Minimal surfaces in the soliton surfaces approach

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Outline

1. Minimal surfaces in $\mathbb{E}^3$
2. Lax pair for the Liouville equation
3. The Weierstrass representation

Outline

Minimal surfaces and their Weierstrass representation

Variation of the area functional

$$A(F+\epsilon\nu) - A(F) = -2\epsilon \int_F H\nu \cdot N \, dA + \dots,$$

F - surface in $\mathbb{E}^3$; ν - deformation field, $\nu_{|\partial F} = 0$
N - the unit normal field to the surface, H - the mean curvature

$H \equiv 0$ – minimal surface; locally

$$F = \mathrm{Re} \int_{z_0}^{z} \left(\frac{1}{2}(1-\psi^2), \frac{i}{2}(1+\psi^2), \psi \right) \eta^2 \, dz$$

ψ, η - holomorphic functions

The Liouville equation

$$u_{,z\bar{z}} = 2\mathrm{e}^{-u}$$

Given a holomorphic function ψ

$$\mathrm{e}^{-u} = \frac{|\psi_{,z}|^2}{(1 + |\psi|^2)^2}$$

Outline

Conformal immersions of surfaces in $\mathbb{H}^3(\lambda)$

$\mathbb{H}^3(\lambda) \subset \mathbb{R}^{3,1}$ hyperboloid $(X|X) = -\lambda^{-2}$ (the induced metric is positive definite and has constant sectional curvature)
$F : \mathcal{R} \to \mathbb{H}^3(\lambda)$ – a conformal immersion of the Riemann surface $\mathcal{R}$
$z = x + iy$ – local complex coordinate

$$(F_{,z}|F_{,z}) = (F_{,\bar{z}}|F_{,\bar{z}}) = 0$$

the vectors F, $F_{,z}$, $F_{,\bar{z}}$, and the unit normal N

$$(F|N) = (F_{,z}|N) = (F_{,\bar{z}}|N) = 0, \qquad (N|N) = 1,$$

form a (Gauss–Weingarten) basis in the (complexified) $\mathbb{R}^{3,1}$.
Define functions u, H and Q by

$$(F_{,z}|F_{,\bar{z}}) = \frac{1}{2}\mathrm{e}^{u}, \qquad (F_{,z\bar{z}}|N) = \frac{1}{2}H\mathrm{e}^{u}, \qquad (F_{,zz}|N) = Q$$

Conformal immersions of surfaces in $\mathbb{H}^3(\lambda)$ – cont.

Gauss-Weingarten equations of the moving frame

$$\begin{aligned} F_{,zz} =& u_{,z} F_{,z} + QN, \\ F_{,z\bar{z}} =& \frac{\lambda^2}{2} \mathrm{e}^u F + \frac{1}{2} H \mathrm{e}^u N, \\ N_{,z} =& - HF_{,z} - 2Q\mathrm{e}^{-u} F_{,\bar{z}} \end{aligned}$$

The Gauss–Mainardi–Codazzi equations

$$u_{,z\bar{z}} + \frac{1}{2}\left(H^2 - \lambda^2\right)\mathrm{e}^u - 2|Q|^2\mathrm{e}^{-u} = 0, \qquad Q_{,\bar{z}} = \frac{1}{2} H_{,z} \mathrm{e}^u$$

Remark

When $H \equiv \lambda$ and $Q \equiv 1$ we have the Liouville equation

Spinors and 2×2 representation of GW equations

Identify the Lorentz space with 2×2 hermitean matrices

$$X = (X_0, X_1, X_2, X_3) \leftrightarrow X^\sigma = X_0 \mathbf{1}_2 + \sum_{k=1}^{3} X_k \sigma_k = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}$$

$$\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We use the homomorphism $\rho : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3,1)$ given by

$$(\rho(a)X)^\sigma = a^+ X^\sigma a.$$

We will be looking for $\Phi \in \mathrm{SL}(2, \mathbb{C})$ which transforms the orthonormal basis $(\mathbf{1}_2, \sigma_1, \sigma_2, \sigma_3)$ into the orthonormal basis

$$\left(\lambda F^\sigma, \mathrm{e}^{-u/2} F^\sigma_{,x}, \mathrm{e}^{-u/2} F^\sigma_{,y}, N^\sigma\right) = \Phi^+ (\mathbf{1}_2, \sigma_1, \sigma_2, \sigma_3) \Phi.$$

2×2 linear problem

Define the $\mathfrak{sl}(2, \mathbb{C})$ valued functions U, V by

$$\Phi_{,z} = U\Phi, \qquad \Phi^{+}_{,z} = \Phi^{+} V, \tag{1}$$

then we also have

$$\Phi_{,\bar{z}} = V^{+}\Phi, \qquad \Phi^{+}_{,z} = \Phi^{+} U^{+}.$$

and the matrices U and V have the following form

$$U = \begin{pmatrix} \frac{1}{4}u_{,z} & -Q\mathrm{e}^{-u/2} \\ \frac{1}{2}\mathrm{e}^{u/2}(\lambda + H) & -\frac{1}{4}u_{,z} \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{1}{4}u_{,z} & Q\mathrm{e}^{-u/2} \\ \frac{1}{2}\mathrm{e}^{u/2}(\lambda - H) & \frac{1}{4}u_{,z} \end{pmatrix}$$

The formula for immersion

Proposition

Given solution (u, Q, H) of the GMC equations, and given $\mathrm{SL}(2,\mathbb{C})$ valued solution of the above linear system, then

$$F^{\sigma} = \frac{1}{\lambda}\Phi^{+}\Phi,$$

represents a conformal immersion in $\mathbb{H}^3(\lambda)$

Problem: In the limit $\lambda \to 0$ we have $\mathbb{H}^3(\lambda) \to \mathbb{E}^3$, but because of λ in the denominator F^{σ} blows up.

Solution: Before taking the limit we shift the origin from the center of the hyperboloid to one of its points

$$\tilde{F}^{\sigma} = \lim_{\lambda\to 0}\frac{1}{\lambda}\left(\Phi^{+}\Phi - \mathbf{1}_2\right)$$

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$H \equiv \lambda$ surfaces

Reduced GMC equations

$$u_{,z\bar{z}} - 2|Q|^2 \mathrm{e}^{-u} = 0, \qquad Q_{,\bar{z}} = 0,$$

Reduced linear problem

$$\Phi_{,z} = \begin{pmatrix} \frac{1}{4}u_{,z} & -Q\mathrm{e}^{-u/2} \\ \lambda \mathrm{e}^{u/2} & -\frac{1}{4}u_{,z} \end{pmatrix} \Phi, \qquad \Phi_{,\bar{z}} = \begin{pmatrix} -\frac{1}{4}u_{,\bar{z}} & 0 \\ \bar{Q}\mathrm{e}^{-u/2} & \frac{1}{4}u_{,\bar{z}} \end{pmatrix} \Phi$$

The same GMC equations as for minimal of surfaces in $\mathbb{E}^3$

Given two arbitrary holomorphic functions η, ψ we obtain general solution of the reduced GMC system

$$\mathrm{e}^{u/2} = \eta\bar{\eta}\left(1 + \psi\bar{\psi}\right), \qquad Q = -\eta^2 \psi_{,z}.$$

Soliton surfaces approach

Start from a Lax pair

$$\Psi_{,x} = U(\lambda)\Psi, \qquad \Psi_{,y} = V(\lambda)\Psi,$$

$U(x, y; \lambda)$, $V(x, y; \lambda)$ take values in a semisimple Lie algebra $\mathfrak{g}$

The corresponding nonlinear system (Zakharov–Shabat equations):

$$U_{,y}(\lambda) - V_{,x}(\lambda) + [U(\lambda), V(\lambda)] = 0$$

For $\Psi(\lambda)$ taking values in the Lie group G of $\mathfrak{g}$, the Sym formula

$$F(x, y; \lambda) = \Psi^{-1}(x, y; \lambda)\Psi(x, y; \lambda)_{,\lambda},$$

for fixed $\lambda \in \mathbb{R}$, gives a surface in $\mathfrak{g}$, provided the tangent vectors

$$F_{,x} = \Psi^{-1} U_{,\lambda}\Psi, \qquad F_{,y} = \Psi^{-1} V_{,\lambda}\Psi$$

are linearly independent

Outline

Solution of the linear problem

After the gauge transform $\Psi = M\Phi$, where

$$M = \frac{1}{(1+\psi\bar{\psi})^{1/2}} \begin{pmatrix} \left(\frac{\eta}{\bar{\eta}}\right)^{1/2}\psi & -\left(\frac{\eta}{\bar{\eta}}\right)^{1/2} \\ \left(\frac{\bar{\eta}}{\eta}\right)^{1/2} & \left(\frac{\bar{\eta}}{\eta}\right)^{1/2}\bar{\psi} \end{pmatrix} \in \mathrm{SU}(2).$$

the function Ψ satisfies the following linear system

$$\Psi_{,z} = \lambda\eta^2 \begin{pmatrix} \psi & -1 \\ \psi^2 & -\psi \end{pmatrix} \Psi, \qquad \Psi_{,\bar{z}} = 0.$$

whose fundamental solution is

$$\Psi(z) = \mathbf{1}_2 + \lambda \int_{z_0}^{z} dz_1 \eta(z_1)^2 \begin{pmatrix} \psi(z_1) & -1 \\ \psi(z_1)^2 & -\psi(z_1) \end{pmatrix} + \ldots$$

$$+\lambda^k \int_{z_0}^{z} dz_k \ldots \int_{z_0}^{z_2} dz_1 \prod_{i=1}^{k} \eta(z_i)^2 \prod_{i=1}^{k-1} (\psi(z_{i+1}) - \psi(z_i)) \begin{pmatrix} \psi(z_1) & -1 \\ \psi(z_1)\psi(z_k) & -\psi(z_k) \end{pmatrix} + \ldots$$

Recovering the Weierstrass representation

We need only first two terms $\Psi = \mathbf{1}_2 + \lambda\Psi_1 + \dots$

$$\tilde{F}^\sigma = \lim_{\lambda\to 0} \frac{1}{\lambda}\left(\Phi^+\Phi - \mathbf{1}_2\right) = \Psi_1 + \Psi_1^+ =$$

$$= \begin{pmatrix} \int^z \eta^2\psi d\zeta + \overline{\int^z \eta^2\psi d\zeta} & -\int^z \eta^2 d\zeta + \overline{\int^z \eta^2\psi^2 d\zeta} \\ \int^z \eta^2\psi^2 d\zeta - \overline{\int^z \eta^2 d\zeta} & -\int^z \eta^2\psi d\zeta - \overline{\int^z \eta^2\psi d\zeta} \end{pmatrix}$$