + p_\chi)^2} \sin \chi + \frac{z}{\rho} p_\chi\right)^2}{2A} + U(\rho, z). \tag{11}$$

Reduction

The reduced Hamiltonian:

$$H = \frac{p_\rho^2}{2} + \frac{p_\chi^2}{2\rho^2} + \frac{p_z^2}{2} + \frac{p_f^2}{2C} + \frac{\left(\sqrt{p_\beta^2 - (p_f + p_\chi)^2}\cos\chi + \rho p_z - z p_\rho\right)^2}{2A} + \frac{\left(\sqrt{p_\beta^2 - (p_f + p_\chi)^2}\sin\chi + \frac{z}{\rho} p_\chi\right)^2}{2A} + U(\rho, z). \tag{10}$$

Properties of Hamiltonian

1. Hamiltonian depends on three non-cyclic variables: ρ, z and χ.
2. The cyclic variables represent all first integrals of motion
   - 2.1 p_α is the third component of the total angular momentum of the system.
   - 2.2 p_β is the magnitude of the total angular momentum of the system.
   - 2.3 p_f is the third component of the angular momentum of the rigid body.

Do it simpler!

New coordinates:

$$\boldsymbol{Z} := [R_1, R_2, R_3, P_1, P_2, P_3, L_1, L_2, L_3]^T = \left[\mathbf{R}^T, \mathbf{P}^T, \mathbf{L}^T\right]^T.$$

where $\mathbf{L} = \mathbf{G} + \mathbf{R} \times \mathbf{P}$. Poisson matrix

$$\mathbf{\Pi}(\mathbf{Z}) := \begin{bmatrix} \mathbf{0} & \mathbf{E} & \mathbf{0} \\ -\mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{L}} \end{bmatrix},$$

equations of motion

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{R} &= \mathbf{R} \times \mathbf{\Omega} + \mathbf{P}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P} &= \mathbf{P} \times \mathbf{\Omega} - \frac{\partial U}{\partial \mathbf{R}}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L} &= \mathbf{L} \times \mathbf{\Omega}, \end{aligned}\right\}$$

where now $\mathbf{\Omega} = \mathbf{I}^{-1}(\mathbf{L} - \mathbf{R} \times \mathbf{P})$. The Hamiltonian of the system is

Do it simpler!

Hamiltonian

$$H := \frac{1}{2}\langle \mathbf{P}, \mathbf{P}\rangle + \frac{1}{2}\langle \mathbf{L} - \mathbf{R}\times\mathbf{P}, \mathbf{\Omega}\rangle + U(\mathbf{R}), \qquad \mathbf{\Omega} = \mathbf{I}^{-1}\left(\mathbf{L} - \mathbf{R}\times\mathbf{P}\right).$$

Levels of the Casimir are described simply as

$$M_L := \left\{(\mathbf{R}, \mathbf{P}, \mathbf{L}) \in \mathbb{R}^9 \mid \ \|\mathbf{L}\| = L\,\right\}.$$

Now, we can introduce new coordinates (φ, I, L) which are defined by the following equations

$$L_1 = \sqrt{L - I}\sin\varphi, \qquad L_2 = \sqrt{L - I}\cos\varphi, \qquad L_3 = I$$

$$\begin{aligned} H = &\frac{1}{2}\langle \mathbf{P}, \mathbf{P}\rangle + \frac{1}{2A}\left(\sqrt{L - I}\sin\varphi - R_2P_3 + R_3P_2\right)^2 + \\ &\frac{1}{2B}\left(\sqrt{L - I}\cos\varphi - R_3P_1 + R_1P_3\right)^2 + \frac{1}{2C}\left(I - R_1P_2 + R_2P_1\right)^2 + U(\mathbf{R}), \end{aligned}$$

H. Kinoshita.
Stationary Motions of an Axisymmetric Body around a Spherical Body and Their Stabilities.
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