Canonical Coordinate Transformations in Quantum Mechanics

Part 2

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Twisted tensor product

The space of states $\mathcal{H} = L^2(\mathbb{R}^2)$ can be written as an appropriate tensor product of $\left(L^2(\mathbb{R})\right)^*$ and $L^2(\mathbb{R})$:

$$\mathcal{H} = \left(L^2(\mathbb{R})\right)^* \otimes_M L^2(\mathbb{R}),$$

$$(\varphi^* \otimes_M \psi)(x,p) = \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}y \, \varphi^* \left(x - \frac{1}{2}y\right) \psi \left(x + \frac{1}{2}y\right) e^{-\frac{i}{\hbar}py},$$

where $\varphi, \psi \in L^2(\mathbb{R})$.

Note, that functions $\varphi^* \otimes_M \varphi$ are the well known Wigner functions, being quasi-probabilistic distribution functions describing pure states of the quantum system in the phase space representation.

States

For every $\Psi \in L^2(\mathbb{R}^2)$ there holds

$$\hat{\Psi} := \sqrt{2\pi\hbar}\Psi \star = \hat{1} \otimes_M \hat{\rho},$$

where $\hat{\rho}$ is some Hilbert-Schmidt operator.

In particular, for $\Psi = \varphi^* \otimes_M \psi$ the corresponding Hilbert-Schmidt operator $\hat{\rho}$ takes the form

$$\hat{\rho} = \langle \varphi | \cdot \rangle_{L^2} \psi.$$

Observables

For every $A \in \mathcal{A}_Q$ there holds

$$\hat{A} = A \star = \hat{1} \otimes_M A_M(\hat{q}, \hat{p}),$$

where $\hat{q} = x$ and $\hat{p} = -i\hbar\partial_x$ are canonical operators of position and momentum, and $A_M(\hat{q}, \hat{p})$ denotes a symmetrically-ordered function of operators $\hat{q}$, $\hat{p}$.

Using the fact that S_T is an isomorphism of the algebra of observables $\mathcal{A}_Q$ onto the transformed algebra of observables $\mathcal{A}'_Q$, and the space of states $\mathcal{H} = L^2(\mathbb{R}^2)$ onto the transformed space of states $\mathcal{H}' = L^2(\mathbb{R}^2, \mu_T)$ we can write $\mathcal{H}'$ as the following twisted tensor product

$$\mathcal{H}' = \left(L^2(\mathbb{R})\right)^* \otimes_{M,S_T} L^2(\mathbb{R}) := S_T\left(\left(L^2(\mathbb{R})\right)^* \otimes_M L^2(\mathbb{R})\right).$$

Also states $\Psi \in \mathcal{H}'$ and observables $A \in \mathcal{A}'_Q$ take the form

$$\hat{\Psi} := \sqrt{2\pi\hbar}\Psi \star'_T \ = \hat{1} \otimes_{M,S_T} \hat{\rho},$$
$$\hat{A} = A \star'_T \ = \hat{1} \otimes_{M,S_T} A_{M,S_T}(\hat{q}', \hat{p}'),$$

where

$$A_{M,S_T}(\hat{q}', \hat{p}') := (S_T^{-1} A)_M(\hat{q}', \hat{p}').$$

The transformation T of coordinates induces a unitary operator $\hat{U}_T \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined on the Hilbert space of states of the ordinary quantum mechanics:

$$(\varphi^* \otimes_M \psi) \circ T =: (\hat{U}_T \varphi)^* \otimes_{M,S_T} \hat{U}_T \psi, \quad \varphi, \psi \in L^2(\mathbb{R}).$$

The operator $\hat{U}_T$ transforms wave functions to a new coordinate system.

For $A \in \mathcal{A}_Q$ there holds

$$A'_{M,S_T}(\hat{q}', \hat{p}') \equiv (S_T^{-1} A')_M(\hat{q}', \hat{p}') = \hat{U}_T A_M(\hat{q}, \hat{p}) \hat{U}_T^{-1},$$

where $A' = A \circ T$ and

$$(\hat{q}\psi)(x) = x\psi(x), \qquad (\hat{p}\psi)(x) = -i\hbar \frac{\mathrm{d}\psi}{\mathrm{d}x}(x),$$

$$(\hat{q}'\psi')(x') = x'\psi'(x'), \qquad (\hat{p}'\psi')(x') = -i\hbar \frac{\mathrm{d}\psi'}{\mathrm{d}x'}(x'),$$

for $\psi, \psi' \in L^2(\mathbb{R})$.

This result shows that applying the Born's quantization rule to a transformed classical observable gives an operator unitarily equivalent with an operator corresponding to an untransformed classical observable, provided that the ordering of $\hat{q}'$, $\hat{p}'$ will be appropriately changed.

From previous result follows that

$$\hat{q}' = \hat{U}_T Q_M(\hat{q}, \hat{p}) \hat{U}_T^{-1},$$
$$\hat{p}' = \hat{U}_T P_M(\hat{q}, \hat{p}) \hat{U}_T^{-1},$$

or from the other side

$$\hat{q} = \hat{U}_T^{-1} q_{M,S_T}(\hat{q}', \hat{p}') \hat{U}_T,$$
$$\hat{p} = \hat{U}_T^{-1} p_{M,S_T}(\hat{q}', \hat{p}') \hat{U}_T,$$

where $T^{-1}(x,p) = (Q(x,p), P(x,p))$ and $T(x',p') = (q(x',p'), p(x',p'))$.

The above result can be understand as follows. An operator of position corresponding to a new coordinate system can be simply defined as $Q_M(\hat{q},\hat{p})$. This operator can be written in a position representation, i.e. as an operator of multiplication by coordinate variable. More precisely, there exist a unitary operator $\hat{U}$ such that

$$\hat{U} Q_M(\hat{q}, \hat{p}) \hat{U}^{-1} = x'.$$

From previous result $\hat{U} = \hat{U}_T$.

Example

For a transformation of coordinates

$$T(x', p') = \left(-\left(f'(x')\right)^{-1} p' - \left(f'(x')\right)^{-1} g'(x'), f(x') \right)$$

generated by a function $F(x, x') = xf(x') + g(x')$, and for a linear transformation the operator $\hat{U}_T$ takes the form

$$\begin{aligned}(\hat{U}_T \varphi)(x') &= \frac{1}{\sqrt{2\pi\hbar}} \int \varphi(x) \sqrt{\left| \frac{\partial^2 F}{\partial x \partial x'}(x, x') \right|} e^{-\frac{i}{\hbar} F(x,x')} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int \varphi(x) \sqrt{|f'(x')|} e^{-\frac{i}{\hbar}(xf(x') + g(x'))} \, \mathrm{d}x.\end{aligned}$$

Example

For a point transformation

$$T(x',p') = \Big(f(x'), \big(f'(x')\big)^{-1}p' - \big(f'(x')\big)^{-1}g'(x')\Big)$$

generated by a function $F(x',p) = -pf(x') - g(x')$ the operator $\hat{U}_T$ takes the form

$$\begin{aligned}(\hat{U}_T\varphi)(x') &= \frac{1}{\sqrt{2\pi\hbar}}\int \tilde{\varphi}(p)\sqrt{\left|\frac{\partial^2 F}{\partial x'\partial p}(x',p)\right|}e^{\frac{i}{\hbar}F(x',p)}\,\mathrm{d}p \\ &= \frac{1}{\sqrt{2\pi\hbar}}\int \tilde{\varphi}(p)\sqrt{|f'(x')|}e^{-\frac{i}{\hbar}(pf(x')+g(x'))}\,\mathrm{d}p \\ &= \sqrt{|f'(x')|}e^{-\frac{i}{\hbar}g(x')}\varphi(f(x')).\end{aligned}$$

Example

Let us again consider a harmonic oscillator after a canonical transformation of coordinates

$$T(x', p') = \begin{cases} (\sqrt{|2x'|}, p'\sqrt{|2x'|}), & x' > 0 \\ (-\sqrt{|2x'|}, p'\sqrt{|2x'|}), & x' < 0 \end{cases}.$$

The Hamiltonian of the oscillator in this new coordinates takes the form

$$H'(x', p') = |x'|p'^2 + \omega^2|x'|.$$

Let us associate to H' an operator in accordance with the Born's quantization rule, remembering to use an appropriate ordering. First, we need to calculate $S_T^{-1}H'$, which gives

$$(S_T^{-1}H')(x', p') = |x'|p'^2 + \omega^2|x'| + \frac{1}{16}\hbar^2|x'|^{-1}.$$

Then $H'_{M,S_T}(\hat{q}', \hat{p}')$ being a symmetrically-ordered operator of the function $S_T^{-1}H'$ is equal

$$H'_{M,S_T}(\hat{q}', \hat{p}') = (S_T^{-1}H')_M(\hat{q}', \hat{p}') = \frac{1}{2}|\hat{q}'|\hat{p}'^2 + \frac{1}{2}\hat{p}'^2|\hat{q}'| + \omega^2|\hat{q}'| + \frac{1}{16}\hbar^2|\hat{q}'|^{-1}.$$

The operators $H_M(\hat{q},\hat{p})$ and $H'_{M,S_T}(\hat{q}',\hat{p}')$ are indeed unitarily equivalent. To check this let us calculate the action of $H'_{M,S_T}(\hat{q}',\hat{p}')$ on $\hat{U}_T\varphi_0$, where $\varphi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega x^2}{2\hbar}\right)$ is a ground state of the harmonic oscillator. One finds that

$$H'_{M,S_T}(\hat{q}',\hat{p}')\hat{U}_T\varphi_0 = \frac{1}{2}\hbar\omega\hat{U}_T\varphi_0,$$

which shows that $\hat{U}_T\varphi_0$ is an eigen-state of the transformed Hamiltonian of the oscillator, corresponding to an energy $\frac{1}{2}\hbar\omega$.

The End
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