The cross ratio and its applications

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Plan

Cross ratio in projective geometry

known also as anharmonic ratio.

s – natural parameter along a line. Cross ratio:

$$(s_1, s_2; s_3, s_4) := \frac{(s_3 - s_1)(s_4 - s_2)}{(s_3 - s_2)(s_4 - s_1)} \equiv \frac{\dfrac{s_3 - s_1}{s_3 - s_2}}{\dfrac{s_4 - s_1}{s_4 - s_2}}$$

α – angle for lines in the corresponding pencil

$$(\alpha_1, \alpha_2; \alpha_3, \alpha_4) := \frac{\sin(\alpha_3 - \alpha_1)\sin(\alpha_4 - \alpha_2)}{\sin(\alpha_3 - \alpha_2)\sin(\alpha_4 - \alpha_1)}$$

Theorem: $(s_1, s_2; s_3, s_4) = (\alpha_1, \alpha_2; \alpha_3, \alpha_4)$

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Projective invariance of the cross ratio

The idea of the proof is simple. Areas of triangles are computed in two ways:

$$P_{jk} = \frac{1}{2}(s_k - s_j)h = \frac{1}{2}r_j r_k \sin(\alpha_k - \alpha_j)$$

Corollary: Cross ratio – invariant of projective transformations.

Cross ratio in Möbius geometry.

Möbius transformations: fractional linear transformations in $\mathbb{C}$.

$$(z_1, z_2; z_3, z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

Cross ratio is invariant with respect to Möbius transformations:

$$w(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0$$

- translations: $w = z + a$
- rotations: $w = e^{i\alpha} z, \quad \alpha \in \mathbb{R}$,
- dilations: $w = \lambda z, \quad \lambda \in \mathbb{R}$,
- reflection: $w = \bar{z}$,
- inversion: $w = \bar{z}^{-1}$

[Reflection and inversion change sign of the cross ratio.]

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Cross ratio in Möbius geometry
Automorphisms of the upper half-plane

Möbius transformations transform circles into circles (a straight line is considered as a degenerated circle, a circle containing $z = \infty$).

For $a, b, c, d \in \mathbb{R}$ Möbius transformation is an automorphism of the upper half-plane (preserving also geodesic lines). Indeed:

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) \equiv \frac{ad-bc}{|cz+d|}\,\mathrm{Im}z$$

Cross ratio identities

There are 4!=24 possible cross ratios of 4 points. Following identities can be directly verified:

$$(x_j, x_k; x_m, x_n) = (x_m, x_n; x_j, x_k) = (x_j, x_k; x_n, x_m)^{-1},$$

$$(x_j, x_k; x_m, x_n) + (x_j, x_m; x_k, x_n) = 1 \ ,$$

There are at most 6 different values:

$$(z_1, z_2; z_3, z_4) \equiv \lambda,$$
$$(z_1, z_2; z_4, z_3) = \lambda^{-1},$$
$$(z_1, z_3; z_4, z_2) = (1 - \lambda)^{-1},$$
$$(z_1, z_3; z_2, z_4) = 1 - \lambda,$$
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Clifford algebra $\mathcal{C}(V)$

generated by a vector space V equipped with a quadratic form $\langle \cdot | \cdot \rangle$

Clifford product satisfies: $\boldsymbol{vw} + \boldsymbol{wv} = 2\langle \boldsymbol{v} | \boldsymbol{w} \rangle \mathbf{1}, \quad (\boldsymbol{v}, \boldsymbol{w} \in V).$

$$\boldsymbol{vw} = \langle \boldsymbol{v} | \boldsymbol{w} \rangle + \boldsymbol{v} \wedge \boldsymbol{w}$$

The algebra $\mathcal{C}(V)$ ("Clifford numbers") is spanned by:

- $\mathbf{1}$ scalars
- $\mathbf{e}_k$ vectors $\mathbf{e}_k^2 = \pm 1, \quad \mathbf{e}_j \mathbf{e}_k = -\mathbf{e}_k \mathbf{e}_j \quad (k \neq j)$
- $\mathbf{e}_j \mathbf{e}_k$ (j < k) bi-vectors
- $\mathbf{e}_{k_1} \ldots \mathbf{e}_{k_r}$ $(k_1 < k_2 < \ldots < k_r)$ multi-vectors
- $\mathbf{e}_1 \mathbf{e}_2 \ldots \mathbf{e}_n$ volume element $n = p + q$

$\dim Cl_{p,q} = 2^{p+q}$, $\mathcal{C}_{p,q} \equiv \mathcal{C}(\mathbb{R}^{p,q})$.

Conformal transformations in $\mathbb{R}^n$ in terms of Clifford numbers

For $N \geqslant 3$ all conformal transformations are generated by

- Euclidean motions (translations, reflections, rotations)
- Dilations, inversions: $\vec{x}' = \lambda \vec{x}$, $\vec{x}' = \dfrac{\vec{x}}{|\vec{x}|^2}$

Translation $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{c}$
Reflection $\boldsymbol{x}' = -\boldsymbol{n}\boldsymbol{x}\boldsymbol{n}^{-1}$ [boldface: Clifford vectors]
Dilation $\boldsymbol{x}' = \lambda \boldsymbol{x}$
Inversion $\boldsymbol{x}' = \boldsymbol{x}^{-1}$
Rotation (by Cartan's theorem) is a composition of reflections:

$$\boldsymbol{x}' = \boldsymbol{n}_k \ldots \boldsymbol{n}_1 \boldsymbol{x} \boldsymbol{n}_1{}^{-1} \ldots \boldsymbol{n}_k{}^{-1}$$

The case $N = 2$: any holomorphic bijective function is conformal.

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Lipschitz group and Spin group

Lipschitz group (Clifford group) $\Gamma(V)$ is the multiplicative group (with respect to the Clifford product) generated by vectors: $\boldsymbol{v_1 v_2 \ldots v_M} \in \Gamma(V)$.

$\Gamma_0(V)$ is generated by even number of vectors.

$\mathrm{Pin}(V)$ subgroup generated by unit vectors

$\mathrm{Spin}(V)$ subgroup generated by even number of unit vectors.

$$\mathrm{Spin}(V) \subset \mathrm{Pin}(V) \subset \Gamma(V) \subset \mathcal{C}(V) ,$$

$$V \subset \mathrm{Pin}(V) , \quad \Gamma_0(V) \subset \Gamma(V) .$$

$\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are double covering of $\mathrm{O}(V)$ and $\mathrm{SO}(V)$, respectively.

The Clifford cross ratio

J. Cieśliński, The cross ratio and Clifford algebras, *Adv. Appl. Clifford Alg.* **7** (1997) 133.

For $X_k \in \mathbb{R}^n \subset \mathcal{C}(\mathbb{R}^n)$ we define:

$$Q(X_1, X_2, X_3, X_4) := (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}.$$

In general, $Q(X_1, X_2, X_3, X_4) \in \Gamma_0(\mathbb{R}^n)$.

For $X_k \in \mathbb{R}^{p,q}$ the definition is not well defined if $X_2 - X_3$ or $X_4 - X_1$ are isotropic (null, non-invertible).

Proposition. $Q(X_1, X_2, X_3, X_4)$ is real (i.e., proportional to **1**) if and only if X_1, X_2, X_3, X_4 lie on a circle or are co-linear.

Therefore the Clifford cross-ratio can be used to characterize discrete analogues of curvature nets, isothermic surfaces etc.

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Conformal covariance of the Clifford cross ratio

$$\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{c} \qquad Q \to Q$$

$$\boldsymbol{x} \to \lambda \boldsymbol{x} \qquad Q \to Q$$

$$\boldsymbol{x} \to -\boldsymbol{n}\boldsymbol{x}\boldsymbol{n}^{-1} \qquad Q \to \boldsymbol{n}Q\boldsymbol{n}^{-1}$$

$$\boldsymbol{x} \to \boldsymbol{x}^{-1} \qquad Q \to X_1 Q X_1^{-1}$$

$$\boldsymbol{x} \to A\boldsymbol{x}A^{-1} \qquad Q \to AQA^{-1}$$

Corollary: Eigenvalues of the Clifford cross ratio are invariant under all conformal transformations.

Cross ratio. Ordering conventions.

$$(x_1, x_2; x_3, x_4) \equiv (x_1 - x_3)(x_3 - x_2)^{-1}(x_2 - x_4)(x_4 - x_1)^{-1}$$

$$Q(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_2 - x_3)^{-1}(x_3 - x_4)(x_4 - x_1)^{-1}$$

$$Q(x_1, x_2, x_3, x_4) = (x_1, x_3; x_2, x_4)$$

We proceed to presenting two different applications of the cross ratio: in electrostatics (van der Pauw method) and, then, in difference geometry (circular nets, isothermic nets).

Assumptions of the van der Pauw method

Van der Pauw method (1958) is a standard method to measure resistivity of flat thin conductors.

- Flat, very thin, conducting sample
- Homogeneous, isotropic
- Arbitraty shape without holes (i.e., simply connected)
- Four point contacts on the circumference: A, B, C, D.

Method easily accessible for undergraduate students (provided that the van der Pauw formula is taken for granted).

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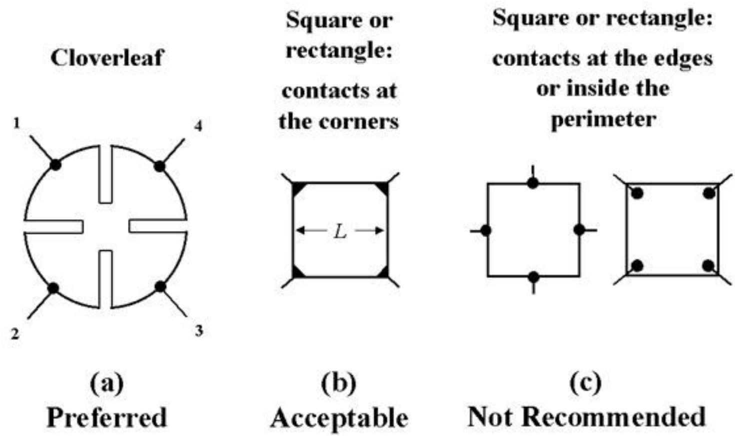
The van der Pauw method. Typical samples.



Measurements, notation, the van der Pauw formula

Two measurements:

- Current J_{AB}, voltage $U_{CD} = \Phi_D - \Phi_C$, $R_{AB,CD} = \dfrac{U_{CD}}{J_{AB}}$,
- Current J_{BC}, voltage $U_{DA} = \Phi_A - \Phi_D$, $R_{BC,DA} = \dfrac{U_{DA}}{J_{BC}}$,

Van der Pauw formula (σ is to be determined):

$$e^{-\pi d \sigma R_{AB,CD}} + e^{-\pi d \sigma R_{BC,DA}} = 1$$

σ conductivity, ρ resistivity , $\sigma = \frac{1}{\rho}$,
d thickness of the sample

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Main idea of van der Pauw

- Computations are easy for (infinite) half-plane [intuitive physics].
- Exact positions of A, B, C, D are not needed, their order is sufficient.
- The result does not depend on the shape of the sample (if simply connected) [proof: advanced mathematics].

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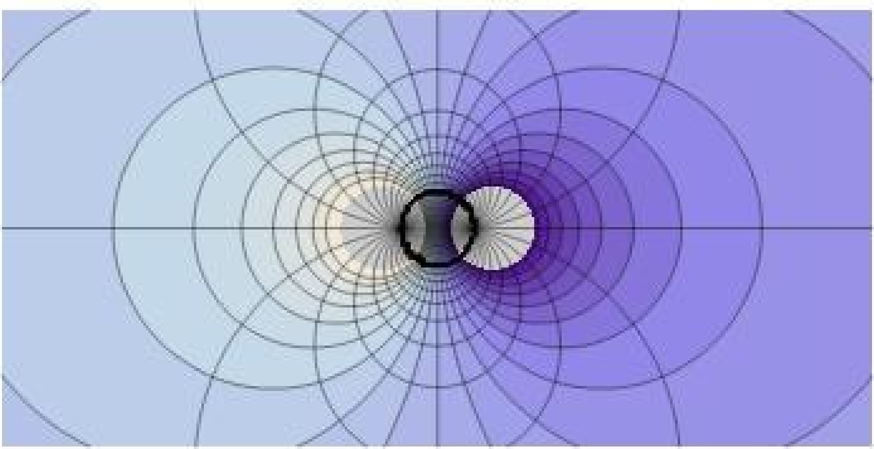
Potential distribution on the conducting plane

(thin, homogeneous, isotropic: current enters at $z = A$ and flows out at $z = B$).

- Ohm's law: $\vec{j} = -\sigma \operatorname{grad}\Phi$, Φ potential.
- Current J entering at $z = A$ flows symmetrically to ∞.
- Conservation of electric charge: $2\pi r\, d\, j = J$, $r = |z - A|$.
- $$\frac{\partial \Phi}{\partial r} = -\rho j \quad \Rightarrow \qquad \Phi_1(z) = -\frac{J\rho}{2\pi d} \ln|z - A|.$$
- For J flowing out at $z = B$, $\Phi_2(z) = \frac{J\rho}{2\pi d} \ln|z - B|$.
- Finally, $\Phi(z) = \Phi_1(z) + \Phi_2(z) = \frac{J\rho}{2\pi d} \ln\left|\frac{z - B}{z - A}\right|$

Equipotential lines and current lines.

Conducting plane. Current flows in at $z = -1$ and flows out at $z = 1$.



Conducting half-plane.

Current J flows in at $z = x_1$ and flows out at $z = x_2$, (where x_1, x_2 are real).

Conducting plane: real axis is a symmetry axis (and a current line).

Potential for conducting half-plane is the same as for the plane with the current $2J$ (dividing equally into two half-planes).

$$\Phi(z) = \frac{2J\rho}{2\pi d} \ln \left| \frac{z - x_2}{z - x_1} \right| .$$

We compute: $R_{12,34} = \frac{\Phi(x_4) - \Phi(x_3)}{J_{12}}$. Hence (van der Pauw):

$$R_{12,34} = \frac{\rho}{\pi d} \ln \left| \frac{x_4 - x_2}{x_4 - x_1} \cdot \frac{x_3 - x_1}{x_3 - x_2} \right| = \frac{\rho}{\pi d} \ln \frac{(a+b)(b+c)}{b(a+b+c)} ,$$

where $a = x_2 - x_1, \; b = x_3 - x_2, \; c = x_4 - x_3.$

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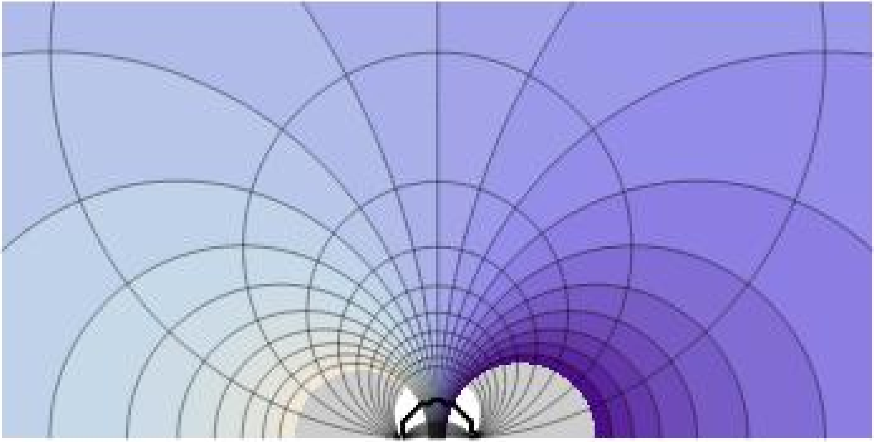
Equipotential lines and current lines.

Conducting half-plane. Current flows in at $z = -1$ and flows out at $z = 1$.



Complex potential is holomorphic

Potencjał $\Phi(z)$ satisfies

$$\Delta\Phi = \rho J \delta(z - x_2) - \rho J \delta(z - x_1),$$

where δ is the Dirac delta. Therefore, we may define

$$F(z) = \Phi(x, y) + i\,\Psi(x, y)$$

where Ψ is determined (up to a constant) from Cauchy-Riemann conditions:

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \quad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}.$$

F is holomorphic outside sigular points x_1, x_2.

Explicit form of the complex potential
for the upper half-plane.

$$\Phi(z) \equiv \mathrm{Re} F(z) = \frac{J\rho}{\pi d} \ln \left| \frac{z - x_2}{z - x_1} \right| , \qquad F(z) = \frac{J\rho}{\pi d} \ln \frac{z - x_2}{z - x_1}$$

$$\Psi(z) = \mathrm{Im} F(z) = \frac{J\rho}{\pi d} \left(\mathrm{Arg}(z - x_2) - \mathrm{Arg}(z - x_1) \right)$$

Boundary conditions:

$$z \in (-\infty, x_1) \cup (x_2, \infty) \quad \Longrightarrow \quad \Psi(z) = 0,$$

$$z \in (x_1, x_2) \quad \Longrightarrow \quad \Psi(z) = \frac{J\rho}{d} = \text{const},$$

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Riemann mapping theorem

Twierdzenie: Any simply connected region of the complex plane (except the whole complex plane) is conformally equivalent to the unit open disc.

Region: an open and connected subset of $\mathbb{C}$.
Conformal map: preserves angles.

Conformal mappings of the complex plane:

biholomorphic functions (i.e., the inverse function is also holomorphic).

Any holomorphic function $w = f(z)$ (i.e., $u + iv = f(x + iy)$) such that $f'(z) \neq 0$ is a conformal map.

(Counter)example: $f(z) = z^2$ is not conformal at $z = 0$, because the angle between lines through $z = 0$ is doubled after this transformation.

$N = 2$: conformal transformations are biholomorphic maps. Of special interest is a subgroup of Möbius transformations.

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Disc and half-plane are conformally equivalent.

Function $w \mapsto z = i\,\dfrac{w+i}{i-w}$ is a biholomorphic map of the unit disc ($|w| \leqslant 1$) onto the upper half-plane ($\mathrm{Im} z \geqslant 0$).

The inverse function $z \mapsto w = i\,\dfrac{z-i}{z+i}$.

w	1	i	-1	$-i$
z	1	∞	-1	0

All conformal maps of the upper half plane onto the unit disc:

$$w = e^{i\theta}\frac{z-a}{z-\bar{a}}, \qquad \mathrm{Im} a > 0, \quad \theta \in \mathbb{R}.$$

Christofell-Schwarz theorem:
Any polygon can be conformally mapped onto the upper half-plane.

Explicit formula for the map of the upper half-plane onto n-gon with angles: $k_1\pi, k_2\pi, \ldots, k_n\pi$ (where $k_1 + k_2 + \ldots + k_n = n - 2$):

$$w(z) = A\int_{z_0}^{z} (\zeta - x_1)^{k_1-1} \ldots (\zeta - x_n)^{k_n-1} d\zeta + B$$

$$\frac{dw}{dz} = A(z - x_1)^{k_1-1} \ldots (z - x_n)^{k_n-1}$$

Note that for $z = x \in (x_j, x_{j+1})$ the phase of $\frac{dw}{dz}$ is constant! Therefore, indeed, this segment is mapped into a line segment: $w(x) = e^{i\theta}\int^{x} |w'(x)|dx$.

Conformal covariance of current lines and equipotential lines

A conformal (= biholomorphic) transformation: $z \mapsto w = f(z)$.

$F(z) = \Phi + i\Psi$ transforms as a scalar, i.e.,

$$\tilde{F}(w) = F(f^{-1}(w))$$

thus $\tilde{F}$ is holomorphic, and $\tilde{\Phi}$, $\tilde{\Psi}$ satisfy a Poisson equation (Laplace equation outside singular points $w_1 = f(x_1)$, $w_2 = f(x_2)$).

The same boundary conditions: $\Psi(z) = 0, \Psi(z) = J\rho/d$.

Corollary: $\tilde{\Phi}$, $\tilde{\Psi}$ yield potential and current lines in the transformed region.

The case non-simply connected is much more difficult.

Typical counterexample: conformal mappings of an annulus.

Annulus: $\{z : r < |z - z_0| < R\}$ has a single “hole”.

Theorem: Two annuli, defined by r_1, R_1 and r_2, R_2, respectively, are conformally equivalent iff $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.

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Van der Pauw's formula can be reformulated in terms of the cross ratio:

$$R_{12,34} = \frac{\rho}{\pi d} \ln \left| (x_1, x_2; x_3, x_4) \right|$$

Van der Pauw does not mention at all the cross ratio. His motivation came from electrodynamics (*reciprocity theorem of passive multipoles*). The same concerns other authors who developed or worked with the van der Pauw method.

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Sign of the cross ratio

for points lying on the real axis, i.e., $z_k \equiv x_k$.

If segments $[x_1, x_2]$ i $[x_3, x_4]$ partially overlap (have a common segment), then

$$(x_1, x_2; x_3, x_4) < 0.$$

If these segments are disjoint or one is contained inside the other one, then

$$(x_1, x_2; x_3, x_4) > 0.$$

If any of these segments degenerates to a point, then

$$(x_1, x_2; x_3, x_4) = 1,$$

which means that $\ln(x_1, x_2; x_3, x_4) = 0$.

A new formula for the van der Pauw method

Following van der Pauw, we assume (to fix an attention):
$x_1 < x_2 < x_3 < x_4$. Then:

$$(x_1, x_2; x_3, x_4) > 0, \quad (x_1, x_4; x_3, x_2) > 0, \quad (x_1, x_3; x_2, x_4) < 0.$$

We recall:

$$(x_1, x_2; x_3, x_4)^{-1} + (x_1, x_4; x_3, x_2)^{-1} = 1, \quad \leftarrow \text{ van der Pauw}$$

$$(x_1, x_2; x_3, x_4) + (x_1, x_3; x_2, x_4) = 1. \quad \leftarrow \text{ new formula?}$$

$$\exp\left(-\frac{\pi d R_{12,34}}{\rho}\right) + \exp\left(-\frac{\pi d R_{14,32}}{\rho}\right) = 1$$

$$\exp\frac{\pi d R_{12,34}}{\rho} - \exp\frac{\pi d R_{13,24}}{\rho} = 1 \quad \leftarrow \text{ new formula!}$$

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Was our “new identity” known to van der Pauw?

From identities: $(z_1, z_2; z_3, z_4)^{-1} + (z_1, z_4; z_3, z_2)^{-1} = 1$, $(z_1, z_2; z_3, z_4) + (z_1, z_3; z_2, z_4) = 1$, it follows

$$(z_1, z_3; z_2, z_4) = -(z_1, z_2; z_3, z_4)(z_1, z_4; z_3, z_2)^{-1}.$$

Rewriting it in terms of $R_{ij,kl}$

$$\exp \frac{\pi d R_{13,24}}{\rho} = \exp \frac{\pi d R_{12,34}}{\rho} \exp \left(-\frac{\pi d R_{14,32}}{\rho} \right)$$

we obtain an identity known to van der Pauw:

$$R_{13,24} = R_{12,34} - R_{14,23}.$$

Some consequences of van der Pauw formulas
(old and new)

$$\exp\left(-\frac{\pi d R_{12,34}}{\rho}\right) + \exp\left(-\frac{\pi d R_{14,32}}{\rho}\right) = 1$$

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Assuming (without loss of generality): $R_{12,34} > R_{14,32}$
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Determining ρ by the Banach fixed point method

J.L.Cieśliński, preprint arXiv (2012).

We use the new formula: $\exp\frac{\pi d R_{12,34}}{\rho} - \exp\frac{\pi d R_{13,24}}{\rho} = 1.$

$$\sigma = \frac{\ln\left(1 + \exp(\pi d R_{13,24}\sigma)\right)}{\pi d R_{12,34}}, \qquad \sigma = \frac{1}{\rho}.$$

Banach fixed point theorem: $\sigma = F(\sigma)$.

$$F'(\sigma) = \frac{k}{1 + \exp\left(-\pi d R_{13,24}\sigma\right)}, \qquad k = \frac{R_{13,24}}{R_{12,34}}$$

$0 \leqslant k < 1$, therefore $|F'(\sigma)| < 1$, i.e., the iteration procedure is convergent (usually fast).

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Conclusions

- Van der Pauw formula can be expressed by the cross ratio.
- We found a modification of the van der Pauw formula, solvable by the fast convergent fixed point iteration.
- A work on van der Pauw method for samples with a hole is in progress (Szymański-Cieśliński-Łapiński).

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Discrete nets

Discrete nets: maps $F : \mathbb{Z}^n \to \mathbb{R}^m$.

The case $n = 2$: discrete surfaces immersed in $\mathbb{R}^m$.

The map $\mathbb{R}^n \to \mathbb{R}^m$, obtained in the continuum limit from a discrete net, corresponds to a specific choice of coordinates on some smooth surface.

Notation. Forward and backward shift:

$$T_j\, f(m^1, \ldots, m^j, \ldots, m^n) = f(m^1, \ldots, m^j + 1, \ldots, m^n) \ ,$$
$$T_j^{-1}\, f(m^1, \ldots, m^j, \ldots, m^n) = f(m^1, \ldots, m^j - 1, \ldots, m^n) \ .$$

Special classes of *integrable* discrete nets

Bobenko-Pinkall, Doliwa-Santini, Cieśliński-Doliwa-Santini, Nieszporski, ...

Discrete asymptotic nets: any point F and its all four neighbours ($T_1 F$, $T_2 F$, $T_1^{-1} F$, $T_2^{-1} F$) are co-planar.

Discrete pseudospherical surfaces: asymptotic, Chebyshev (segments joining the neighbouring points have equal lengths).

Discrete conjugate nets: planar elementary quadrilaterals. Conjugate nets: the second fundamental form is diagonal.

Circular nets (every quadrilateral is inscribed into a circle) correspond to curvature lines (fundamental forms are diagonal).

Discrete isothemic nets: the cross-ratio for any elementary quadrilateral is a negative constant. Isothermic immersions: curvature lines admit conformal parameterization.

Circular nets have scalar cross ratios

Elementary quadrilateral: four points $F, \ T_k F, \ T_j F, \ T_k T_j F$.

Sides of the quadrilateral: $D_k F$, $D_j F$, $T_k D_j F$, $T_j D_k F$, where $D_k F := T_k F - F$. We define

$$Q_{kj}(F) := Q(F, T_k F, T_{kj} F, T_j F) = (D_k F)(T_k D_j F)^{-1}(T_j D_k F)(D_j F)^{-1}$$

Proposition. The net $F = F(m^1, \ldots, m^n)$ is a circular net if and only if $Q_{kj}(F) \in \mathbb{R}$ for any $k, j \in \{1, \ldots, n\}$.

Circular nets by the Sym formula

We consider the Clifford algebra $\mathcal{C}(V \oplus W)$, where $\dim V = q, \dim W = r$. Let Ψ satisfies:

$$T_j \Psi = U_j \Psi \ , \quad (j = 1, \ldots, n)$$

where $n \leqslant q$, and $U_j = U_j(m^1 \ldots, m^n, \lambda) \in \Gamma_0(V \oplus W)$ have the following Taylor expansion around a given λ_0:

$$U_j = \mathbf{e}_j B_j + (\lambda - \lambda_0)\mathbf{e}_j A_j + (\lambda - \lambda_0)^2 C_j \ldots \ ,$$

$$A_j \in W, \quad B_j \in V, \quad \mathbf{e}_j \in V \ ,$$

A_j, B_j are invertible, and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are orthogonal unit vectors.

Circular nets by the Sym formula (continued)

Proposition. Discrete net F, defined by the Sym-Tafel formula

$$F = \Psi^{-1}\Psi_{,\lambda}|_{\lambda=\lambda_0} \ ,$$

(where Ψ solves the above linear problem and mild technical assumptions that at least at a single point $m_0^1, \ldots, m_0^n$ we have: $\Psi(m_0^1, \ldots, m_0^n, \lambda_0) \in \Gamma_0(V)$, $\Psi(m_0^1, \ldots, m_0^n, \lambda) \in \Gamma_0(V \oplus W)$), can be identified with a circular net in $V \wedge W$ provided that

$$A_k(T_k A_j)^{-1}(T_j A_k)A_j^{-1} \in \mathbb{R} \ .$$

Namely: $(F(m^1, \ldots, m^n) - F(m_0^1, \ldots, m_0^n)) \in V \wedge W$.

Special projections generalizing Sym's approach.

Let $P : W \to \mathbb{R}$ is a projection ($P^2 = P$).

We extend its action on $V \wedge W$ (in order to get $P : V \wedge W \to V$) in a natural way. Namely, if $\mathbf{v}_k \in V$ and $\mathbf{w}_k \in W$, then

$$P(\sum_k \mathbf{v}_k \mathbf{w}_k) := \sum_k P(\mathbf{w}_k)\mathbf{v}_k \ .$$

Proposition. Let P is a projection and F is defined by the Sym-Tafel formula. Then $P(F)$ is a circular net.

Discrete isothermic surfaces in $\mathbb{R}^q$

$$U_j = \mathbf{e}_j B_j + \lambda \mathbf{e}_j A_j \qquad (j = 1, 2) \; ,$$

$$A_j \in W \simeq \mathbb{R}^{1,1}, \qquad B_j \in V \simeq \mathbb{R}^q, \qquad \mathbf{e}_j \in V \; ,$$

$$(C_j = 0 \; , \quad \lambda_0 = 0) \; ,$$

projection: $P(\mathbf{e}_{q+1}) = 1 \; , \quad P(\mathbf{e}_{q+2}) = \pm 1 \; .$

Smooth isothermic immersions admit isothermic (isometric) parameterization of curvature lines. In these coordinates $ds^2 = \Lambda(dx^2 + dy^2)$ and the second fundamental form is diagonal.

Discrete Guichard nets in $\mathbb{R}^q$

Conjecture:

$$U_j = \mathbf{e}_j B_j + \lambda \mathbf{e}_j A_j \ ,$$

$$A_j \in W \simeq \mathbb{R}^{2,1}, \qquad B_j \in V \simeq \mathbb{R}^q, \qquad \mathbf{e}_j \in V \ ,$$

$$(C_j = 0 \ , \quad \lambda_0 = 0) \ ,$$

$$P(\mathbf{e}_{q+1}) = \cos\varphi_0, \ P(\mathbf{e}_{q+2}) = \sin\varphi_0, \ P(\mathbf{e}_{q+3}) = \pm 1.$$

Guichard nets in $\mathbb{R}^3$ are characterized by the constraint $H_1^2 + H_2^2 = H_3^2$, where H_j are Lamé coefficients, i.e., $ds^2 = H_1^2 dx^2 + H_2^2 dy^2 + H_3^2 dz^2$.

Discretization of some class of orthogonal nets in $\mathbf{R}^n$

$$U_j = \mathbf{e}_j B_j + \lambda \mathbf{e}_j A_j \ ,$$

$$A_j \in W \simeq \mathbb{R}^n, \qquad B_j \in V \simeq \mathbb{R}^n, \qquad \mathbf{e}_j \in V \ ,$$

$$(C_j = 0 \ , \quad \lambda_0 = 0) \ ,$$

$$P(\mathbf{e}_{n+k}) = 1, \quad P(\mathbf{e}_{n+j}) = 0 \ \ (j \neq k), \quad k\text{– fixed}$$

This class in the smooth case is defined by the constraint $H_1^2 + \ldots + H_n^2 = \text{const}$, where $ds^2 = H_1^2 (dx^1)^2 + \ldots + H_n^2 (dx^n)^2$.

Discrete Lobachevsky n-spaces in $\mathbb{R}^{2n-1}$

$$U_j = \mathbf{e}_j \left(\frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) A_j + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) P_j + Q_j \right) ,$$

$$V = V_1 \oplus V_2 , \quad V_1 \simeq \mathbb{R}^n , \quad V_2 \simeq \mathbb{R}^{n-1} , \quad W \simeq \mathbb{R} , \quad \lambda_0 = 1 ,$$

$$\mathbf{e}_j \in V_1 , \quad Q_j \in V_1 , \quad P_j \in V_2 , \quad A_j \in W , \quad P_j + Q_j = B_j .$$

In the continuum limit we get immersions with the constant negative sectional curvature (Lobachevsky spaces).

Conclusions and open problems

- Cliford cross ratio is a convenient tool to study circular nets (discrete analogue of curvature lines).
- Open problem: find purely geometric characterization of discrete nets generated by the Sym formula
- Open problem: Clifford cross ratio identities.
- Plans: Darboux-Bäcklund transformations and special solutions.

Thank you for attention!

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Original sources on the van der Pauw method

Leo J. van der Pauw,
A method of measuring specific resistivity and Hall effect of discs of arbitrary shape,
Philips Research Reports 13 (1958) 1-9.

Leo J. van der Pauw,
A method of measuring the resistivity and Hall coefficient on lamellae of arbitrary shape,
Philips Technical Review 20 (1958) 220-224.

Both papers available through Wikipedia. Moreover, perpaps:

L.V. Bewley.
Two-dimensional fields in electrical engineering.
MacMillan, New York 1948.

Generalization of the Riemann mapping theorem.

Uniformization theorem [Riemann-Poincaré-Koebe]

Uniformization theorem extends the Riemann mapping theorem on Riemann surfaces:

Any simply connected Riemann surface is conformally equivalent (biholomorphically isomorphic) to

- unit disc $|z| < 1$
- complex plane $\mathbb{C}$
- Riemann sphere $\mathbb{C}P(1)$