

Laplace-Moutard-Darboux
transformations, geometry and
applications

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Laplace transformations. The general Laplace equation (LE)

$$\phi_{xy} + \alpha(x, y)\phi_x + \beta(x, y)\phi_y + \gamma(x, y)\phi = 0$$

goes to

$$\psi_{xy} + a\psi_y + b\psi = 0 \quad (1)$$

after gauge transformation $\phi = g\psi$. Laplace transformations (LT)

$$a \rightarrow a_{-1} = a - \partial_x \ln(b - a_y), \quad b \rightarrow b_{-1} = b - a_y,$$

$$\psi \rightarrow \psi_{-1} = \psi_x + a\psi,$$

$$a \rightarrow a_1 = a + \partial_x \ln b, \quad b \rightarrow b_1 = b + \partial_y (a + \partial_x \ln b),$$

$$\psi \rightarrow \psi_1 = \frac{\psi_y}{b}$$

can be taken as a starting point in the theory of soliton equations in 2+1 dimensions. The LT is a kind of "dressing", it leads to a "partial" factorization of the operator of (2) and in the case of zero *Laplace invariants* at some step of LT iterations allows us to build explicit solutions.

Moutard and Goursat equations

A constraint for the coefficients a and b of LE

$$\psi_{xy} + a\psi_y + b\psi = 0 \quad (2)$$

fixes a particular class of equations which we are interesting in. Namely, the condition

$$a = 0, \quad b = u, \quad (3)$$

yields the *Moutard equation* (ME)

$$\psi_{xy} + u(x, y)\psi = 0, \quad (4)$$

while

$$a = -\frac{1}{2}\partial_x \ln \lambda, \quad b = -\lambda \quad (5)$$

leads to the equation

$$\psi_{xy} = \frac{1}{2}(\ln \lambda)_x \psi_y + \lambda \psi \quad (6)$$

Which, by the substitution $\psi = \sqrt{\zeta_x}$, $\chi = \sqrt{\zeta_y}$ and

$$\psi_y = \sqrt{\lambda}\chi, \quad \chi_x = \sqrt{\lambda}\psi$$

relates to the *Goursat equation* (GE)

$$\zeta_{xy} = 2\sqrt{\lambda \zeta_x \zeta_y} . \quad (7)$$

In a context of the DT theory, more exact,

combined Darboux–Laplace (CDL) transformations

$$a \rightarrow a_1 = a - \partial_x \ln(a + \sigma) , \quad b \rightarrow b_1 = b + \sigma_y , \quad (8)$$

$$\psi \rightarrow \psi_1 = \psi_x - \sigma \psi , \quad (9)$$

$$a \rightarrow {}_1a = -(\sigma + b\rho) , \quad b \rightarrow {}_1b = b - (b\rho)_y , \quad (10)$$

$$\psi \rightarrow {}_1\psi = \rho\psi_y - \psi . , \quad (11)$$

where $\sigma = \sigma(x, y) = \phi_x/\phi$, $\rho = \phi/\phi_y$, and ψ and ϕ are particular solutions of (2) with predetermined a and b . We refer to ϕ as *the support function* of the DT.

The functions u and λ may be considered as solutions of the special equations which we call the *reduction equations*. E.g., the family of systems ($C_{1,2} \in C$ -parameters):

$$\phi_{xy} = \phi_y [F_x + 2C_1 \phi \exp F] , \quad F_y \phi_y = C_2 \phi .$$

Its integrability with respect to the combined transformations (CDL) is established, see:

E.Doktorov S. Leble, Dressing method in mathematical physics, Springer 2007.

Moutard transformation

The Moutard transformation (Moutard, Th.F., C.R. Acad. Sci 1875) is a map that connects solutions and the coefficient $u(x, y)$ of the equation (4), then the solution of the twin equation with $\psi \rightarrow \psi[1]$ and $u(x, y) \rightarrow u[1](x, y)$ can be constructed by the solution of the system

$$(\psi[1]\varphi)_x = -\varphi^2(\psi\varphi^{-1})_x,$$

$$(\psi[1]\varphi)_y = \varphi^2(\psi\varphi^{-1})_y.$$

In other terms,

$$\psi[1] = \psi - \varphi\Omega(\varphi, \psi)/\Omega(\varphi, \varphi), \quad (12)$$

where Ω is the integral of the exact differential form

$$d\Omega = \varphi\psi_x dx + \psi\varphi_y dy. \quad (13)$$

and if φ and ψ are different solutions of (4). The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log \varphi)_{xy} = -u + \varphi_x\varphi_y/\varphi^2.$$

Darboux transformation as reduction: ignore y-dependence.

From old

Euler, L. (1780) Methodus Nova Investigandi omnes casus cuibus hanc aequationem differentio-differentialen, M.S. Academiae exhibit aie Ianuarii 1780 Institutiones calculi integralis 4, 533-543.

to recent

De Rham complex point of view: (A Prykarpatskij, CEJM 2005), with beautiful title:

"The generalized de Rham-Hodge theory aspects of Delsarte-Darboux type transformations in multidimension".

The important feature of the MT is general/basic for DT: the transform is parameterized by a pair of solutions of the equation and the transform vanishes if the solutions coincide. The Moutard equation is transformed to 2-dimensional Schrödinger equation and studied in connection with the central problems of the classical differential geometry (e.g. recent book Gu et al).

In the soliton theory the ME enters the Lax pairs for nonlinear equations such as the KP, 2+1 KdV equations.

In the case of GE:

We refer to λ as the potential function. The reduction (5) is valid only for special types of potentials if the form of the Laplace equation is supported while transformations are performed. Our interest to the GE is caused by applications of this equation in geometry and in the soliton theory.

An interesting application of the GE is a link to the **two-dimensional Dirac equation** (Leble, S.B. and Yurov, A.V. JMP 2002

Classical geometry

As regards *geometry*, let x be the complex coordinate, $y = -\bar{x}$, $\sqrt{\lambda}$ is the real-valued function and ψ or χ as solutions of (6) are complex-valued functions. An original Weierstrass formulae starts with two arbitrary holomorphic functions of the complex variables $z = x' + iy'$, $\bar{z} \in C$ and leads to minimal surfaces. Its generalization use three real-valued functions X_i , $i = 1, 2, 3$ which are the coordinates of a surface in \mathbb{R}^3 (Konopelchenko 1993, equivalent to Kenmotsu 1979, but linear in ψ, χ):

$$X_1 + iX_2 = 2i \int_{\Gamma} (\overline{\psi}^2 dy' - \overline{\chi}^2 dx') ,$$

$$X_1 - iX_2 = -2i \int_{\Gamma} (\psi^2 dy' - \chi^2 dx') , \quad (14)$$

$$X_3 = -2 \int_{\Gamma} (\overline{\psi}\chi dy' + \overline{\chi}\psi dx') ,$$

where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K and the mean curvature H yield:

$$ds^2 = 4U^2 dx dy , \quad K = \frac{1}{U^2} \partial_x \partial_y \ln U , \quad H = \frac{\sqrt{\lambda}}{U} .$$

Here $U = |\psi|^2 + |\chi|^2$ and any analytic surface in \mathbb{R}^3 can be globally represented by (14). (Leble, S.B. and Yurov, A.V. JMP 2002)

Soliton 2D-MKdV equation

As an example of soliton equations, consider the system of the 2D-MKdV equations introduced by Boiti, Leon, Martina, and Pempinelli:

$$\begin{aligned} &4\lambda^2(\lambda_t - A\lambda_x + B\lambda_y - \lambda_{xxx} - \lambda_{yyy}) + \\ &4\lambda^3 [(2\lambda + B)_y + (2\lambda - A)_x] \\ &+ 6\lambda(\lambda_y\lambda_{yy} + \lambda_x\lambda_{xx}) - 3(\lambda_x^3 + \lambda_y^3) = 0, \quad (15) \end{aligned}$$

$$B_x = 3\lambda_y - \lambda_x, \quad A_y = \lambda_y - 3\lambda_x.$$

Here $\lambda = \lambda(x, y, t)$, $A = A(x, y, t)$ and $B = B(x, y, t)$.

If we introduce the function $u = \sqrt{\lambda}$ then we can rewrite (15) in the more customary form:

$$u_t + 2u^2(u_x + u_y) + \frac{1}{2}(B_y - A_x)u + Bu_y - Au_x - u_{3y} - u_{3x} = 0$$

$$B_x = (3\partial_y - \partial_x)u^2, \quad A_y = (\partial_y - 3\partial_x)u^2. \quad (16)$$

The reduction conditions $A = -B = -2u^2$ and $u_y = u_x$ lead to the MKdV equation,

$$u_t + 12u^2u_x - 2u_{3x} = 0,$$

so one calls (16) either BLMP, or the 2D-MKdV equations.

The 2D-MKdV (BLMP) equations (16) are the compatibility condition of the linear system comprising (6) and

$$\psi_t = \psi_{3x} + \psi_{3y} - \frac{3\lambda_y}{2\lambda}\psi_{yy} + \left[\frac{3}{4} \left(\frac{\lambda_y}{\lambda} \right)^2 - \lambda - B \right] \psi_y$$

$$+ (A - \lambda)\psi_x + \frac{1}{2}(A_x - \lambda_x)\psi .$$

Resembles Niznik-Veselov-Novikov eq. We will study (16) in next Section.

Moutard- Goursat transformations We employ the MGT and binary MGT for a construction of explicit solutions of the GE. These transformations allow us to obtain new solutions of the GE without solving reduction equation. We also discuss the transformation for Laplace invariants.

Theorem (Leble, S.B. and Yurov, A.V. JMP 2002) Let the transform $\psi[1]$ is introduced by the relations

$$\begin{aligned} (z_1\psi[1]/\psi_1)_x &= z_1(\psi_2/\psi_1)_x , \\ (z_1\psi[1]/\psi_1)_y &= [z_1z_{1xy} - 2z_{1x}z_{1y}/z_{1xy}](\psi_2/\psi_1)_y , \end{aligned} \quad (17)$$

where $z_{1,2}$ are solutions of (7), $\psi_{1,2} = \sqrt{z_{1,2x}}$ solve (6), then $\psi[1]$ is a solution of the (transformed) (6) with the potential

$$\lambda[1] = \lambda - (\ln z_1)_{xy}$$

and the transform $z[1]$ is found by a quadrature from

$$\begin{aligned} z[1]_x &= \psi[1]^2, \\ z[1]_y &= (\psi[1]_y)^2/\lambda[1] . \end{aligned} \quad (18)$$



Remark

The transformed function $\psi[1]$ and potential $\lambda[1]$ are extracted by quadratures.

This transformation preserves the form of the Laplace–Goursat equation (6), e.g., the Eq. possesses **the covariance property**.

Binary Moutard-Goursat transformation

Next we introduce a binary DT for the GE with the same property.

We introduce new variables $\xi = x + y$ and $\eta = x - y$ and rewrite (6) in the matrix form,

$$\Psi_\eta = \sigma_3 \Psi_\xi + U \Psi, \quad (19)$$

where

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \chi_1 & \chi_2 \end{pmatrix}, \quad U = \sqrt{\lambda} \sigma_1, \quad (20)$$

$\psi_k = \psi_k(\xi, \eta)$, $\chi_k = \chi_k(\xi, \eta)$, $k = 1, 2$ are particular solutions of (6) with some $\lambda(\xi, \eta)$, $\sigma_{1,3}$ are the Pauli matrices.

Let Ψ_1 is some solution of (19) and $\Psi \neq \Psi_1$. We define a matrix function $\sigma \equiv \Psi_{1,\xi} \Psi_1^{-1}$. Equation (19) is covariant with respect to the classical DT:

$$\Phi[1] = \Phi_\xi - \sigma\Phi, \quad U[1] = U + [\sigma_3, \sigma]. \quad (21)$$

It is a particular case of the general classical Matveev (non-Abelian) formula.

Remark It is not difficult to check that the DT (21) is the superposition formula for two simpler Darboux transformations.

Equation (20) is the spectral problem for the Davey–Stewartson (DS) equation. LT produces an explicitly invertible Bäcklund autotransformations for the DS equation. It is shown that these transformations permit to construct solutions to the DS equation that fall off in all directions in the plane according to exponential and algebraic laws.

Let us next consider *a closed 1-form*

$$d\Omega = d\xi \Phi \Psi + d\eta \Phi \sigma_3 \Psi , \quad \Omega = \int d\Omega ,$$

where a 2×2 matrix function Φ solves the equation

$$\Phi_\eta = \Phi_\xi \sigma_3 - \Phi U . \quad (22)$$

We apply the DT for (20). It can be verified by immediate substitution that (22) is covariant with respect to the transformation

$$\Phi[+1] = \Omega(\Phi, \Psi_1) \Psi_1^{-1} .$$

We can alternatively affect U by the following transformation:

$$U[+1, -1] = U + [\sigma_3, \Psi_1 \Omega^{-1} \Phi] .$$

The particular solution of (22) has the form

$$\Phi_1 = \begin{pmatrix} s_1 \psi_1 + s_2 \psi_2 & -s_1 \chi_1 - s_2 \chi_2 \\ s_3 \psi_1 + s_4 \psi_2 & -s_3 \chi_1 - s_4 \chi_2 \end{pmatrix} , \quad (23)$$

where $s_k = \text{const}$ ($k = 1, \dots, 4$). It is convenient to choose Φ_1 in the form

$$\Phi_1 = \Psi_1^t \sigma_3 , \quad (24)$$

where the superscript "t" stands for the transpose.

Equation (24) is the particular case of (23).

In this case

$$U[+1, -1] = U - 2A_F , \quad (25)$$

where A_F is the off-diagonal part of the matrix $A = \Psi_1 \Omega^{-1} \Psi_1^t$, $\Omega = \Omega(\Phi_1, \Psi_1)$ and

$$A_F^t = A_F = f\sigma_1 , \quad (26)$$

where $f = f(\xi, \eta)$ is some function.

Using (21), (25) and (26), we arrive at important

Observation

$U[+1, -1]$ has the same form that the initial matrix U :

$$U[+1, -1] \equiv \begin{pmatrix} 0 & \sqrt{\lambda[+1, -1]} \\ \sqrt{\lambda[+1, -1]} & 0 \end{pmatrix} = \\ \begin{pmatrix} 0 & \sqrt{\lambda} - 2f \\ \sqrt{\lambda} - 2f & 0 \end{pmatrix},$$

thus the reduction restriction is valid without the reduction equations.

The new function $\Phi[+1, -1]$ has the form

$$\Phi[+1, -1] = \Phi - \Omega(\Phi, \Psi_1) (\Omega(\Phi_1, \Psi_1))^{-1} \Phi_1, \quad (27)$$

where Φ is an arbitrary solution of the spectral problem (22).

Using binary DT (25) and (27), we can construct new solution of the GE by means of dressing a particular solution. As a result we get the following theorem (returning to the former variables x and y):

Theorem

Let

$$\psi_{k,y} = \sqrt{\lambda}\chi_k, \quad \chi_{k,x} = \sqrt{\lambda}\psi_k,$$

$$\alpha_{k,y} = -\sqrt{\lambda}\beta_k, \quad \beta_{k,x} = -\sqrt{\lambda}\alpha_k,$$

where $k = 1, 2$. Then new functions

$$\alpha'_1 = \alpha_1 - \frac{A_1\psi_1 + A_2\psi_2}{D}, \quad \beta'_1 = \beta_1 + \frac{A_1\chi_1 + A_2\chi_2}{D}$$

are solutions of the equations

$$\alpha'_{1,y} = \sqrt{\lambda'}\beta'_1, \quad \beta'_{1,x} = \sqrt{\lambda'}\alpha'_1,$$

where

$$\sqrt{\lambda'} = -\sqrt{\lambda} + \frac{\psi_1\chi_1\Omega_{22} + \psi_2\chi_2\Omega_{11} - (\psi_1\chi_2 + \psi_2\chi_1)\Omega_{12}}{D}$$

and

$$\Omega_{11} = \int dx \psi_1^2 + dy \chi_1^2 ,$$

$$\Omega_{12} = \Omega_{21} = \int dx \psi_1 \psi_2 + dy \chi_1 \chi_2 ,$$

$$\Omega_{22} = \int dx \psi_2^2 + dy \chi_2^2 , \quad D = \Omega_{11} \Omega_{22} - \Omega_{12}^2 ,$$

$$\Lambda_{11} = \int dx \alpha_1 \psi_1 + dy \beta_1 \chi_1 ,$$

$$\Lambda_{12} = \int dx \alpha_1 \psi_2 + dy \beta_1 \chi_2 ,$$

$$\Lambda_{21} = \int dx \alpha_2 \psi_1 + dy \beta_2 \chi_1 ,$$

$$\Lambda_{22} = \int dx \alpha_2 \psi_2 + dy \beta_2 \chi_2 ,$$

$$A_1 = \Lambda_{11} \Omega_{22} - \Lambda_{12} \Omega_{12} , \quad A_2 = \Lambda_{12} \Omega_{11} - \Lambda_{11} \Omega_{12} .$$

Here $f = \int_{\Gamma}$, where Γ is an arbitrary path of integration in the plane. The explicit expressions for the functions α'_2 and β'_2 are obtained by the direct picking up of the indicated relations. ■

Thus the binary DT allows us to construct explicit solutions of the GE without solving reduction equation.

Geometry

Zero range potentials and dressing

History starts from Fermi δ potential (1936)

Yu. Demkov V. Ostrovski book "ZRP method..."

mathematical aspect: from **F. Berezin**, L. Faddeev observation DAN 1961, A note on Schrodinger equation with singular potential



Extension theory: Neumann, Krein (*widen space*), ...

B. Pavlov, M. Faddeev: - go to **Infinite-dim space**

Our **first observation** shows that generalized ZRPs (see Huang, Yang 1957, corrected: Derevianko 2005) appear as a result of **ZRP Darboux transformations (Leble, Yalunin 2002)**. In order to demonstrate it we consider a radial Schrödinger equation for partial wave ψ_l with orbital momentum l . The atomic units are used

$$\left(-\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{l(l+1)}{2r^2} + u_l - E \right) \psi_l(r) = 0, \quad (28)$$

where u_l are potentials for the partial waves with the following asymptotics at infinity

$$\psi_l(r) \sim \frac{\sin(kr - \frac{l\pi}{2} + \delta_l)}{kr}. \quad (29)$$

The equation (28) describes scattering of a particle with energy E and momentum $k = \sqrt{2E}$.

In the absence of the potential, partial shifts $\delta_l = 0$ and partial waves can be expressed via Bessel functions with half-integer indices. Let us demonstrate that generalized ZRP can be introduced by DT. For our purpose it is convenient to use a chain of DTs (Crum formulas with the wave and prop functions multiplied by r), which for our equation look like

$$\psi_l^{(1)} = \text{const} \cdot \frac{W(r\psi_l, r\phi_1, \dots, r\phi_{2l+1})}{rW(r\phi_1, \dots, r\phi_{2l+1})}, \quad (30)$$

$$u_l^{(1)} = u_l - (\ln W(r\phi_1, \dots, r\phi_{2l+1}))'', \quad (31)$$

where W is Wronskian, and

$$\phi_m = \sqrt{\frac{\pi}{2\kappa_m r}} H_{l+1/2}^{(1)}(\kappa_m r), \quad m = \overline{1, 2l+1} \quad (32)$$

and κ_m satisfy algebraical equation $\kappa_m^{2l+1} = i\alpha_l$ with real α_l . Here we assume $u_l = 0$. The transformation (30) combines the solution ψ_l and functions ϕ_m . The Crum formulas result from the replacement of a chain of $2l + 1$ first order transformations by a single $(2l + 1)$ th order transformation, which happens to be more efficient in practical calculations. Direct substitution of (32) to Wronskian shows

that

$$W(r\phi_1, \dots, r\phi_{2l+1}) = \text{const.} \quad (33)$$

Second important observation.

It means that dressed potential $u_l^{(1)}(r > 0) = 0$. The transformation (31) allows to calculate potential in range $r > 0$. We state that DTs also yield a generalized ZRP at $r = 0$. In order to prove this we perform transformation (30) and show that $\psi_l^{(1)}$ is a solution for a generalized ZRP. Since potential is equal zero in the region $r > 0$, it is enough to determine asymptotic behavior of the wave function. Substituting $\psi_l = \sqrt{\frac{\pi}{2kr}} J_{l+1/2}(kr)$ to the Crum formulas, and using well-known properties of Bessel functions, we obtain

$$\psi_l^{(1)} \sim \text{const} \cdot \left[(-i)^l \frac{e^{ikr}}{kr} \frac{\Delta(ik, \kappa_1, \dots, \kappa_{2l+1})}{\Delta(\kappa_1, \dots, \kappa_{2l+1})} - i^l \frac{e^{-ikr}}{kr} \frac{\Delta(-ik, \kappa_1, \dots, \kappa_{2l+1})}{\Delta(\kappa_1, \dots, \kappa_{2l+1})} \right],$$

where Δ is Wandermont determinant.

Considering one as the product

$$\Delta(ik, \kappa_1, \dots, \kappa_{2l+1}) = \text{const} \cdot \prod_{m=1}^{2l+1} (\kappa_m - ik) \quad (34)$$

we obtain an asymptotics, which coincides with the asymptotics of the solution:

$$\psi_l^{(1)} = \text{const} \cdot (H_{l+1/2}^{(1)}(kr)e^{2i\delta_l} - H_{l+1/2}^{(2)}(kr)), \quad (35)$$

where

$$\exp(2i\delta_l) = \prod_{m=1}^{2l+1} \frac{\kappa_m - ik}{\kappa_m + ik}.$$

Theorem The wave function (35) describes a scattering by generalized ZRP with orbital momentum l . One is conventionally represented as the boundary condition at $r = 0$ on the wavefunction.

$$\left. \frac{\left(\frac{d}{dr}\right)^{2l+1} r^{l+1} \psi_l^{(1)}}{r^{l+1} \psi_l^{(1)}} \right|_{r=0} = -\frac{2^l l! \alpha_l}{(2l-1)!!}, \quad (36)$$

where α_l – inverse scattering length for the partial wave with orbital momentum l . ■

It can be verified by direct substitution to the boundary condition for generalized ZRP:

Recall that at low-energies $\tan(\delta_l) \sim -a_l k^{2l+1}$ for a short-range potential, where a_l is the scattering length. In the special case of $l = 0$ we obtain $(\ln r\psi)' = -\alpha$. This generalized boundary condition can be extracted from asymptotics of the wave function at the vicinity of zero, which was used by some authors . Let us consider the scattering matrix on the complex k -plane. Each element $\exp(2i\delta_l)$ has $2l + 1$ poles at the points $k = i\kappa_m$, which lie on a circle of the complex plane. Since the bound states correspond to the poles on the imaginary positive semi-axis on complex k -plane, a bound state exists only if either $\alpha_l > 0$ and l is odd number or $\alpha_l < 0$ and l is even. Otherwise ZRP has an antibound state.

Dressing in a multi-center problem

The **second observation** is principal, it allows to built a zero-range potential eigenfunction in the multi-center problem. In a more general situation one can consider a system with a smooth potential plus a number of ZRP. If one knows the Green function for the smooth potential, then one can provide a solution for the problem with the ZRPs added. Generalization to the case with an arbitrary number of ZRP is straightforward. On the contrary, our general idea is to "dress" a multicenter system without Green function consideration. This procedure gives simple formulas for partial phases and their corrections at low energies.

Let us consider scattering problem for a non-spherical potential U :

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\hat{L}^2}{2r^2} + \hat{U} - E \right) \psi(\vec{r}) = 0, \quad (37)$$

where \hat{L}^2 is square of angular momentum operator, E describes the energy of particle. The asymptotic of wave function $\psi(\vec{r})$ looks like

$$\psi(\vec{r}) \stackrel{r \rightarrow \infty}{\sim} \exp(i\vec{k} \cdot \vec{r}) + f(\theta) \frac{e^{ikr}}{r}, \quad (38)$$

where $f(\theta)$ is scattering amplitude, which depends on scattering angle θ .

The operator \hat{L}^2 commutes with all radial derivatives, in particular with $\partial = \partial/\partial r$. The first order DT for Schrödinger equation (37) is

$$\begin{aligned}\psi^{(1)} &= (\partial - \hat{s})\psi, \\ \hat{U}^{(1)} &= \hat{U} + 1/r^2 - \hat{s}',\end{aligned}\tag{39}$$

and \hat{s} must be assumed as function of the operator variable \hat{L}^2 . The formula (39) gives non-local (over angles) potential which depends on \hat{L}^2 . In order to find operator \hat{s} we can use covariance principle for the equation (37). The covariance principle formally yields explicit constraint for \hat{s} , which gives

$$\hat{s}' + \frac{2}{r}\hat{s} + \hat{s}^2 = \frac{\hat{L}^2}{r^2} + 2\hat{U} + K^2,\tag{40}$$

It is supposed that the constant of integration (by r) $K = \sum_{n=0}^{\infty} K_n \hat{L}^{2n}$ is the analytical function of \hat{L}^2 . The operator \hat{s} can be found as series $\sum_{n=0}^{\infty} s_n \hat{L}^{2n}$ where the coefficients s_n depend only on r . It is easy to show that the equation leads to recursion relations for the coefficients s_n .

Thus, we have the algorithm that determine the operator \hat{s} and a dressed potential via the operator K . The choice $\hat{s}' = 0$ at the infinity that corresponds the desirable case $U(\hat{1}) \rightarrow 0$ (39) yields $\hat{s}(\infty) = K$. For our purpose (cross section evaluation) we need only partial phases or scattering amplitude related to operator K . In order to find the partial phases for a dressed potential we need to apply the DT to wave function. However, we have one trouble: in general DT modifies the plane wave $\exp(i\vec{k} \cdot \vec{r})$. Thus, DT applied to wave function $\psi(\vec{r})$ with asymptotic (38) gives an another asymptotic. In some particular cases, special choice of the operator K allows to avoid this problem. Indeed, consider the partial wave asymptotics for a non-spherical potential.

$$\psi_J(\vec{r}) \sim \frac{1}{2ikr} (e^{ikr+i\delta_J} \Lambda_J(\vec{n}) - e^{-ikr-i\delta_J} \Lambda_J(-\vec{n})), \quad (41)$$

where \vec{n} is unit vector directed as \vec{r} , δ_J denote partial shifts, and $\Lambda_J(\vec{n})$ are normalized eigenvectors of S-matrix operator (partial harmonics). The most simple formulas for the shifts $\delta_J^{(1)}$ for the potential $\hat{U}^{(1)}$ result when partial harmonic Λ_J are also eigenvectors of operator K .

For example, suppose all partial harmonic Λ_J are eigenvector of K but only Λ_0 has nonzero eigenvalue κ

$$K\Lambda_0(\vec{n}) = \kappa\Lambda_0(\vec{n}). \quad (42)$$

The asymptotic dressing is reduced to the action of the operator $\partial - K$ on asymptotic (41). It is easy to show by using expression

$$\ln \left(\frac{\kappa - ik}{\kappa + ik} \right) = -2i \arctan(k/\kappa), \quad (43)$$

for real-valued variables k, κ , that DT changes only the partial shift δ_0 as

$$\delta_0^{(1)} = \delta_0 - \arctan(k/\kappa). \quad (44)$$

In this special case we add only one additional parameter. In the region $k \gg |\kappa|$ the second term of the equation (44) practically does not contribute to the partial cross section

$$\sigma_J = \frac{4\pi}{k^2} \sin^2 \delta_J. \quad (45)$$

One observes an important contribution to the cross section when $k \approx |\kappa|$ and hence it can be considered as a correction at low energies. One of most important problems of solvable models is the problem fitting them to some physically meaning parameters.

Plane Pseudopotentials via Moutard transformations

Consider again the Moutard equation

$$\psi_{\sigma\tau} + u(\sigma, \tau)\psi = 0. \quad (46)$$

The Moutard transformation connects solutions and the coefficient

$u(\sigma, \tau)$ of the equation (46) so that if φ and ψ are different solutions of it (46), then the solution of the twin equation with $\psi \rightarrow \psi[1]$ and $u(\sigma, \tau) \rightarrow u[1]$ can be constructed by the system

$$(\psi[1]\varphi)_{\sigma} = -\varphi^2(\psi\varphi^{-1})_{\sigma},$$

$$(\psi[1]\varphi)_{\tau} = \varphi^2(\psi\varphi^{-1})_{\tau}.$$

In other words,

$$\psi[1] = \psi - \varphi \Omega(\varphi, \psi) / \Omega(\varphi, \varphi) , \quad (47)$$

where Ω is the integral of the exact differential form

$$d\Omega = \varphi \psi_{\sigma} d\sigma + \psi \varphi_{\tau} d\tau . \quad (48)$$

The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log \varphi)_{\sigma\tau} \quad (49)$$

Changing variables by the complex substitution $\sigma = x + iy$, $\tau = x - iy$ transforms (46) to a 2-dimensional Schrödinger equation for x, y for potentials linked by $U(x, y) = -u(\sigma, \tau) + E$

$$-\frac{1}{4}[\psi_{xx} + \psi_{yy}] + U(x, y)\psi = E\psi . \quad (50)$$

The explicit form of the ZRP depends on a choice of symmetry. For a cylindric symmetry [?], passing to polar coordinates $x = \rho \cos \phi, y = \rho \sin \phi$ and separating variables $\exp[i\nu\phi]R$ yields either R as the modified Bessel equation for $E = k^2 > 0$, or the Bessel equation for $E = -\kappa^2 < 0$. The case may be treated almost identically as in Sec. 2 by means of an iterated (multi-kink) MT, see the Wronskian formulas in [?] .

We, however, develop the theory by the MT, extending it to more general symmetry, rewriting the (??) in polar coordinates

$$U[1] = U + \frac{1}{2} \Delta(\log \varphi) = U + \frac{1}{2} \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \frac{1}{\rho^2} \frac{d^2}{d\phi^2} \right] (\log \varphi) \quad (51)$$

while $\psi[1]$ is the ψ transform by (47) with

$$\int d\Omega = \frac{1}{2} \int_{0,0}^{\rho,\phi} \left[(\psi\varphi)_\rho - i \frac{\varphi^2}{\rho} \left(\frac{\psi}{\varphi} \right)_\phi \right] d\rho + \left[(\psi\varphi)_\phi + i\rho\varphi^2 \right] \quad (52)$$

For $E = 0$, the Euler equation case in the ρ variable is obtained, and

a general solution is

$$\psi = \sum_{\nu=-\infty}^{+\infty} c_n \exp[i\nu\phi] \rho^\nu.$$

To demonstrate it by an example, let us substitute the

particular solutions $\varphi = \exp[i\nu\phi] \rho^\nu$ into the MT

formulas. Direct differentiation prove a potential

invariance $U[1] = U$. The same result gives the special case of

$$\nu = 0, \quad \varphi = C \ln \rho + A.$$

Consider the Hilbert space $H = L_2$ and a manifold of continuous functions $\psi \in M \subset H$. Applying Gauss theorem yields for a disk S inside a circumference L of small radius ϵ ,

$$\begin{aligned} \lim_{L \rightarrow 0} \int_S \Delta \psi dS + 2 \int_S \alpha \delta_2(\rho, \phi) \psi \rho d\rho d\phi = \\ \lim_{L \rightarrow 0} \int_L (\vec{n} \cdot \nabla \psi) dL + 2\alpha \int_0^{2\pi} \psi(0, \phi) d\phi, \end{aligned} \quad (53)$$

by definition of $\delta_2(\rho, \phi)$.

Generalizing to functions with possible singularity in $\rho = 0$, we arrive at a boundary condition for the solution (46) with zero potential of the form

$$\lim_{L \rightarrow 0} \frac{\int_L (\vec{n} \cdot \text{grad} \psi) \rho d\phi}{\int_0^{2\pi} \psi(\epsilon, \phi) d\phi} = 2\alpha. \quad (54)$$

Now we can formulate the approach to ZRP in two dimensions by the following algorithm. It is known that the set of iterated MT has an explicit link to Ribokur transformations. This defines solutions of the Lamé equations for coordinate systems

Generalizing (54), let us build a closed curve L as a coordinate line $\exists \epsilon > 0, a = a_0 \in [0, \epsilon], b \in [0, 1]$ by means of such a construction and define the action of $\delta_2(a, b)$ by

Lemma. 1 *The relation $\int_S \delta_2(a, b) \psi(a, b) dS = \int_0^1 \psi(0, b) db$ determines a distribution $\delta_2(a, b) \in D$, if L bounds a domain S (interior of L).*

For the proof it is enough to recall the isoperimetric inequality and the Jordan theorem; the functional linearity and continuity is obvious. Going to the set of coordinate systems a_n, b_n , numbered by the MT iteration number yields the

Theorem. 2 (Main) *The set of distributions defined by*

$$\lim_{\epsilon \rightarrow 0} \frac{\int_0^1 (\vec{n} \cdot \text{grad} \psi) db_n}{\int_0^1 \psi(a_n, b_n) db_n} = 2\alpha \quad (55)$$

is dense in a vicinity of 0.

The proof is based on the lemma and the theorem of Ganzha on local completeness of iterated Moutard transformations [E. Ganzha *On the approximation of solutions of some 2+1-dimensional integrable systems.* Sibirsk. Mat. Zh. **41:3** (2000), 541-553.]

Goursat equation, matrix ZRP and geometry of surfaces

Let us consider the Laplace equation

$$\psi_{\sigma\tau} + a(\sigma, \tau) \psi_{\sigma} + b(\sigma, \tau) \psi = 0. \quad (56)$$

The system

$$\psi_{\sigma} = p\chi, \quad \chi_{\tau} = p\psi, \quad (57)$$

is related directly to the Goursat equation

$$\psi_{\sigma\tau} = \frac{p_{\tau}}{p} \psi_{\sigma} + p^2 \psi, \quad (58)$$

with the obvious constraint between a, b in (56); , where a covariance with respect to a generalized MT was established.

In [Leble, Yurov], the matrix form of the problem for $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \chi_1 & \chi_2 \end{pmatrix}$ was introduced in the variables ξ and η as:

$$\partial_{\sigma} = \partial_{\eta} - \partial_{\xi}, \quad \partial_{\tau} = \partial_{\eta} + \partial_{\xi},$$

and rewritten (57) in the form of 2x2 Dirac system:

$$\Psi_\eta = \sigma_3 \Psi_\xi + U\Psi, \quad (59)$$

where $U = p(\xi, \eta)\sigma_1$. The functions $\psi_k = \psi_k(\xi, \eta)$, $\chi_k = \chi_k(\xi, \eta)$ with $k=1,2$ are particular solutions of (59) with some $p(\xi, \eta)$, and $\sigma_{1,3}$ are the Pauli matrices. Let $\Psi_1 \neq \Psi$ be a solution of the equation (59). We define a matrix function $\Xi \equiv \Psi_{1,\xi} \Psi_1^{-1}$. The equation (59) is covariant with respect to DT:

$$\Phi[1] = \Phi_\xi - \Xi\Phi, \quad U[1] = U + [\sigma_3, \Xi]. \quad (60)$$

Let us consider a closed 1-form

$$d\Omega = \Phi\Psi d\xi + \Phi\sigma_3\Psi d\eta.$$

Lemma. 3 *The form is exact if Ψ satisfies (59) and a 2×2 matrix function Φ solves the conjugate equation:*

$$\Phi_\eta = \Phi_\xi\sigma_3 - \Phi U. \quad (61)$$

The proof is by direct cross differentiation.

Theorem. 4 (Leble.Yurov) *One can verify by a substitution that (61) is covariant with respect to the transform if*

$$\Phi[+1] = \Omega(\Phi, \Psi_1)\Psi_1^{-1}. \quad (62)$$

Now we can alternatively affect U , by the following transformation:

$$U[+1, -1] = U + [\sigma_3, \Psi_1 \Omega^{-1} \Phi]. \quad (63)$$

The relations (62,63) we call a binary generalized Moutard transformation (BGMT).

Such a formalism gives a new possibility to define **ZRP for Dirac equation** via Darboux (60) or BGMT (62) transformation. The construction starts from a solution with a matrix potential U which directly relates to the equation (58) with constant p . Therefore we can use the solutions ψ_k of the Schrödinger equation (50) with $E = p^2$, constructed in the previous section. The matrices Ψ, Φ , are built from solutions ψ_k and $\chi_k = p^{-1}\psi_k$

Back to differential geometry

As **geometry** is concerned, the original Weierstrass formulas start with two arbitrary holomorphic functions of complex variables $z, \bar{z} \in C$. They yield an approach for constructing minimal surfaces. Generalization to the arbitrary mean curvature case was given by Kenmotsu and Konopelchenko in complex coordinates as in (46), $\tau, \sigma = -\bar{\tau}$. Here p is a real-valued function and ψ or χ as solutions of (57) are complex-valued functions. We define three real-valued functions $X_i, i = 1, 2, 3$ which are the coordinates of a surface in

\mathbb{R}^3 : $X_1 + iX_2 = 2i \int_{\Gamma} (\overline{\psi^2} d\sigma' - \overline{\chi^2} d\tau')$, $X_3 = -2 \int_{\Gamma} (\overline{\psi\chi} d\sigma')$ where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K and the mean curvature H yield:

$$ds^2 = 4N^2 d\tau d\sigma, \quad K = \frac{1}{N^2} \partial_{\tau} \partial_{\sigma} \ln N, \quad H = \frac{\sqrt{p}}{N}. \quad (64)$$

Here $N = |\psi|^2 + |\chi|^2$. Any analytic surface in \mathbb{R}^3 can be globally represented by X_i . As it is seen from the solutions nonzero N may yields zero p and hence zero mean curvature on a punctured surface [P. Exner, K. Yoshitomi].

Eq. (59) is a spectral problem for the Davey-Stewartson (DS) and Boiti-Martina-Leon-Pempinelli (BMLP) equations and produce explicitly invertible Bäcklund auto-transformations. It also induces deformations of the correspondent surfaces.

Discussion and Conclusion

The importance in applications of the pseudopotentials, introduced as distributions, lies in the possibility to solve multicenter scattering or eigenvalue problems in low energies.

The dressing procedure also may be applied to such multicenter pseudopotential.

This gives additionally ability to approximate real interaction. Technically it is applied to a combination of Green functions of the Schrödinger equation, i.e. $\psi = \sum C_i G(|\vec{r} - \vec{r}_i|)$ and, next, substituting the result, to boundary conditions in each center ($\vec{r} = \vec{r}_i$). The result is a set of algebraic equations.

One of the interesting problems is related to quantum dots, randomly distributed by place and size to be modeled by a generalized ZRPs. The theorem about a dense cover of the distribution space in a vicinity of a given point opens a way to develop new representations in potential theory. The problem of the matrix ZRP introduction is solved in an example of a two-dimensional Dirac equation. The idea of a dressing scheme is naturally generalized to other matrix problems as multi-channel scattering [Leble, Yalunin] or 4x4 matrix Dirac eigenvalue problem [Szmytkowski].

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$$F = \Psi^{-1} \Psi_{,\lambda} |_{\lambda=1}$$

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$$h(X) = XA + AX, [A, X] \neq 0, \text{ (Euler top - Manakov pair)}$$

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Conclusion The transformations of DT type give a good possibility to construct potentials together with linear problems solutions, explicit solutions of nonlinear problems being the compatibility conditions of jointly covariant linear problems, each corresponds to some geometric objects.

Problems:

1. Three-dimensional case. Tsarev S - via Laplace-like chains
2. Random Schrodinger operator models (Anderson).
3. Further development allows to include GZRP.

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Further development allows to include the additional evolution variable 'y'. As it was shown, the value of the parameter α depends on interpretation of σ . The case $\alpha = 0$ corresponds to $\sigma = \phi_x \phi^{-1}$, ϕ is the eigenfunction of the operator with eigenvalue μ ; it differs from that for $\alpha \neq 0$. For general statements and some applications see .